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A TAUBERIAN REMAINDER THEOREM WITH APPLICATIONS TO LAMBERT SUMMABILITY

SIN PHING MOO STRUBE

1. Introduction.

The term "Tauberian theorems" owes its name to Tauber [9] who in 1897 proved a conditional converse of the well-known theorem of N. Abel [4, p. 10]. Later J. Littlewood [6] obtained the same conclusion with a much weaker condition. The special Tauberian theorems applied in the study of divergent series and summability methods were generalized by N. Wiener's "General Tauberian theorem" [12] in 1932. A Tauberian remainder theorem is a quantitative version of a Tauberian theorem. Beurling [1] was the first to study the size of the remainder term corresponding to Wiener's theorem, using the theory of Fourier analysis. The remainder theorem corresponding to the Littlewood theorem was proved by Freud [2] and Korevaar [5], using approximation theory. Ganelius [3] used the method of Fourier analysis to generalize these results.

If $K \in L^1(-\infty,\infty)$, we denote the Fourier transform of K by

$$\hat{K}(x) = \int_{-\infty}^{\infty} K(t)\exp(-2\pi itx)\,dt, \qquad x \in (-\infty,\infty) .$$

The kernels K considered in many known remainder theorems satisfy the condition that the reciprocal of the Fourier transform of K has an analytic continuation to a strip about the real axis or to the whole complex plane, and which satisfies a certain growth condition in that region. Lyttkens [7] studied the case where $\hat{K}(x)^{-1}$ has an analytic continuation to a strip below the real axis. (Professor Harold S. Shapiro informed me that Lyttkens has obtained further results as yet unpublished.)

We have obtained results for the class of kernels with no assumption of analyticity but with conditions on the growth of $\hat{K}(x)^{-1}$ and its second derivative on the real line. This enables us to study the remainder term corresponding to the Lambert summability method. The proof of our main theorem is partly based on those discussed in [3].

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2. The main theorem.

NOTATION. Let $K \in L^1(-\infty,\infty)$, $\varphi \in L^\infty(-\infty,\infty)$. The convolution of K and φ

$$K * \varphi(x) = \int_{-\infty}^{\infty} K(x-t)\varphi(t)\,dt$$

is defined for each $x \in (-\infty,\infty)$.

Throughout this paper let c denote a constant which need not be the same at each occurrence.

We say a function g belongs to S if g is monotonic increasing, $g(x) = O(x)$ and $g(2x) = O[g(x)]$.

THEOREM 1. *Let $\hat{K}(t)^{-1} = f(t)$, $\varphi \in L^\infty(-\infty,\infty)$. Suppose that*

$$t[|f(t)| + |f''(t)|] = O[H(|t|)], \quad |t| \to \infty \tag{1}$$

where $H(|t|)/|t|$ is monotonic increasing and $H(|t|) = O[\exp(ct^2)]$,

$$\psi(x) = K * \varphi(x) = O[W(x)^{-1}], \quad x \to \infty \tag{2}$$

where $W \in S$.

Furthermore let φ satisfy the tauberian condition

$$\sup_{x \leq y \leq x+V(x)^{-1}} [\varphi(y) - \varphi(x)] = O[V(x)^{-1}], \quad x \to \infty \tag{3}$$

where $V \in S$. Then

$$\varphi(x) = O(1) \inf_{R<V(x)} [R^{-1} + H(2R+1)W(x)^{-1}]$$

where $O(1)$ is a constant independent of R and x.

COROLLARY. *Under the assumptions of Theorem 1, with $H(t) = \exp|t|$, and $W(x) < \exp(V(x))$, we have*

$$\varphi(x) = O([\log W(x)]^{-1}) \quad x \to \infty .$$

PROOF. Theorem 1 implies that

$$\varphi(x) = O(1) \inf_{R<V(x)} [R^{-1} + W(x)^{-1} \exp(2R+1)] .$$

By choosing $3R = \log W(x)$, we obtain the result.

3. Proof of Theorem 1.

We first state and prove a simple lemma which we shall apply in the final stage of the proof.

LEMMA. *Let $\varphi \in L^\infty(-\infty,\infty)$ and let $V(x)$ be a monotonic increasing function such that $V(x+1) \leqq cV(x)$, $c>1$. Suppose*

$$\sup_{x\leqq y\leqq x+V(x)^{-1}}[\varphi(y)-\varphi(x)] \leqq cV(x)^{-1} \quad x\to\infty . \tag{4}$$

Then

$$\sup_{x\leqq y\leqq x+R^{-1}}[\varphi(y)-\varphi(x)] = O(R^{-1}), \quad x\to\infty$$

for all $R<V(x)$ where "O" is a constant independent of R and x.

PROOF. It is only necessary to consider large values of R and x. Let x be fixed and define $n=[cV(x)R^{-1}]$ where $[a]$ denotes the greatest integer in a. Define x_k recursively by $x_0=x$,

$$x_k = x_{k-1}+V(x_{k-1})^{-1}, \quad (k=1,2,\ldots,n+1) .$$

Then it is easily seen that for large values of R and x we have

$$x_{n+1} \geqq x+R^{-1} \quad \text{and} \quad x_n \leqq x+cR^{-1} .$$

Hence for $x\leqq y\leqq x+R^{-1}$

$$\begin{aligned}\varphi(y)-\varphi(x) &\leqq c\textstyle\sum_{m=1}^{n+1} V(x_{m-1})^{-1}\\ &= c(x_{n+1}-x)\\ &= c(x_n+V(x_n)^{-1}-x)\\ &\leqq c(c+1)R^{-1}\end{aligned}$$

if $R<V(x)$, and lemma is proved.

We shall begin the proof of Theorem 1. We may assume that

$$H(|t|) \leqq \exp(\pi t^2/8) .$$

NOTATION. Let

$$E(y) = \exp(-\pi y^2), \quad \delta(y) = R(\sin\pi Ry)^2(\pi Ry)^{-2} .$$

Then

$$\hat{E}(u) = \exp(-\pi u^2), \quad \hat{\delta}(u) = 1-|u|R^{-1} .$$

Define

$$\hat{Q}(t) = f(t)\int_{-\infty}^{\infty}\hat{E}(t-u)\hat{\delta}(u)\exp(-2\pi iu\eta)\,du \tag{5}$$

where η is a real number. With a suitably chosen η the following inequality holds:

$$|\varphi(x)| \leqq 6|Q*\varphi(x)| + \tag{6}$$
$$+4\,\sup_{v\leqq\xi\leqq v+2R^{-1}}[\varphi(x-v)E(v)-\varphi(x-\xi)E(\xi)]\,.$$

This is the same inequality as formula (1) on page 20 in [3]. It is proved in the same way, the only difference being that the integrability of Q in our case follows from the formula

$$Q(x) = (2\pi x)^{-2}\int_{-\infty}^{\infty}\hat{Q}''(t)\exp(2\pi ixt)\,dt\,.$$

We now denote the two terms on the right of (6) by I_1 and I_2 and we start by estimating I_1. For all y we have

$$\begin{aligned}|Q(y)| &= \left|\int_{-\infty}^{\infty}\hat{Q}(t)\exp(2\pi ity)\,dt\right|\\ &\leqq \int_{-\infty}^{\infty}|f(t)|\int_{-R}^{R}\hat{E}(t-u)\hat{\delta}(u)\,du\,dt\\ &\leqq \int_{-R}^{R}\int_{-\infty}^{\infty}|f(t)|\hat{E}(t-u)\,dt\,du\,.\end{aligned}$$

Now

$$\begin{aligned}&\int_{-\infty}^{\infty}|f(t)|\hat{E}(t-u)\,dt\\ &\quad\leqq \int_{|t|\leqq R}|f(t+u)|\,dt+\int_{|t|>R}|f(t+u)|\hat{E}(t)\,dt\\ &\quad= O[H(2R)/R]\end{aligned}$$

uniformly for all $|u|<R$, since $tf(t)=O[H(t)]$ and $H(|t|)\leqq\exp(\pi t^2/8)$. Thus for all y,

$$Q(y) = O[H(2R)]\,. \tag{7}$$

Next for all $y\neq 0$, we obtain by integrating the following by parts twice that

$$\begin{aligned}&\left|\int_{-\infty}^{\infty}f(t)\hat{E}(t-u)\exp(2\pi ity)\,dt\right|\\ &\qquad\leqq c|y|^{-2}\int_{-\infty}^{\infty}|[f(t)\hat{E}(t-u)]''|\,dt\,.\end{aligned}$$

It is clear from condition (1) and by an application of Taylor's Theorem that $|tf'(t)|=O[H(|t|+1)]$. Similarly to the discussion leading to identity (7), we have that for all $y\neq 0$,

$$Q(y) = O[H(2R+1)/y^2]\,. \tag{8}$$

It follows from (7) and (8) that for all values of y

$$Q(y) = O[H(2R+1)/(y^2+1)]\,. \tag{9}$$

Finally, for all large x, we have (in view of identity (9) and condition (2)) that

$$
\begin{aligned}
(10) \qquad |Q * \psi(x)| &= O[H(2R+1)\textstyle\int_{-\infty}^{\infty} |\psi(x-y)|/(y^2+1)dy] \\
&= O\big[H(2R+1)\,[\textstyle\int_{2|y|<x} |\psi(x-y)|/(y^2+1)dy + \\
&\quad + \textstyle\int_{2|y|\geqq x} |\psi(x-y)|/(y^2+1)dy]\big] \\
&= O[H(2R+1)W(x)^{-1}]\,,
\end{aligned}
$$

which gives an estimate for I_1.

The final stage is to estimate I_2, which can be rewritten as

$$I_2 = \sup_\xi([\varphi(x-v)-\varphi(x-\xi)]E(v)+[E(v)-E(\xi)]\varphi(x-\xi))\,.$$

Since the derivative of E is bounded, we have for all v

$$\sup_\xi[E(v)-E(\xi)]\varphi(x-\xi) = O(R^{-1})\,.$$

As for the remaining term in I_2, we consider two cases separately.

CASE 1. Suppose $|v| > x/2$. Since φ is bounded

$$\sup_\xi[\varphi(x-v)-\varphi(x-\xi)]E(v) = O[E(x/2)] = O(R^{-1})$$

if $R \leqq \exp(x^2/4)$.

CASE 2. Suppose $|v| \leqq x/2$. This is the case where we need the Tauberian condition (3) on φ. Clearly

$$
\begin{aligned}
&\sup_\xi[\varphi(x-v)-\varphi(x-\xi)]E(v) \\
&= O(1)\sup_\xi[\varphi(x-v)-\varphi(x-\xi)] \\
&= O(1)\sup_{X-2R^{-1}\leqq Y\leqq X}[\varphi(X)-\varphi(Y)]
\end{aligned}
$$

where $X = x-v$, $Y = x-\xi$, and it is noted that $x \leqq 2X \leqq 3x$, $x-2 \leqq 2Y \leqq 3x$ if $R > 2$.

Now it follows from condition (3) and the above Lemma that

$$\sup_{X-2R^{-1}\leqq Y\leqq X}[\varphi(X)-\varphi(Y)] = O(R^{-1})$$

if $R < V(Y)$. Thus

$$\sup_\xi[\varphi(x-v)-\varphi(x-\xi)] = O(R^{-1})$$

if $R < V(x)$. We therefore have

$$(11) \qquad I_2 = O(R^{-1})$$

if $R < V(x)$. Combining (6), (10), and (11) we conclude that as $x \to \infty$,

$$\varphi(x) = O(1)\inf_{R<V(x)}(R^{-1}+H(2R+1)W(x)^{-1})$$

which is the required result.

4. Further result.

By weakening the Tauberian condition on $\varphi(x)$, we may still be able to obtain a remainder term for $\varphi(x)$. In the study of summability, this would imply that a weaker Tauberian condition on a bounded sequence $\{s_n\}$ will yield convergence provided the transformed sequence converges at a suitable rate.

THEOREM 2. *Under the assumptions of Theorem* 1, *replacing condition* (3) *by*

$$\sup_{x \leq y \leq x+V(x)^{-1}} [\varphi(y)-\varphi(x)] = O[V(x)^{-1}U(x)] \tag{3'}$$

where $U \in S$ and $U(x)=o(V(x))$, we have

$$\varphi(x) = O(1) \inf_{R<V(x)} (R^{-1}U(x)+H(2R+1)W(x)^{-1}) .$$

PROOF OF THEOREM 2. Since identity (10) is still valid, it is only necessary to consider I_2. Similar to the previous section leading to an estimate for I_2, together with a slight variation of the Lemma and the above condition (3′) we obtain

$$I_2 = O[R^{-1}U(x)]$$

if $R<V(x)$.

5. Application to the Lambert summability.

The Tauberian theorem for the Lambert summability method is used in the proof of the prime number theorem. In using this method, the essential information needed is that the zeta function $\zeta(s)$ does not vanish on the line $\mathrm{Re}\{s\}=1$. The aim of this section is to obtain a remainder theorem for this method. We cannot apply general remainder theorems which require the analytic continuation of the kernel under consideration since this would presume the knowledge that $\zeta(s)$ has no zeros in a strip $|\mathrm{Re}\{s\}-1|<\varepsilon$. However our theorem is applicable since the rate of growth of $\zeta(s)$ and its second derivative is known on the line $\mathrm{Re}\{s\}=1$.

DEFINITION [4] [8]. Let $\sum_{n=0}^{\infty} a_n$ be an infinite series and let

$$\sigma(u) = (1-u)\sum_{n=1}^{\infty} \frac{na_n u^n}{1-u^n}, \quad |u| < 1 . \tag{12}$$

We say that a sequence $\{s_n\}$ is limitable by the Lambert method to s if $\sigma(u)$ is convergent for all $|u|<1$ and $\lim_{u\to 1-}\sigma(u)=s$, where $a_n=s_n-s_{n-1}$.

It is known [4] that the Tauberian condition for the Lambert method is $a_n \leqq n^{-1}$. Our theorem is:

THEOREM 3. *Let $W \in S$. Suppose*

(i) $$\sigma(u) = O(1/W[-\log(-\log u)]), \quad u \to 1^-$$
and

(ii) $$a_n \leqq n^{-1}.$$
Then

$$s_n = O(1/\log[W(\log n)]), \quad n \to \infty.$$

In particular if $W(x) = x$ then under the above conditions, we have

$$s_n = O(1/\log(\log n)), \quad n \to \infty.$$

PROOF OF THEOREM 3. By a change of variables, we obtain as $y \to \infty$

$$\sigma(e^{-1/y}) = [1+O(y^{-1})]y^{-1}\int_0^\infty \frac{te^{-t/y}}{1-e^{-t/y}}\,ds(t)$$

where $s(t) = \sum_{n \leqq t} a_n$, $(a_0 = 0)$. Let $G(t) = te^{-t}/(1-e^{-t})$, $t > 0$. By a further change of variables, we have as $x \to \infty$,

(13) $$\sigma(\exp[-\exp(-x)]) = \big(1+O(\exp(-x))\big)(K * \varphi(x))$$

where $K(x) = \exp(-x)G'[\exp(-x)]$ and $\varphi(x) = s(\exp x)$. Now it is not difficult to show [8, 10] that

$$\hat{K}(x) = -2\pi ix\Gamma(1+2\pi ix)\zeta(1+2\pi ix)$$

and $f(x) = \hat{K}(x)^{-1} = O(\exp(c|x|))$ as $x \to \infty$. Moreover f' and f'' have order less than or equal to $\exp(c|x|)$. This follows from known results [11] that, as $|x| \to \infty$, the functions $1/\zeta(1+ix)$, $\zeta(1+ix)$, $\zeta'(1+ix)$ and $\zeta''(1+ix)$ are all bounded by certain powers of $\log|x|$. Hence condition (i) of Theorem 1 is satisfied with $H(|t|) = \exp(c|t|)$. Next, condition (i) of Theorem 3 and relation (13) imply that

$$K * \varphi(x) = O[W(x)^{-1}], \quad x \to \infty.$$

Finally, we have by hypothesis (ii) that

$$\begin{aligned}\varphi(y) - \varphi(x) &= s(\exp y) - s(\exp x)\\ &\leqq (\exp(y) - \exp(x) + 1)(\exp(-x))\\ &= (\exp(y-x) - 1 + \exp(-x)).\end{aligned}$$

Hence

$$\sup_{x < y \leqq x + W(x)^{-1}}[\varphi(y) - \varphi(x)] = O[W(x)^{-1}], \quad x \to \infty.$$

It follows from corollary of Theorem 1 that

$$\varphi(x) = O([\log W(x)]^{-1}), \quad x \to \infty .$$

That is,

$$s(n) = O(1/\log[W(\log n)]), \quad n \to \infty .$$

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF WESTERN ONTARIO
LONDON, ONTARIO N6A 3K7
CANADA