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REMARK ON THE DUAL OF AN INTERPOLATION SPACE

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0. Introduction.

Let $\vec{A} = \{A_0, A_1\}$ be a Banach couple (that is A_0 and A_1 are two Banach spaces both continuously embedded in some Hausdorff topological vector space $\mathscr{A}$). The interpolation space ("K-space") $\vec{A}_{\theta q} = (A_0, A_1)_{\theta q}$, where $0 < \theta < 1$, $0 < q < \infty$, is defined as the subspace of $\Sigma = \Sigma(\vec{A}) = A_0 + A_1$ given by the condition

$$(0.1) \qquad \|a\|_{\vec{A}_{\theta q}} = \left(\int_0^\infty (K(t,a)/t^\theta)^q \, dt/t\right)^{1/q} < \infty ,$$

with

$$(0.2) \qquad K(t,a) = \inf_{a = a_0 + a_1} (\|a_0\|_{A_0} + t\|a_1\|_{A_1})$$

(cf. Butzer & Berens [1, chapter III, in particular p. 167]). If $1 \leqq q \leqq \infty$ $\vec{A}_{\theta q}$ is a Banach space (with the norm given by (0.1)) but if $0 < q < 1$ in general only a quasi-Banach space (see Section 1). If

$$(*) \qquad \Delta = \Delta(\vec{A}) = A_0 \cap A_1 \text{ is dense in both } A_0 \text{ and } A_1$$

one can consider the dual couple $\vec{A}' = \{A_0', A_1'\}$, and it is possible to make the identification:

$$(0.3) \qquad (\vec{A}_{\theta q})' \approx (A')_{\theta q'} \quad \text{if } 1 \leqq q < \infty, \text{ with } 1/q + 1/q' = 1$$

(cf. [1, p. 214]; the first version of this result is due to Lions [6]). If $q = \infty$ we have the following substitute for (0.3)

$$(0.4) \qquad (\vec{A}^{\Delta}_{\theta\infty})' \approx (\vec{A}')_{\theta 1}$$

where, generally speaking, E^{Δ} denotes the closure of Δ in E (cf. Scherer [7, p. 17]). In this note we shall be dealing with the case $0 < q < 1$. Indeed we shall show that (Section 2):

$$(0.5) \qquad (\vec{A}_{\theta q})' \approx (\vec{A}')_{\theta\infty} \quad \text{if } 0 < q < 1 .$$

We shall also give (Section 3) two simple illustrations of the relation (0.5).

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1. Preliminaries on quasi-Banach spaces.

By a quasi-Banach space we mean a complete topological vector space E (over the reals, say) with the topology given by a quasi-norm $\|x\|$, that is we have

$$\|x+y\| \leqq \varkappa(\|x\|+\|y\|) \quad \text{for some } \varkappa \geqq 1$$

(quasi-triangle inequality),

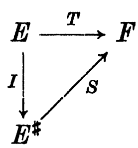
$$\|\lambda x\| = |\lambda|\,\|x\|\,,$$

$$\|x\| \geqq 0,\ \|x\| = 0 \Leftrightarrow x = 0\,.$$

With any quasi-Banach space E we associate a Banach space $E^{\#}$ as follows. In E we define a semi-norm $\|x\|^{\#}$ by the formula

$$\|x\|^{\#} = \inf\{\textstyle\sum_{\nu=1}^{n} \|x_\nu\| \mid x=\sum_{\nu=1}^{n} x_\nu\}\,.$$

Let N be the subspace of E defined by $\|x\|^{\#}=0$. Then $E^{\#}$ is the completion of the quotient space E/N with the quotient norm induced by $\|x\|^{\#}$. Clearly E has the following universal property: If $T: E \to F$ is a continuous linear mapping, F being a Banach space, then there exists a continuous linear mapping $S: E^{\#} \to F$ such that we have the factorisation



$I: E^{\#} \to E$ being the canonical quotient mapping. In particular taking F to be the scalar field we see that

$$(E^{\#})' \approx E'\,. \tag{1.1}$$

It is this relation that will be exploited in what follows.

EXAMPLE 1.1 (cf. Haaker [4]). If $E=L_p$, $0<p<1$, we have $E^{\#}=0$ and consequently (by (1.1)) $E'=0$ (Day [2]). If $E=l_p$, $0<p<1$, we have $E^{\#}=l_1$ and consequently $E'=l_\infty$.

2. A variant of the Lions duality theorem.

We return to our Banach couple $\vec{A}$ (see Section 0), maintaining notably the density assumption (*). We can now prove

THEOREM 2.1. *If* $0<q<1$ *we have* $(\vec{A}_{\theta q})^{\#} \approx \vec{A}_{\theta 1}$.

In particular (using (1.1) and (0.4) with $q=1$) we see that (0.5) holds true.

PROOF. Since $\vec{A}_{\theta q} \subset \vec{A}_{\theta 1}$ if $0<q<1$ we have

$$(\vec{A}_{\theta q})^{\#} \subset (\vec{A}_{\theta 1})^{\#} = A_{\theta 1} .$$

There remains thus only the opposite inequality

$$\vec{A}_{\theta 1} \subset (\vec{A}_{\theta q})^{\#} . \tag{2.1}$$

To prove (2.1) we first note the convexity inequality (cf. [1, p. 176] for $q \geqq 1$; the proof is the same if $0<q<1$)

$$\|a\|_{\vec{A}_{\theta q}} \leqq C \|a\|_{A_0}{}^{1-\theta} \|a\|_{A_1}{}^{\theta} .$$

Now the same inequality holds also with $\|a\|_{\vec{A}_{\theta q}}$ replaced by $\|a\|_{(\vec{A}_{\theta q})^{\#}}$. And this clearly implies (2.1), by the following well-known lemma (cf. [1, pp. 176–177]).

LEMMA 2.1. *Let E be any Banach space (contained in Σ) satisfying for some θ, $0<\theta<1$,*

$$\|a\|_E \leqq C \|a\|_{A_0}{}^{1-\theta} \|a\|_{A_1}{}^{\theta}$$

Then $\vec{A}_{\theta 1} \subset E$.

3. Two examples.

We give two simple illustrations of (0.5). These special cases can also easily be treated directly (see Haaker [4] and Flett [3][1]) respectively.

EXAMPLE 3.1. (Lorentz space). If L_{pq}, where $0<p\leqq\infty$, $0<q\leqq\infty$, are the Lorentz spaces (say, on [0, 1]) we have as is well-known

$$L_{pq} = (L_1, L_\infty)_{\theta q} \quad \text{if } \theta = 1-1/p,\ 1<p<\infty,\ 0<q\leqq\infty .$$

By the Lions theorem (see (0.3)) we have

$$L'_{pq} \approx L_{p'q'} \quad \text{if } 1<p<\infty,\ 1\leqq q<\infty \tag{3.1}$$

and thus in particular

$$L'_{p1} \approx L_{p'\infty} \quad \text{if } 1<p<\infty .$$

Using (0.5) we now can complement (3.1) by

$$L'_{pq} \approx L_{p'\infty} \quad \text{if } 1<p<\infty,\ 0<q<1 . \tag{3.2}$$

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[1] I owe this reference to professor H. Triebel.

EXAMPLE 3.2. (Besov space). If B_p^{sq}, where $-\infty < s < \infty$, $1 \leqq p \leqq \infty$, $0 < q \leqq \infty$) are the Besov (=Lipschitz) spaces (say, on R^n) we have, using any of the many equivalent definitions,

$$B_p^{sq} = (L_p, W_p^m)_{\theta q} \quad \text{if } \theta = s/m,\ 0 < s < m \text{ integer},\ 1 < p < \infty, \quad 0 < q \leqq \infty,$$

where W_p^m denote the Sobolev spaces. This thus for $s > 0$. A corresponding formula holds for $s < 0$. Also the case $s = 0$ can be incorporated. It follows from (0.3) that

$$(3.3) \qquad (B_p^{sq})' \approx B_{p'}^{-sq'} \quad \text{if } 1 < p < \infty,\ 1 \leqq q < \infty,$$

which is well-known. Using (0.5) we get the following complement of (3.3):

$$(3.4) \qquad (B_p^{sq})' \approx B_{p'}^{-s\infty} \quad \text{if } 1 < p < \infty,\ 0 < q < 1.$$

REMARK 3.1. The spaces B_p^{sq} in the case $0 < q < 1$ appear also already in classical analysis. Let $\mathscr{F}$ be the Fourier transform on R^n. Then a theorem by Zygmund (see [8, vol. 1, p. 242]) may be restated as follows:

$$(3.5) \qquad \mathscr{F}:\ W_1^n \cap B_\infty^{0,1/q} \to L_1.$$

More generally one may prove that

$$(3.6) \qquad \mathscr{F}:\ B_{p_0}^{s_0 q_0} \cap B_{p_1}^{s_1 q_1} \to L_1$$

with

$$(1-\theta)/q_0 + \theta/q_1 = 1,$$

$$s_0(1-\theta) + s_1\theta = n/2, \quad (1-\theta)/p_0 + \theta/p_1 = \tfrac{1}{2},$$

where clearly must hold either $q_0 < 1$ or $q_1 < 1$ (cf. Izumi & Izumi [5] for results in this sense).

Summary.

It is shown that $(A_0, A_1)_{\theta_q} \approx (A_0', A_1')_{\theta\infty}$ if $0 < q < 1$. This result is applied to the dual of the spaces L_{pq} and B_p^{sq} in the case $0 < q < 1$.

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