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DIVISIBLE AND CODIVISIBLE MODULES

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In this paper divisible and codivisible modules are studied relative to a torsion theory on Mod R. It is shown, for example, that if the torsion theory is hereditary, then the following are equivalent; Every module is codivisible. Every torsionfree module is injective. $R/T(R)$ is semi-simple Artinian. In addition, codivisible covers are investigated and it is shown that if every module has a codivisible cover, then $R/T(R)$ is perfect. A partial converse to this is obtained by showing that if $R/T(R)$ is perfect, then every torsionfree module has a codivisible cover. This leaves open the question of whether or not codivisible covers universally exist when $R/T(R)$ is perfect.

1. Preliminaries.

Throughout this paper R will denote an associative ring with identity and our attention will be confined to the category Mod R of unital right R-modules.

In [4], Dickson defined a torsion theory on Mod R to be a pair $(\boldsymbol{A},\boldsymbol{B})$ of non-empty classes of modules such that:

(a) $\boldsymbol{A} \cap \boldsymbol{B} = 0$
(b) If $A \to A^* \to 0$ is exact with $A \in \boldsymbol{A}$, then $A^* \in \boldsymbol{A}$.
(c) If $0 \to B^* \to B$ is exact with $B \in \boldsymbol{B}$, then $B^* \in \boldsymbol{B}$.
(d) For each module M in Mod R, there is an exact sequence $0 \to A \to M \to B \to 0$ with $A \in \boldsymbol{A}$ and $B \in \boldsymbol{B}$.

Modules in $\boldsymbol{A}$ are called torsion and those in $\boldsymbol{B}$ torsionfree. If $(\boldsymbol{A},\boldsymbol{B})$ is a torsion theory, then $\boldsymbol{A}$ is closed under isomorphic images, factor modules, extensions and direct sums, while $\boldsymbol{B}$ is closed under isomorphic images, submodules, extensions and direct products [4, p. 226, Theorem 2.3]. By saying that a class $\boldsymbol{C}$ of modules is closed under extensions, we mean that $M \in \boldsymbol{C}$ whenever N is a submodule of M such that N and M/N are in $\boldsymbol{C}$. If $(\boldsymbol{A},\boldsymbol{B})$ is a torsion theory such that $\boldsymbol{B}$ is closed under injective hulls [5], then $(\boldsymbol{A},\boldsymbol{B})$ is called hereditary. It is known that

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$(\boldsymbol{A},\boldsymbol{B})$ is hereditary if and only if $\boldsymbol{A}$ is closed under submodules [4, p. 227, Theorem 2.9]. Uniquely associated with each hereditary torsion theory there is a topologizing idempotent right ideal filter

$$\boldsymbol{F} = \{I \subseteq R \mid I \text{ is a right ideal of } R \text{ and } R/I \text{ is torsion}\}$$

(see Gabriel [6]).

An object functor $T:\mathrm{Mod}R \to \mathrm{Mod}R$ is said to be a radical if $T(M) \subseteq M$, $f:M \to N$ implies that $f(T(M)) \subseteq T(N)$ and $T(M/T(M)) = 0$. A radical T such that $T(T(M)) = T(M)$ for all modules M is called idempotent. It is well-known that $(\boldsymbol{A},\boldsymbol{B})$ is a torsion theory if and only if there exists an idempotent radical T on $\mathrm{Mod}R$ such that

$$\boldsymbol{A} = \{A \in \mathrm{Mod}R \mid T(A) = A\}$$

and

$$\boldsymbol{B} = \{B \in \mathrm{Mod}R \mid T(B) = 0\}$$

[8, p. 2, Proposition 0.1]. Furthermore, this correspondence is one-to-one [10, p. 6, Proposition 2.3]. Hence a module M is torsion if and only if $T(M) = M$ and torsionfree if and only if $T(M) = 0$. $T(M)$ is usually referred to as the torsion submodule of M and it can be described as that (necessarily unique) torsion submodule of M which contains every torsion submodule of M. If $(\boldsymbol{A},\boldsymbol{B})$ is a torsion theory with associated idempotent radical T, then $(\boldsymbol{A},\boldsymbol{B})$ is hereditary if and only if T is a left exact functor [10, p. 8, Proposition 2.6]. (In this context if $f:M \to N$, then $T(f)$ is the restriction of f to $T(M)$). When T is left exact, it is not difficult to show that $T(N) = N \cap T(M)$ for every submodule N of M. If $(\boldsymbol{A},\boldsymbol{B})$ is a torsion theory such that R is torsionfree, then $(\boldsymbol{A},\boldsymbol{B})$ is said to be faithful.

A projective module $P(M)$ together with an epimorphism $\pi:P(M) \to M$ with small kernel is said to be a projective cover of M. A submodule K of M is said to be small in M if $N = M$ whenever N is a submodule of M such that $K + N = M$. We will call any epimorphism with small kernel minimal. A ring R is (semi-) perfect if every (cyclic) module has a projective cover. (See [2] for several characterizations of perfect and semi-perfect rings). Torsion theories for which every torsion module has a projective cover have been studied in [3].

In the discussion which follows, $(\boldsymbol{A},\boldsymbol{B})$ will denote a torsion theory on $\mathrm{Mod}R$ with idempotent radical T. When $(\boldsymbol{A},\boldsymbol{B})$ is hereditary, $\boldsymbol{F}$ will denote the associated idempotent right ideal filter.

2. Divisible and codivisible modules.

Following Lambek [8, p. 8] we call a module M divisible provided that every diagram of the form

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r, "f"] \arrow[d] & N \\ & M & \end{tikzcd}$$

where coker f is torsion can be completed to a commutative diagram. Dually, M is said to be codivisible if every diagram

$$\begin{tikzcd} & M \arrow[d] & \\ N \arrow[r, "f"] & L \arrow[r] & 0 \end{tikzcd}$$

where ker f is torsionfree can be completed to a commutative diagram The usual argument now shows that a (direct sum) direct product of modules is (codivisible) divisible if and only if each factor is (codivisible) divisible. The following result is well-known. We include a proof for completeness.

LEMMA 2.1. *If $(\boldsymbol{A},\boldsymbol{B})$ is hereditary, then for any module M,*

$$T(M) = \{x \in M \mid (0:x) \in \boldsymbol{F}\}.$$

($(0:x)$ denotes the right annihilator of x in R). In particular, $T(R)$ is an ideal of R.

PROOF.

$$x \in T(M) \Leftrightarrow xR \subseteq T(M) \Leftrightarrow xR \text{ is torsion}$$

$$\Leftrightarrow R/(0:x) \text{ is torsion} \Leftrightarrow (0:x) \in \boldsymbol{F}.$$

Thus

$$T(M) = \{x \in M \mid (0:x) \in \boldsymbol{F}\}.$$

If $x \in T(R)$ and $r \in R$, then $(0:x) \subseteq (0:rx)$ implies that $(0:rx) \in \boldsymbol{F}$. Hence $rx \in T(R)$ and so $T(R)$ is an ideal of R.

THEOREM 2.2. *If $(\boldsymbol{A},\boldsymbol{B})$ is hereditary, then the following are equivalent:*

(a) *Every module is codivisible.*
(b) *Every torsionfree module is injective.*
(c) *$R/T(R)$ is a semi-simple Artinian ring.*

PROOF. (c) $\Rightarrow$ (b). Let M be a torsionfree R-module. If $m \in M$ and $x \in T(R)$, then $(0:x) \subseteq (0:mx)$ and so $(0:mx) \in \boldsymbol{F}$ since $(0:x) \in \boldsymbol{F}$. Thus it follows that $MT(R) \subseteq T(M) = 0$ and so M is an injective $R/T(R)$-module. Next let I be a right ideal of R and suppose that $f: I \to M$ is R-linear. If

$$x \in T(I) = I \cap T(R),$$

then $f(x) = 0$ since $f(T(I)) \subseteq T(M) = 0$. Hence

$$g:(I + T(R))/T(R) \to M \text{ defined by } x + T(R) \to f(x)$$

is a well-defined R-linear mapping which is easily seen to be $R/T(R)$-linear. Consequently, by Baer's criterion [1], there is an $m \in M$ such that

$$g(x + T(R)) = m(x + T(R))$$

for all $x + T(R)$ in $(I + T(R))/T(R)$. But this yields $f(x) = mx$ for all $x \in I$ and so M is R-injective.

(b) $\Rightarrow$ (a). Consider the diagram

$$\begin{tikzcd} & M \arrow[d, "g"] & \\ L \arrow[r, "f"] & N \arrow[r] & 0 \end{tikzcd}$$

where $K = \ker f$ is torsionfree. Since K is injective, $L = K \oplus N'$ and $h = f|N': N' \to N$ is an isomorphism. Hence if $\varphi = h^{-1} \circ g$, then φ completes the diagram commutatively.

(a) $\Rightarrow$ (c). Let $I/T(R)$ be a right ideal of $R/T(R)$. Since $I/T(R)$ is torsionfree, the R-exact sequence

$$0 \to I/T(R) \to R/T(R) \to R/I \to 0$$

splits. Hence $I/T(R)$ is an R-direct summand and consequently an $R/T(R)$-direct summand of $R/T(R)$. Thus $R/T(R)$ is semi-simple Artinian.

Notice that from the proof of (a) $\Rightarrow$ (c) we see that (a) can be replaced by: Every cyclic module is codivisible.

COROLLARY 2.3 *If $(\boldsymbol{A}, \boldsymbol{B})$ is a faithful, hereditary torsion theory and every module is codivisible, then R is semi-simple Artinian.*

The next theorem has a proof which is similar to that of theorem 2.2. The implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c) are straightforward while (c) $\Rightarrow$ (a) can be proven by using the generalized Baer's criterion for divisible modules [7, p. 20, Proposition 3.2].

THEOREM 2.4. *If* $(\boldsymbol{A},\boldsymbol{B})$ *is hereditary, then the following are equivalent*:

(a) *Every module is divisible.*
(b) *Every torsion is projective.*
(c) *Every* $I \in \boldsymbol{F}$ *is a direct summand of* R.

If $f:M \to N$ is R-linear, then we will call f free if $\ker f$ is torsionfree. The following lemma will prove useful.

LEMMA 2.5. *If* $f:M \to N$ *is a minimal, free epimorphism, then* f *is an isomorphism whenever* N *is codivisible.*

PROOF. Since $K = \ker f$ is torsionfree, the exact sequence

$$0 \longrightarrow K \longrightarrow M \xrightarrow{f} N \longrightarrow 0$$

splits whenever N is codivisible. Hence $M = K \oplus N'$ where N' is a submodule of M isomorphic with N. But K is small in M and so $K = 0$.

The following lemma can be found in [9, p. 189, Hilfssatz 3.1].

LEMMA 2.6. *If* $f:M \to N$ *is R-linear and* A *is small in* M, *then* $f(A)$ *is small in* N.

THEOREM 2.7. *Let* $K = \ker f$ *where* $f:C \to M$ *is an epimorphism and* C *is codivisible. Then* M *is codivisible if* K *is torsion. Conversely, if* M *is codivisible and* K *is small in* C, *then* K *is torsion.*

PROOF. Consider the diagram

$$\begin{array}{ccccc} & & M & & \\ & & \downarrow{\scriptstyle g} & & \\ L & \xrightarrow{h} & N & \longrightarrow & 0 \end{array}$$

where $\ker h$ is torsionfree. Since C is codivisible $g \circ f$ factors through L. Suppose $\varphi:C \to L$ is such that $h \circ \varphi = g \circ f$. Now $\varphi:K \to \ker h$ and so if K is torsion, then

$$\varphi(K) = \varphi(T(K) \subseteq T(\ker h)) = 0\,.$$

Hence we have an induced mapping $\varphi^*: M \to L$ which completes the diagram commutatively.

Conversely, suppose that M is codivisible and K is small in C. Then $f^*: C/T(K) \to M$ defined by

$$f^*(x+T(K)) = f(x)$$

is a minimal, free epimorphism ($\ker f^* = K/T(K)$ is small in $C/T(K)$ by lemma 2.6.). Hence f^* must be an isomorphism by lemma 2.5 and so it follows that $K = T(K)$.

The following theorem shows that a necessary and sufficient condition for every module to be codivisible can be found which is intrinsic to R is semi-perfect.

THEOREM 2.8. *If (A,B) is hereditary and R is semi-perfect, then every module is codivisible if and only if $J(R) \subseteq T(R)$. ($J(R)$ = the Jacobson radical of R.)*

PROOF. Due to our observation following the proof of (a) $\Rightarrow$ (c) in theorem 2.2, we can confine our attention to cyclic modules. Note also that a right ideal I of R is contained in $J(R)$ if and only if I is small in R [2, p. 472, Lemma 2.4]. Hence if $I \subseteq J(R)$ is a right ideal of R and every cyclic module is codivisible, then

$$0 \to I \to R \to R/I \to 0$$

is exact with R/I codivisible. Thus it follows from theorem 2.7 that I is torsion.

Conversely, suppose that $J(R) \subseteq T(R)$. If I is any right ideal of R, then R/I has a projective cover

$$0 \to K \to P \to R/I \to 0 .$$

Now we can assume (without loss of generality) that P is a direct summand of R and so since K is small in P, it follows that K is small in R. Hence $K \subseteq J(R)$ and so K is torsion. Consequently, by theorem 2.8, R/I is codivisible.

3. Codivisible covers.

A codivisible module $C(M)$ together with a minimal, free epimorphism $\mu: C(M) \to M$ is said to be a codivisible cover of M. As in the case of projective covers, it can be shown that if

$$\{\mu_\alpha: C(M_\alpha) \to M_\alpha\} \quad (\alpha \in \Delta)$$

is a finite family of codivisible covers, then

$$\oplus\mu_\alpha \colon \oplus C(M_\alpha) \to \oplus M_\alpha$$

is a codivisible cover. If R/I is a factor ring of R, then an R/I-module M will be considered to be R/I-codivisible if every diagram

$$\begin{tikzcd} & M \arrow[d] & \\ L \arrow[r, "f"] & N \arrow[r] & 0 \end{tikzcd}$$

of R/I-modules and R/I-module homomorphisms can be completed to a commutative diagram where $\ker f$ is torsionfree when viewed as an R-module. Similar remarks hold for free R/I-homomorphisms and R/I-codivisible covers.

The following two theorems also appear in [3] though with slightly different proofs. We include them here for convenience of the reader and for the sake of continuity.

THEOREM 3.1. *A codivisible cover (when it exists) is unique up to an isomorphism.*

PROOF. Let $\mu: C(M) \to M$ and $\mu^*: C(M)^* \to M$ be codivisible covers of M. Then the diagram

$$\begin{tikzcd} & C(M) \arrow[d, "\mu"] & \\ C(M)^* \arrow[r, "\mu^*"] & M \arrow[r] & 0 \end{tikzcd}$$

can be completed to a commutative diagram by a homomorphism φ since μ^* is free. Hence it follows that

$$C(M)^* = \operatorname{Im}\varphi + \ker\mu^* .$$

But μ^* is minimal and so φ is an epimorphism. Now $\ker\varphi \subseteq \ker\mu$ and consequently φ is minimal and free. Hence by lemma 2.5 φ must be an isomorphism.

THEOREM 3.2. *If $\pi: P(M) \to M$ is a projective cover of M, then*

$$\mu: P(M)/T(\ker\pi) \to M$$

is a codivisible cover of M where μ is the mapping induced by φ.

PROOF. Since $\ker\mu \cong \ker\pi/T(\ker\pi)$, then μ is free. Notice again by lemma 2.6 that $\ker\mu$ is small in $P(M)/T(\ker\pi)$. Hence μ is a minimal, free epimorphism. That $P(M)/T(\ker\pi)$ is codivisible follows directly since factor modules of codivisible modules by torsion submodules are codivisible, theorem 2.7.

We also see by theorem 2.7 that if $\pi: P(M) \to M$ is a projective cover of M, then M is codivisible if and only if $\ker\pi$ is torsion. Notice that since the class of torsionfree modules is closed under extensions, then $C(M)$ (if it exists) is torsionfree whenever M is torsionfree.

The above theorem shows that every (cyclic) module has a codivisible cover when R is a (semi-) perfect ring. For the sake of brevity, we will call R a (t-ring) T-ring if every (cyclic) module has a codivisible cover.

THEOREM 3.3. *If R is a (t-ring) T-ring, then every factor ring of R is a (t-ring) T-ring.*

PROOF. Let R/I be a factor ring of R and suppose that M is an R/I-module. If $\mu: C(M) \to M$ is an R-codivisible cover of M, then

$$C^* = C(M)/C(M)I$$

is an R/I-module. Since $\mu(C(M)I) = MI = 0$, we have an induced mapping $\mu^*: C^* \to M$ which is a minimal epimorphism. Now C^* is R/I-codivisible and so it follows that $C^*/T(\ker\mu^*)$ together with the obvious induced mapping is an R/I-codivisible cover of M.

An identical proof holds for t-rings.

If R is torsionfree and M is codivisible then

$$0 \longrightarrow K \longrightarrow F \xrightarrow{f} M \longrightarrow 0$$

splits where $K = \ker f$ and $f: F \to M$ is a free module om M. It follows then that M is (projective and) torsionfree. If every codivisible module is torsionfree, then R is clearly torsionfree. Hence we have the following:

THEOREM 3.4. *Every codivisible module is (projective and) torsionfree if and only if $(\boldsymbol{A},\boldsymbol{B})$ is faithful.*

The above theorem shows that over faithful torsion theories the class of (t-rings) T-rings and the class of (semi-) perfect rings coincide. From the work completed the following theorem is now evident.

THEOREM 3.5. *If R is a (t-ring) T-ring, then every torsionfree factor ring of R is (semi-) perfect. In particular, if $(\boldsymbol{A},\boldsymbol{B})$ is hereditary, then $R/T(R)$ is (semi-) perfect.*

In conclusion, we obtain a partial converse to theorem 3.5.

THEOREM 3.6. *If* (A,B) *is hereditary and* $R/T(R)$ *is (semi-)perfect, then every torsionfree (cyclic) module has a codivisible cover.*

PROOF. Suppose that $R/T(R)$ is perfect and let M be a torsionfree R-module. Then as shown in the proof of theorem 2.2, M is an $R/T(R)$-module. Let $\pi : P(M) \to M$ be an $R/T(R)$-projective cover of M. If Δ is a set of $R/T(R)$-generators for $P(M)$, then there is a cononical $R/T(R)$-epimorphism

$$\eta : \oplus_{\alpha \in \Delta} R_\alpha/T(R_\alpha) \to P(M)$$

which is R-linear ($R_\alpha = R$ for each $\alpha \in \Delta$). Now $P(M)$ is an $R/T(R)$-direct summand and therefore an R-direct summand of $\oplus_{\alpha \in \Delta} R_\alpha/T(R_\alpha)$. Thus $P(M)$ is R-codivisible and torsionfree since it is the direct summand of an R-codivisible, torsionfree R-module. It follows then, that $\pi : P(M) \to M$ is an R-codivisible cover of M.

The semi-perfect case has a similar proof.

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