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L^p ESTIMATES FOR CONVOLUTION OPERATORS DEFINED BY COMPACTLY SUPPORTED DISTRIBUTIONS IN R^n

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Introduction.

Let R^n be the n-dimensional euclidean space where $x=(x_1\ldots x_n)$ are coordinate vectors and consider also a dual copy of R^n with coordinate vectors $\xi=(\xi_1\ldots\xi_n)$. Let $|x|=(x_1^2+\ldots+x_n^2)^{\frac{1}{2}}$ be the euclidean length and $(x,\xi)=x_1\xi_1+\ldots+x_n\xi_n$ the scalar product.

If $v\in\mathscr{E}'(\mathsf{R}^n)$, that is if v is a distribution with a compact support in R^n, then its Fourier Transform

$$\hat{v}(\xi) = v_x(e^{-i(x,\xi)})$$

exists. Associated with v is the convolution operator T_v, where

$$T_v(f)(x) = (2\pi)^{-n}\int e^{i(x,\xi)}\hat{v}(\xi)\hat{f}(\xi)\,d\xi = (f*v)(x)$$

is defined for every $f\in C_0^\infty(\mathsf{R}^n)$.

Let $\|\cdot\|_p$ be the norm in $L^p(\mathsf{R}^n)$.

THEOREM 1. *Let $v\in\mathscr{E}'(\mathsf{R}^n)$ and suppose that $|\hat{v}(\xi)|\leqq(1+|\xi|^2)^{-\alpha}$ is valid for some $0<\alpha<n/4$. Then*

$$\|T_v(f)\|_p \leqq C(v,\alpha,n)\|f\|_p$$

for every $f\in C_0^\infty(\mathsf{R}^n)$ and where $p=2n/(n+4\alpha)$.

The proof is based upon a consideration of the limit case when $\alpha=n/4$ and a fairly explicit estimate of $C(v,\alpha,n)$ arises from the proof. Observe that when $\alpha>n/4$ then $\int|\hat{v}(\xi)|^2d\xi<\infty$ and hence v already exists as a compactly supported L^2-function and the result in Theorem 1 becomes trivial and is even true when $p=1$.

THEOREM 2. *There exists a constant A_n such that if $v\in\mathscr{E}'(\mathsf{R}^n)$ satisfies*

$$|\hat{v}(\xi)| \leqq (1+|\xi|^2)^{-n/4}$$

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and if

$$\delta(v) = \sup\{|x-y| : x,y \in \operatorname{supp}(v)\},$$

then

$$\|T_v(f)\|_p \leqq A_n(1+\delta(v))^{n(1/p-1/2)}(p-1)^{-1}\|f\|_p,$$

for every $1<p<2$.

The two results above turn out to be easy consequences of the powerful methods developed in [1]. We prove (an inproved version of) the limit case first and deduce Theorem 1 by Complex Interpolation. In Section 3 the corresponding periodic case is described and we prove by explicit examples that Theorem 1 is sharp.

1. The case when $\alpha = n/4$.

We refer to [1] for the definition and the basic properties of the two spaces $H^1(\mathsf{R}^n)$ and $\mathrm{BMO}(\mathsf{R}^n)$. The result below is proved in [1, p. 149, Corollary 1].

LEMMA 1.1 (Fefferman-Stein). *Let* $v \in \mathscr{E}(\mathsf{R}^n)$ *and suppose that*

$$\|v*f\|_{\mathrm{BMO}} \leqq A_v\|f\|_\infty$$

for every $f \in C_0^\infty(\mathsf{R}^n)$ *and some constant* A_v. *Then*

$$\|v*f\|_{H^1} \leqq C_n A_v \|f\|_{H^1}$$

for an absolute constant C_n.

The (well-known) result below is easy to verify directly.

LEMMA 1.2. *Let* $J_{n/2}$ *be the Bessel Potential of order* $n/2$, *that is*

$$J_{n/2}(\xi) = (1+|\xi|^2)^{-n/4}.$$

Then

$$\|J_{n/2}*f\|_{\mathrm{BMO}} \leqq A_n\|f\|_2$$

for every f.

Let us put

$$\Delta = \{v \in \mathscr{E}(\mathsf{R}^n) : \operatorname{supp}(v) \subset B^n = \{x \in \mathsf{R}^n : |x| \leqq 1\}$$
$$\text{and } |\hat{v}(\xi)| \leqq (1+|\xi|^2)^{-n/4} \text{ for every } \xi\}.$$

THEOREM 1.1. *There exists a constant* A_n *such that if* $v \in \Delta$ *then*

$$\|v*f\|_{H^1} \leqq \|f\|_{H^1} \quad \text{for all } f \in C_0^\infty(\mathbf{R}^n) .$$

The proof requires a result which takes care of "an error" in [1, p. 143, line 17–23].

LEMMA 1.3. *Let* $v \in \mathscr{E}(\mathbf{R}^n)$ *be such that* $\operatorname{supp}(v) \subset B^n$ *and* $\|\hat{v}\|_\infty \leqq 1$. *If* Q *is an open cube in* $\mathbf{R}^n$, *whose axes are parallel to the coordinate axes and centered at the origin while its n-dimensional volume* $|Q| \geqq 1$, *then*

$$|Q|^{-1} \int_Q |v*f(x)|\, dx \leqq 3^{n/2}\|f\|_\infty \quad \text{for } f \in C_0^\infty .$$

PROOF. Set $f_1(y) = f(y)$ when each $|y_\nu| < 1 + \delta/2$ and let $f_1 = 0$ otherwise. Here $\delta^n = |Q|$ and clearly $v*f(x) = v*f_1(x)$ for every $x \in Q$.

By Schwarz Inequality

$$\begin{aligned}\int_Q |v*f(x)|\, dx &\leqq |Q|^{\frac12}\Big[\int_{\mathbf{R}^n} |v*f_1(x)|^2 dx\Big]^{\frac12} \\ &\leqq |Q|^{\frac12}\|f_1\|_2 \leqq |Q|^{\frac12}(\delta+2)^{n/2}\|f\|_\infty ,\end{aligned}$$

and since $\delta \geqq 1$ the result follows.

Using Lemma 1.1. and Lemma 1.3. and the fact that BMO-norms are translation invariant we see that Theorem 1.1. follows if we can prove:

Let $v \in \Delta$ and let Q be an open cube, centered at the origin while $|Q| < 1$ and let $f \in C_0^\infty$, then there exists a scalar λ such that

$$|Q|^{-1} \int_Q |v*f(x) - \lambda|\, dx \leqq A_n\|f\|_\infty ,$$

where A_n is an absolute constant.

To prove this, set $f_1(y) = f(y)$ when $|y_\nu| < 2$ for every $\nu = 1 \ldots n$, and let $f_1 = 0$ otherwise. Now $v*f(x) = v*f_1(x)$ for every $x \in Q$ and we consider $g(x) = J_{n/2}*v*f_1(x)$.

Since $v \in \Delta$ we see that $\|g\|_2 \leqq \|f_1\|_2 \leqq 4^n\|f\|_\infty$ and finally Lemma 1.2. gives that

$$\|J_{-n/2}*g\|_{\mathrm{BMO}} \leqq 4^n A_n\|f\|_\infty .$$

The result follows since $J_{n/2}*g = v*f$ in Q.

REMARK. The idea to insert the Bessel Potential occurs already in [1, Theorem 1].

Before Theorem 2 is proved we insert some remarks about $\mathscr{M}(H^1) =$ the multiplier algebra over $H^1(\mathbf{R}^n)$. Using the fact that

$$H_0^1 = \{g \in C^\infty(\mathbf{R}^n) : \hat{g} \in C_0^\infty(\mathbf{R}^n \setminus \{0\})\}$$

is a dense subspace of H^1, it follows easily that if T is a translation invariant operator on H^1 then for each $f \in H_0{}^1$ we get

$$Tf(x) = (2\pi)^{-n/2} \int e^{i(x,\xi)} \hat{f}(\xi) m(\xi) d\xi ,$$

where m is a continuous function in $\mathsf{R}^n \setminus \{0\}$ and locally the Fourier transform of a measure with a finite total mass, that is when $\psi \in C_0{}^\infty(\mathsf{R}^n \setminus \{0\})$ then $\psi(\xi)m(\xi) = \hat{\mu}(\xi)$ for some $\mu \in M(\mathsf{R}^n)$. In particular the point-evaluation $m \to m(\xi)$ is a complex-valued homomorphism on the Banach algebra $\mathscr{M}(H^1)$ for every $\xi \in \mathsf{R}^n \setminus \{0\}$. It follows by Gelfand Theory that $|m(\xi)| \leqq$ the operator norm of T over H^1. Hence m is bounded and continuous in $\mathsf{R}^n \setminus \{0\}$, that is $\mathscr{M}(H^1)$ can be identified with an algebra of bounded continuous functions in $\mathsf{R}^n \setminus \{0\}$. In particular the hypothesis in [1, p. 159, Theorem 7] is redundant. Finally we have interpolation between $\mathscr{M}(H^1)$ and the multiplier algebras over L^p, $1 < p \leqq 2$, and we conclude the following.

PROPOSITION 1.4. *If $m \in \mathscr{M}(H^1)$ then $m \in \mathscr{M}(L^p)$ for every $1 < p \leqq 2$ and here*

$$\|m\|_{\mathscr{M}(H^1)} \leqq A_n(p-1)^{-1}(\|m\|_{\mathscr{M}(H^1)})^a(\|m\|_\infty)^{1-a} ,$$

where $a = 2/p - 1$ and $\|m\|_\infty \leqq \|m\|_{\mathscr{M}(H^1)}$ always holds.

PROOF OF THEOREM 2. Translating v if necessary we may assume that the origin belongs to $\operatorname{supp}(v)$. Using Theorem 1.1. and Prop. 1.4. we have the result when $\delta(v) < 1$. Suppose now that $\delta(v) \geqq 1$ and let $v_1 \in \mathscr{E}(\mathsf{R}^n)$ satisfy $\hat{v}_1(\xi) = \hat{v}(\delta(v)\xi)$ which gives that $\operatorname{supp}(v_1) \subset B^n$. Since the norms in $\mathscr{M}(L^p)$ (and in $\mathscr{M}(H^1)$) are invariant under dilatations we see that Theorem 2 follows if we can prove that

$$\|\hat{v}_1\|_{\mathscr{M}(H^1)} \leqq A_n(1 + \delta(v))^{n/2} .$$

This last estimate follows exactly as in Theorem 1.1. if we instead employ the dilated Bessel Potential J_δ satisfying

$$\hat{J}_\delta(\xi) = (1 + \delta(v)^2|\xi|^2)^{-n/4} ,$$

where the companion to Lemma 1.2. gives that

$$\|J_\delta * f\|_{\mathrm{BMO}} \leqq A_n(\delta(v))^{n/2}\|f\|_2 .$$

2. The case when $0<\alpha<n/4$.

Methods and results from [1, p. 156] are used. We treat a normalized case and set

$$\Delta(\alpha) = \{v \in \mathscr{E}(\mathrm{R}^n) : \operatorname{supp}(v) \subset B^n \text{ and } |\hat{v}(\xi)| \leqq (1+|\xi|^2)^{-\alpha}\},$$

where $0<\alpha<n/4$.

If $v \in \Delta(\alpha)$ and if $z=x+iy$, $0 \leqq x \leqq 1$, we define the operator

$$T_z(f)(x) = \int (1+|\xi|^2)^{A(z)} e^{i(x,\xi)} \hat{v}(\xi) \hat{f}(\xi)\, d\xi ,$$

where $A(z)=(z-1)n/4+\alpha$. We get immediately that

$$\|T_{1+iy}(f)\|_2 \leqq \|f\|_2 \quad \text{for every (real) } y . \tag{2.1}$$

Now we wish to establish that

$$\|T_{iy}(f)\|_{H^1} \leqq (1+|y|)^{n+1} C(v,\alpha,n) \|f\|_{H^1} . \tag{2.2}$$

We set

$$Sf(x) = \int (1+|\xi|^2)^{-n/4} e^{i(x,\xi)} \hat{v}(\xi) \hat{f}(\xi)\, d\xi ,$$

so that $T_{iy}(f)=\mathscr{M}_{iy}(Sf)$, where

$$\mathscr{M}_{iy}(g)(x) = \int (1+|\xi|^2)^{iny/4} e^{i(x,\xi)} \hat{g}(\xi)\, d\xi .$$

It is well-known that

$$\|\mathscr{M}_{iy}(g)\|_{H^1} \leqq A_n (1+|y|)^{n+1} \|g\|_{H^1}$$

for every real y.

So it remains only to estimate $S(f)$. Let us put $\beta=n/2-2\alpha$ and consider the Bessel Potential J_β. Let Q be an open cube, centered at the origin. We must establish that

$$|Q|^{-1} \int_Q |J_\beta * v * f(x) - \lambda|\, dx \leqq C(v,\alpha,n) \|f\|_\infty \tag{2.3}$$

for some complex scalar λ.

If $|Q|>1$ then Lemma 1.3. works because $\|J_\beta\|_1=1$ and it is then sufficient to choose $C(v,\alpha,n)=3^{n/2}$ and $\lambda=0$.

Let then $|Q|<1$ and put $f_1(y)=f(y)$ when $|y_v|<4$ for every v, while $f_1=0$ otherwise. Now we have that $\|J_{-2\alpha} * v * f_1\|_2 \leqq 16^n \|f\|_\infty$ and since $J_\beta=J_{n/2} * J_{-2\alpha}$, it follows from Lemma 1.2. that there is a scalar λ such that

$$|Q|^{-1} \int_Q |J_\beta * v * f_1(x) - \lambda|\, dx \leqq 16^n A_n \|f\|_\infty . \tag{2.4}$$

Finally, set $f_2=f-f_1$ and observe that f_2 has its support well away from the origin. Hence $\tilde{J}_\beta * v * f_2 = J_\beta * v * f_2$ in Q, where $\tilde{J}_\beta=\psi(x) J_\beta(x)$ and $\psi \in C^\infty(\mathrm{R}^n)$ is such that $\psi(x)=0$ for $|x|<2$ and $\psi(x)=1$ for $|x| \geqq 3$.

Recall now that J_β is a beautiful rapidly decreasing C^∞ function if we avoid a neighborhood of the origin. Since v has a compact support it follows that $\check{J}_\beta * v$ is a C^∞ function, rapidly decreasing at infinity.

But then we can conclude that

$$\|\check{J}_\beta * v * f_2\|_\infty \leqq C(v, \alpha, n)\|f_2\|_\infty$$

where we can use $C(v,\alpha,n) = \|\check{J}_\beta * v\|_1$ and this number is easy to estimate when $v \in \Delta(\alpha)$.

Since $\|g\|_{\mathrm{BMO}} \leqq \|g\|_\infty$ we have now obtained (2.2) and at this stage we only have to put $z = t$, where $(1-t)n = 4\alpha$ and use [1, Corollary 1, p. 156]. This gives an absolute constant A_n such that

$$\|T_v(f)\|_p \leqq A_n(1 + C(v,\alpha,n))^t \|f\|_p ,$$

if $p = 2n/(n+4\alpha)$ and $C(v,\alpha,n)$ is a constant which makes (2.3) valid.

Finally, using dilatations we can easily analyse the constant $C(v,\alpha,n)$ in Theorem 1 even when v has a large compact support.

3. The periodic case.

Let T^1 be the unit circle and let Z be the set of integers. From Theorem 1 we obtain the result below.

THEOREM 3. *Let* $0 < \alpha < \frac{1}{2}$ *and set* $p = 2/(1+2\alpha)$. *If now* $L = \{\lambda_n\}$ *is a sequence of complex numbers such that* $|\lambda_n| \leqq (1+|n|)^{-\alpha}$ *for every* n, *then we have that*

$$\|\sum a_n \lambda_n e^{int}\|_p \leqq C(L)\|\sum a_n e^{int}\|_p$$

for every trigonometric polynomial $\sum a_n e^{int}$ *and where* $\|\cdot\|_p$ *is the norm in* $L^p(\mathsf{T})$.

Using the proof in [2, p. 478–479] we can verify that Theorem 3 is sharp.

For let $0 < \alpha < \frac{1}{2}$ be given. If $k \geqq 2$ is an integer we choose a (Rudin–Shapiro) polynomial

$$p_k(e^{it}) = \sum \varepsilon_v(k) e^{ivt} ,$$

where $\varepsilon_v(k) = 0$ when v is outside $(2^k, 2^{k+1}]$ and $\varepsilon_v(k) = +1$ or -1 when $2^k < v \leqq 2^{k+1}$, while $|P_k|_q \leqq 4 2^{k/2}$ for all $2 \leqq q \leqq \infty$.

Now we set $\lambda_v(k) = \varepsilon_v(k)$ when $v \in (2^k, 2^{k+1}]$ and $L = \{2^{-k\alpha}\lambda_v\}$ where $\lambda_v = \lambda_v(k)$ if $2^k < v \leqq 2^{k+1}$ and $\lambda_v = 0$ if $v \leqq 4$. Let T_L be the convolution operator defined by L and let $1 < p < 2$ and $1/p + 1/q = 1$.

We get

$$\|T_L(P_k)\|_q = 2^{-k\alpha}\|\textstyle\sum_k e^{int}\|_q ,$$

where $\sum_k$ is taken over $(2^k, 2^{k+1}]$. The last term is greater than $A_0 2^{-k\alpha} 2^{k/p}$ for a fixed positive constant A_0.

So if $\|T_L(P)\|_p \leqq C(L)\|P\|_p$ for all trigonometric polynomials P, then we must have

$$A_0 2^{-k\alpha} 2^{k/p} \leqq C(L) 4 2^{k/2} \quad \text{for } k \geqq 2 .$$

Hence $p \geqq 2/(2\alpha+1)$ and Theorem 3 is sharp by M. Riesz' Convexity Theorem.

4. Added in proof.

Theorem 2 is sharp because the following example can be given. Let $g(x) \in C_0^\infty(\mathsf{R}^n)$ be a fixed test function such that $g(x)=0$ when some $|x_v| > \frac{1}{3}$ while $g(x) \geqq 0$ and $\int g(x)dx = 1$. Let $k \geqq 1$ be an integer and consider the 2^{nk} lattice points $(v_1, \ldots, v_n)$, where each v_i is an integer between 0 and $2^k - 1$ and let $s_1 \ldots s_N$, with $N = 2^{nk}$, be some enumeration of these points. Choose next numbers $\{\varepsilon_v\}_1^N$, $\varepsilon_v = +1$ or -1, in such a way that the periodic polynomial

$$P(\xi_1 \ldots \xi_n) = \sum \varepsilon_v e^{i(s_v, \zeta)}$$

satisfies $|P(\xi)| \leqq 2^{n+nk/2}$ for every ξ.

Put $G(x) = 2^{-n-nk/2} \sum \varepsilon_v g(x-s_v)$ and observe that $|\hat{G}(\xi)| \leqq |\hat{g}(\xi)|$ for every ξ. Hence $\hat{G}$ is majorized by the rapidly decreasing function $\hat{g}$ and this holds independently of k. The diameter of the support of $G(x)$ is roughly 2^k and we can estimate the norm of the convolution operator $G*$ from below over $L^p(\mathsf{R}^n)$ when $1 < p < 2$ as follows.

Fix some $f \in C_0^\infty(\mathsf{R}^n)$ so that $f(x) \geqq 0$ and $\int f(x)dx = 1$ and finally f has a small support close to the origin. Then the functions

$$h_v(x) = h_0(x-s_v) = \int g(x-s_v-y)f(y)dy$$

have pairwise disjoint supports as $v = 1, \ldots, N$, and we conclude that

$$\|G*f\|_p = (2^{-n-nk/2})N^{1/p}\|g*f\|_p$$

holds. Hence the norm of the operator $T_G = G*$ over L^p is at least $2^{-n}a_0(2^k)^{n(1/p-1/2)}$ where

$$a_0 = \|g*f\|_p/\|f\|_p$$

is a fixed positive constant.

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