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ON THE APPROXIMATION OF QUATERNIONS

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1. Introduction.

For any irrational quaternion ξ the approximation constant is defined as

$$C(\xi) = \limsup (|q|\,|\xi q - p|)^{-1},$$

the limsup being extended over all $p,q \in \mathsf{H}\,(q \neq 0)$, H being the ring of quaternion integers (in the sense of Hurwitz).

In a previous paper [3] I showed that $C(\xi) \geqq (2.51)^{\frac{1}{2}}$, unless ξ is equivalent to either

$$\xi_0 = \tfrac{1}{2} + \tfrac{1}{4}(1+5^{\frac{1}{2}})i + \tfrac{1}{4}(1-5^{\frac{1}{2}})j$$

or

$$\xi_1 = \tfrac{1}{2} + \tfrac{1}{4}(1-5^{\frac{1}{2}})i + \tfrac{1}{4}(1+5^{\frac{1}{2}})j \,.$$

For $\xi \sim \xi_0$ or $\xi \sim \xi_1$ we have $C(\xi) = (5/2)^{\frac{1}{2}}$.

In this note I shall prove the following

THEOREM. *For any* $\alpha \in \mathsf{R} \setminus \mathsf{Q}$, *the quaternion*

$$\varphi(\alpha) = \tfrac{1}{2}(1+j+(i-j)\alpha)$$

has the approximation constant

$$C(\varphi(\alpha)) = 2^{-\frac{1}{2}} M(\alpha)\,,$$

where

$$M(\alpha) = \limsup (|\varrho|\,|\alpha\varrho - \pi|)^{-1}\,,$$

the limsup *being extended over all* $\pi,\varrho \in \mathsf{Z}\,(\varrho \neq 0)$, *is the real approximation constant of* α.

REMARK. Notice that by the approximation theorem of Markoff-Hurwitz (cf. [1]) we have

$$\{C(\varphi(\alpha)) \mid \alpha \in \mathsf{R}\setminus\mathsf{Q}\} \cap\,]0, 3\cdot 2^{-\frac{1}{2}}[\; = \{(\tfrac{1}{2}(9-4\cdot Q^{-2}))^{\frac{1}{2}} \mid Q \in \mathscr{M}\}\,,$$

where

$$\mathscr{M} = \{1,2,5,13,29,\ldots\}$$

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is the set of Markoff numbers. In particular, $3 \cdot 2^{-\frac{1}{2}}$ is an accumulation point of the approximation spectrum of quaternions.

Notice also that by the result of M. Hall, Jr. [2], every number $c \geqq 5.1007 \cdot 2^{-\frac{1}{2}}$ belongs to the approximation spectrum of quaternions.

2. Two lemmas.

For convenience we put

$$C_1(\varphi(\alpha)) = \limsup (|q|\,|\varphi(\alpha)q-p|)^{-1},$$

the lim sup being extended over all $p,q \in \mathsf{H}\,(q \neq 0)$ with $pq^{-1} \in \varphi(\mathsf{Q})$. Similarly we put

$$C_2(\varphi(\alpha)) = \limsup (|q|\,|\varphi(\alpha)q-p|)^{-1},$$

the lim sup being extended over all $p,q \in \mathsf{H}\,(q \neq 0)$ with $pq^{-1} \notin \varphi(\mathsf{Q})$.

Lemma 1. *For any* $\alpha \in \mathsf{R} \setminus \mathsf{Q}$ *we have*

$$C_1(\varphi(\alpha)) = 2^{-\frac{1}{2}} M(\alpha).$$

Proof. Let π/ϱ, where $\pi,\varrho \in \mathsf{Z}, \varrho \neq 0$, be any irreducible fraction. Then

$$\varphi(\pi/\varrho) = \tfrac{1}{2}(1+j+(i-j)\pi/\varrho) = (\varrho+\pi i+(\varrho-\pi)j)(2\varrho)^{-1},$$

where

$$N(\varrho+\pi i+(\varrho-\pi)j) = 2(\pi^2-\pi\varrho+\varrho^2), \qquad N(2\varrho) = 2\cdot 2\varrho^2.$$

Since g.c.d. $(\pi,\varrho)=1$, also g.c.d. $(\pi^2-\pi\varrho+\varrho^2, 2\varrho^2)=1$, hence putting

$$p = (\varrho+\pi i+(\varrho-\pi)j)(1-i)^{-1}, \qquad q = \varrho(1+i),$$

we find that

$$p,q \in \mathsf{H}, \quad \text{g.c.d.}(N(p),N(q)) = 1, \quad \varphi(\pi/\varrho) = pq^{-1}.$$

Finally, for any $\alpha \in \mathsf{R} \setminus \mathsf{Q}$

$$\begin{aligned}(|q|\,|\varphi(\alpha)q-p|)^{-1} &= (|q|^2|\varphi(\alpha)-\varphi(\pi/\varrho)|)^{-1}\\ &= (N(q)|\tfrac{1}{2}(i-j)(\alpha-\pi/\varrho)|)^{-1}\\ &= (2\varrho^2 2^{-\frac{1}{2}}|\alpha-\pi/\varrho|)^{-1}\\ &= 2^{-\frac{1}{2}}(|\varrho|\,|\alpha\varrho-\pi|)^{-1}.\end{aligned}$$

Altogether this proves Lemma 1.

Lemma 2. *For any* $\alpha \in \mathsf{R} \setminus \mathsf{Q}$ *we have*

$$C_2(\varphi(\alpha) \leqq (8/3)^{\frac{1}{2}}.$$

PROOF. From the results of [3] it follows easily that there is a unique chain of Farey simplices

$$FS^{(0)}, FS^{(1)}, \ldots, FS^{(n)}, \ldots$$

containing $\varphi(\alpha)$. Also, with a suitable enumeration of the vertices of $FS^{(n)}$,

$$FS^{(n)} = FS\big(p_1^{(n)}(q_1^{(n)})^{-1}, \ldots, p_5^{(n)}(q_5^{(n)})^{-1}\big)$$

has the following properties for all $n \in \mathsf{N}_0$,

(i) $p_1^{(n)}(q_1^{(n)})^{-1} \in \varphi(\mathsf{Q})$,

(ii) $N(q_2^{(n)}) = \ldots = N(q_5^{(n)})$,

(iii) $\frac{1}{4}\big(p_2^{(n)}(q_2^{(n)})^{-1} + \ldots + p_5^{(n)}(q_5^{(n)})^{-1}\big) \in \varphi(\mathsf{Q})$,

(iv) $\varphi(\mathsf{R}) \perp \operatorname{aff}\big(p_2^{(n)}(q_2^{(n)})^{-1}, \ldots, p_5^{(n)}(q_5^{(n)})^{-1}\big)$.

Consequently

$$\begin{aligned}|\varphi(\alpha) - p_l^{(n)}(q_l^{(n)})^{-1}| &> |\tfrac{1}{4}\big(p_2^{(n)}(q_2^{(n)})^{-1} + \ldots + p_5^{(n)}(q_5^{(n)})^{-1} - p_l^{(n)}(q_l^{(n)})^{-1}| \\ &= (3/8)^{\frac{1}{2}}|q_l^{(n)}|^{-2}\end{aligned}$$

for $l = 2, \ldots, 5$.

By Theorem 2 of [3], this proves Lemma 2.

3. Proof of Theorem.

Clearly

$$C(\varphi(\alpha)) = \max\big(C_1(\varphi(\alpha)), C_2(\varphi(\alpha))\big) .$$

Hence in case $C_1(\varphi(\alpha)) \geqq (8/3)^{\frac{1}{2}}$, it follows by Lemmas 1, 2 that

$$C(\varphi(\alpha)) = C_1(\varphi(\alpha)) = 2^{-\frac{1}{2}}M(\alpha) .$$

In case $C_1(\varphi(\alpha)) < (8/3)^{\frac{1}{2}}$, it follows by Lemma 1 and the theorem of Markoff-Hurwitz that $M(\alpha) = 5^{\frac{1}{2}}$ and that $\alpha \sim \frac{1}{2}(1 + 5^{\frac{1}{2}})$ (in the real sense). It follows easily that either $\varphi(\alpha) \sim \xi_0$ or $\varphi(\alpha) \sim \xi_1$, hence

$$C(\varphi(\alpha)) = (5/2)^{\frac{1}{2}} = 2^{-\frac{1}{2}}M(\alpha) .$$

REFERENCES

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