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THE MAXIMAL RING OF QUOTIENTS OF A TRIANGULAR MATRIX RING

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1. Introduction.

The problem we consider in this note is to determine the maximal ring of quotients of a ring of the form

$$R = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix},$$

where A and B are rings and M is an $A-B$-bimodule, and where addition and multiplication is defined as is customary for matrices. The most convenient definition of the maximal right ring of quotients for this purpose is the one due to Lambek [3], namely as the bicommutator of an injective envelope of R (as a right R-module).

2. Preliminaries.

We need some known basic facts concerning injective modules over the ring

$$R = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$$

(we refer to [2] for details).

A right R-module is a pair of modules X_A and Y_B, together with a B-linear map

$$\alpha : X \otimes_A M \to Y .$$

We write it as a row vector $(XY)_\alpha$, and R then operates upon it as

$$(x \;\; y) \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} = (xa \;\; \alpha(x \otimes m) + yb) .$$

Instead of α it may be more convenient to use the corresponding A-linear map

$$\tilde{\alpha} : X \to \mathrm{Hom}_B(M, Y) .$$

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Similarly, a left R-module may be considered as a column vector

$$\begin{pmatrix} X \\ Y \end{pmatrix}_\beta,$$

where $\beta: M \otimes_B Y \to X$ is A-linear.

The ring monomorphism $\varkappa: A \times B \to R$ induces a forgetful functor

$$\varkappa_*: \text{Mod}-R \to \text{Mod}-(A \times B)$$

of the categories of right modules. As usual, $\varkappa_*$ has a right adjoint $\varkappa^*$ (extension by scalars) and a left adjoint $\varkappa^!$. Explicitly,

$$\varkappa^!(X \times Y) = (X \oplus \text{Hom}_B(M, Y) \;\; Y)_\alpha$$

with α given by the evaluation map. From [2] we have:

LEMMA. *The injective envelope of an R-module $(X \; Y)_\alpha$ is*

$$\varkappa^!(E(\text{Ker}\,\tilde{\alpha}) \times E(Y)).$$

We need to know the endomorphism ring of an object of the form $\varkappa^!(X \times Y)$. By adjointness we have

$$\begin{aligned} \text{Hom}_R(\varkappa^!(X \times Y), \varkappa^!(X \times Y)) &\cong \text{Hom}_{A\times B}(\varkappa_*\varkappa^!(X \times Y), X \times Y) \\ &\cong \text{Hom}_{A\times B}((X \oplus \text{Hom}_B(M, Y)) \times Y, X \times Y) \\ &\cong \begin{pmatrix} \text{End}_A(X) & \text{Hom}_A(\text{Hom}_B(M, Y), X) \\ 0 & \text{End}_B(Y) \end{pmatrix}, \end{aligned}$$

and one may check that the multiplcation in $\text{End}_R(\varkappa^!(X \times Y))$ coincides with the matrix multiplication in the last ring, which we for brevity denote by

$$\hat{R} = \begin{pmatrix} \hat{A} & \hat{M} \\ 0 & \hat{B} \end{pmatrix}.$$

As a left $\hat{R}$-module, $\varkappa^!(X \times Y)$ takes the form

$$\begin{pmatrix} X \\ \text{Hom}_B(M, Y) \end{pmatrix}_\beta \oplus \begin{pmatrix} 0 \\ Y \end{pmatrix}_0,$$

where $\beta: \hat{M} \otimes_{\hat{B}} \text{Hom}_B(M, Y) \to X$ is the evaluation map. From this one may in principle compute the bicommutator of $\varkappa^!(X \times Y)$, although the formulas become rather messy.

3. The maximal ring of quotients.

We now choose in particular the R-module $R = (A\ M \oplus B)_\alpha$, where

$$\tilde{\alpha}: A \to \mathrm{Hom}_B(M, M \oplus B) \cong \mathrm{End}_B(M) \oplus \mathrm{Hom}_B(M, B)$$

maps $a \in A$ to the endomorphism of M given by left multiplication with a. Hence we obtain from the Lemma that

$$E(R) = \varkappa^!\big(E(\mathrm{Ann}({}_A M)) \times E(M \oplus B)\big) ,$$

where $\mathrm{Ann}({}_A M)$ denotes the annihilator of the A-module M.

We now assume that M is a faithful A-module. If we write $Y = E(M) \oplus E(B)$, then we simply have

$$E(R) = \varkappa^!(0 \times Y) = \begin{pmatrix} 0 \\ \mathrm{Hom}_B(M, Y) \end{pmatrix}_0 \oplus \begin{pmatrix} 0 \\ Y \end{pmatrix}_0$$

as a left module over the ring

$$\mathrm{End}_R(E(R)) = \begin{pmatrix} 0 & 0 \\ 0 & \mathrm{End}_B(Y) \end{pmatrix} .$$

In this case it is easy to compute the bicommutator of $E(R)$, and we obtain:

THEOREM. *If ${}_A M$ is faithful, then the maximal right ring of quotients of $R = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ is*

$$Q_{\max}(R) = \begin{pmatrix} \mathrm{End}_{\hat{B}}(\mathrm{Hom}_B(M, Y)) & \mathrm{Hom}_{\hat{B}}(\mathrm{Hom}_B(M, Y), Y) \\ \mathrm{Hom}_{\hat{B}}(Y, \mathrm{Hom}_B(M, Y)) & \mathrm{Bic}_B(Y) \end{pmatrix} ,$$

where $Y = E(M) \oplus E(B)$ (as right B-modules) and $\hat{B} = \mathrm{End}_B(Y)$.

Note that $R = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ imbeds in the matrix ring $Q_{\max}(R)$ in a canonical way.

4. An example.

Let B be an arbitrary ring, and let A be a subring of the full matrix ring $M_n(B)$. Then the free right B-module B^n is a faithful left A-module, and we consider the ring

$$R = \begin{pmatrix} A & B^n \\ 0 & B \end{pmatrix} .$$

In this case we have

$$Y = E(B)^{n+1} \quad \text{and} \quad \hat{B} = M_{n+1}(\mathrm{End}_B(E(B))) .$$

Furthermore we have $\mathrm{Hom}_B(M, Y) \cong E(B)^{n(n+1)}$. Clearly we obtain from the Theorem that

$$Q_{\max}\left(\begin{pmatrix} A & B^n \\ 0 & B \end{pmatrix}\right) = \begin{pmatrix} M_n(Q_{\max}(B)) & Q_{\max}(B)^n \\ Q_{\max}(B)^n & Q_{\max}(B) \end{pmatrix} = M_{n+1}(Q_{\max}(B)) .$$

The imbedding $\begin{pmatrix} A & B^n \\ 0 & B \end{pmatrix} \to M_{n+1}(Q_{\max}(B))$ is given by

$$\begin{pmatrix} a & (b_1,\ldots,b_n) \\ 0 & b \end{pmatrix} \rightarrowtail \begin{bmatrix} a & & & b_1 \\ & \cdot & 0 & \cdot \\ & & \cdot & \cdot \\ 0 & & \cdot & \cdot \\ & & a & b_n \\ 0 & \cdot\;\cdot & \cdot\, 0 & b \end{bmatrix}.$$

Two special cases:

(i) $$R = \begin{pmatrix} K & K^n \\ 0 & K \end{pmatrix}, \quad \text{where } K \text{ is a field.}$$

Then the maximal right ring of quotients of R is $Q^r_{\max}(R) = M_{n+1}(K)$, and by symmetry also the left maximal ring of quotients of R is $Q^l_{\max}(R) = M_{n+1}(K)$. However, R is imbedded in $Q^l_{\max}(R)$ by the map

$$\begin{pmatrix} a & (b_1,\ldots,b_n) \\ 0 & b \end{pmatrix} \rightarrowtail \begin{bmatrix} a & b_1 & \cdot & \cdot & \cdot & b_n \\ 0 & b & & & & \\ \cdot & & \cdot & & 0 & \\ \cdot & & & \cdot & & \\ \cdot & 0 & & & \cdot & \\ 0 & & & & & b \end{bmatrix}.$$

Hence $Q^r_{\max}(R)$ and $Q^l_{\max}(R)$ are isomorphic rings, but they do not coincide as over-rings of R (cf. Cateforis [1]).

(ii) $R = T_n(A)$ is the ring of upper triangular matrices over an arbitrary ring A. Then

$$T_n(A) = \begin{pmatrix} T_{n-1}(A) & A^n \\ 0 & A \end{pmatrix},$$

and we obtain $Q_{\max}(T_n(A)) = M_n(Q_{\max}(A))$.

5. Another example.

Consider the ring

$$\begin{pmatrix} K & V \\ 0 & K \end{pmatrix},$$

where K is a field and V is an infinite-dimensional vector space. In this case one finds that the maximal right ring of quotients is

$$\begin{pmatrix} \mathrm{End}_K(V) & V \\ \mathrm{Hom}_K(V,K) & K \end{pmatrix}.$$

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