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CONVOLUTION SEMIGROUPS OF LOCAL TYPE

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Every vaguely continuous convolution semigroup $(\mu_t)_{t>0}$, of positive measures of total mass $\int d\mu_t \leqq 1$, on a locally compact abelian group G, determines — by convolution — a contraction semigroup $(P_t)_{t>0}$ on an appropriate Banach space of functions defined on G.

We shall study necessary and sufficient conditions for the infinitesimal generator (A,D) of $(P_t)_{t>0}$ to be a local operator, in the sense that for all $f \in D$ the support of Af is contained in the support of f. In particular, a positive measure μ on $G \setminus \{o\}$ is constructed corresponding to the semigroup $(\mu_t)_{t>0}$, and it is shown that A is a local operator if and only if μ vanishes.

The main steps are contained in Lemma 5 and Lemma 11, and they might have some independent interest.

1. NOTATION. Let G be a locally compact abelian group with dual group Γ, and let dx and $d\gamma$ be Haar measures on G and Γ, normalized in the usual way.

Let $\mathscr{K} = \mathscr{K}(G)$ be the space of complex, continuous functions on G with compact support, equipped with the usual topology, and let C_b (respectively C_0) denote the Banach space of continuous functions on G, which are uniformly continuous and bounded (respectively tend to zero at infinity), the norm being the supremum norm.

The Hilbert space of square integrable (dx) functions on G is denoted $L^2 = L^2(G,dx)$.

Let $(\mu_t)_{t>0}$ be a *vaguely continuous convolution semigroup* (that is

$$\mu_t * \mu_s = \mu_{t+s} \quad \text{for } t,s > 0$$

and

$$\lim_{t \to 0} \mu_t = \varepsilon_o$$

vaguely, where ε_o is the Dirac measure at the neutral element o of G) of positive measures with total mass $\int d\mu_t \leqq 1$ on G. The set of such semigroups is in one-to-one correspondence with the set of *continuous negative*

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definite functions on the dual group. Let $\psi: \Gamma \to \mathbf{C}$ be the function corresponding to $(\mu_t)_{t>0}$, i.e. the function ψ satisfying (cf. Deny [4])

$$\hat{\mu}_t(\gamma) = e^{-t\psi(\gamma)} \quad \text{for } t>0 \text{ and } \gamma \in \Gamma,$$

where the ^ denotes the Fourier transformation.

Let E denote any one of the Banach spaces C_0, C_b and L^2. The semigroup $(\mu_t)_{t>0}$ induces a family $(P_t)_{t>0}$ of operators on E by the definition

$$P_t f = \mu_t * f \quad \text{for } f \in E \text{ and } t>0,$$

and $(P_t)_{t>0}$ is a *strongly continuous, contraction semigroup* on E (cf. Meyer [8]).

Let $(V_\lambda)_{\lambda>0}$ denote the resolvent of $(P_t)_{t>0}$. For all $\lambda>0$, the operator V_λ is given on E, as convolution with the positive bounded measure

$$\varrho_\lambda = \int_0^\infty e^{-\lambda t} \mu_t dt \quad \text{(vaguely)}.$$

The *infinitesimal generator* (A,D) of the semigroup $(P_t)_{t>0}$ on E is defined by

$$Af = \lim_{t\to 0} t^{-1}(P_t f - f) \quad \text{for } f \in D,$$

where D is the set of elements in E such that this limit exists in E. We shall later use the fact that $V_\lambda(E) \subseteq D$ for all $\lambda>0$. It is clear that the operators $(P_t)_{t>0}$, $(V_\lambda)_{\lambda>0}$ and (A,D) all commute with the translations of G.

Let (A_0,D_0) (respectively (A_b,D_b), respectively (A_2,D_2)) denote the infinitesimal generator for $(P_t)_{t>0}$, considered as contraction semigroup on C_0 (respectively C_b, respectively L^2).

The following simple characterization of (A_2,D_2) in terms of the associated negative definite function ψ, can be found in [1].

2. Lemma. *The domain D_2 of A_2 is given by*

$$D_2 = \{f \in L^2(G) \mid \psi \hat{f} \in L^2(\Gamma, d\gamma)\}$$

and

$$(A_2 f)^\wedge = -\psi \hat{f} \quad \textit{for } f \in D_2.$$

Here the ^ denotes the Fourier–Plancherel transformation from $L^2(G,dx)$ onto $L^2(\Gamma,d\gamma)$.

3. Definition. The semigroup $(\mu_t)_{t>0}$ determines a sesquilinear form $\beta: D_2 \times D_2 \to \mathbf{C}$ by the definition

$$\beta(f,g) = (-A_2 f, g) \quad \text{for } f,g \in D_2,$$

(inner product in $L^2(G)$). By Lemma 2 we can write

$$\beta(f,g) = (\psi\hat{f},\hat{g}) \quad \text{for } f,g \in D_2 ,$$

(inner product in $L^2(\Gamma)$).

A family $(\beta_\lambda)_{\lambda>0}$ of sesquilinear forms $\beta_\lambda: L^2 \times L^2 \to \mathsf{C}$ is defined by putting

$$\beta_\lambda(f,g) = \left(\frac{\lambda\psi}{\lambda+\psi}\hat{f},\hat{g}\right) \quad \text{for } f,g \in L^2 .$$

This is well-defined, in view of the inequality

$$|\lambda\psi/(\lambda+\psi)| \leqq \lambda \quad \text{for } \lambda>0 ,$$

which is easily established, using that $\operatorname{Re}\psi \geqq 0$.

The sesquilinear form β — the *form associated with the semigroup* $(\mu_t)_{t>0}$ — is approximated by the sesquilinear forms $(\beta_\lambda)_{\lambda>0}$ in the following way.

4. Lemma. *For all $f,g \in D_2$ we have*

$$\beta(f,g) = \lim_{\lambda\to\infty}\beta_\lambda(f,g) .$$

Proof. Let $f,g \in D_2$. For all $\lambda>0$ we have the inequality

$$|\lambda\psi/(\lambda+\psi)| \leqq |\psi| ,$$

and since

$$\lim_{\lambda\to\infty}\lambda\psi/(\lambda+\psi) = \psi \quad \text{pointwise on } \Gamma ,$$

the result follows by dominated convergence.

5. Lemma. *The set $\mathscr{K} \cap D_0 \cap D_2$ is dense in $\mathscr{K}$.*

Proof. Let $\lambda>0$ be fixed. The measure ϱ_λ (cf. 1) is a (bounded) Hunt kernel, hence a "noyau associé" in the sense of Deny (cf. Deny [2] and [3]). It follows that there exists for every $\omega \in \mathscr{N}$, where $\mathscr{N}$ is the set of open, relatively compact neighbourhoods of o, a positive measure μ_ω, of total mass $\int d\mu_\omega \leqq 1$, such that the measure $\varrho_\lambda*(\varepsilon_o-\mu_\omega)$ is non-negative, non-vanishing and has support contained in $\bar{\omega}$. The set

$$\mathscr{P} = \{\varrho_\lambda*(\varepsilon_0-\mu_\omega)*\varphi \mid \varphi \in \mathscr{K},\ \omega \in \mathscr{N}\}$$

is therefore dense in $\mathscr{K}$, and since $\mathscr{P}$ consists of differences of ϱ_λ-potentials generated by functions from $C_0 \cap L^2$ (μ_ω is bounded) we have (cf. 1) that

$$\mathscr{P} \subseteq \varrho_\lambda*(C_0 \cap L^2) \subseteq D_0 \cap D_2 .$$

6. REMARK. It is clear that the set $\mathscr{K} \cap D_0 \cap D_2$ is dense in C_0 and L^2 with their respective topologies.

7. PROPOSITION. *There exists a positive measure* μ *on* $G \setminus \{o\}$ *such that*

$$(*) \qquad \mu(\check{f} * \bar{g}) = -\beta(f,g)$$

for all $f,g \in \mathscr{K} \cap D_2$ *satisfying* $\operatorname{supp} \check{f} * \bar{g} \subseteq G \setminus \{o\}$.

PROOF. Suppose that $f,g,f_1,g_1 \in \mathscr{K} \cap D_2$ satisfies

$$\check{f} * \bar{g} = \check{f}_1 * \bar{g}_1 .$$

For every $\lambda > 0$ we then have

$$\begin{aligned} \beta_\lambda(f,g) &= \lambda(\hat{f},\hat{g}) - \lambda^2((\lambda+\psi)^{-1}\hat{f},\hat{g}) \\ &= \lambda \check{f} * \bar{g}(o) - \lambda^2 \varrho_\lambda(\check{f} * \bar{g}) \\ &= \beta_\lambda(f_1,g_1) \end{aligned}$$

and Lemma 4 now gives, that

$$\beta(f,g) = \beta(f_1,g_1) .$$

This shows that $\mu(\check{f} * \bar{g})$ is well-defined by (*), and an analogous reasoning gives that the mapping

$$\check{f} * \bar{g} \longmapsto \mu(\check{f} * \bar{g})$$

extends by linearity to a linear form, also denoted μ, on

$$\mathscr{K}^* = \operatorname{span}\{\check{f} * \bar{g} \mid f,g \in \mathscr{K} \cap D_2, \operatorname{supp} \check{f} * \bar{g} \subseteq G \setminus \{o\}\}$$

Consider $h \in \mathscr{K}^*$ and suppose that $h \geqq 0$. We can write

$$h = \textstyle\sum_{i=1}^n a_i \check{f}_i * \bar{g}_i$$

where $a_i \in \mathbf{C}$ and $f_i, g_i \in \mathscr{K} \cap D_2$ satisfies $\operatorname{supp} \check{f}_i * \bar{g}_i \subseteq G \setminus \{o\}$, and we find

$$\begin{aligned} \mu(h) &= \textstyle\sum_{i=1}^n a_i \, \mu(\check{f}_i * \bar{g}_i) \\ &= \textstyle\sum_{i=1}^n a_i \lim_{\lambda \to \infty}(-\beta_\lambda(f_i,g_i)) \\ &= \textstyle\sum_{i=1}^n a_i \lim_{\lambda \to \infty}(\lambda^2 \varrho_\lambda(\check{f}_i * \bar{g}_i)) \\ &= \lim_{\lambda \to \infty} \lambda^2 \varrho_\lambda(h) \geqq 0 . \end{aligned}$$

The linear form μ is thus positive on $\mathscr{K}^*$, and since $\mathscr{K}^*$ by Lemma 5 is dense in $\mathscr{K}(G \setminus \{o\})$, we see that μ determines a positive measure on $G \setminus \{o\}$.

8. REMARKS. a) In analogy with the situation of regular, translation invariant Dirichlet spaces, we will say that the measure μ from Proposition 7, is the *singular measure* associated with the semigroup $(\mu_t)_{t>0}$ (cf. Itô [7]).

b) From the proof of Proposition 7 we see that μ is vague limit of the measures $\lambda^2\varrho_\lambda | \complement\{o\}$, that is, for all $\varphi \in \mathscr{K}$ such that $\operatorname{supp}\varphi \subseteq G \setminus \{o\}$ we have

$$\mu(\varphi) = \lim_{\lambda\to\infty}\lambda^2\varrho_\lambda(\varphi) .$$

Likewise we have for all such φ, that

$$\mu(\varphi) = \lim_{t\to 0}t^{-1}\mu_t(\varphi) .$$

To see this, it is enough (by Lemma 5) to consider functions φ of the form $\varphi = \check{f} * \bar{g}$ with $f, g \in \mathscr{K} \cap D_2$ such that $\operatorname{supp}\check{f} * \bar{g} \subseteq G \setminus \{o\}$. We then find

$$\begin{aligned} \lim_{t\to 0}t^{-1}\mu_t(\check{f} * \bar{g}) &= \lim_{t\to 0}t^{-1}(\mu_t * f - f, g) \\ &= (A_2 f, g) \\ &= -\beta(f, g) \\ &= \mu(\check{f} * \bar{g}) . \end{aligned}$$

c) It is rather easy to see that the measure μ is the measure constructed by Harzallah (cf. [6]) starting from the negative definite function ψ. So (at least) in the symmetric case (ψ real), μ is the measure of the Lévy–Khinchine representation of ψ.

9. PROPOSITION. *The following three conditions are equivalent.*

(i) *For all* $f \in D_b$: $\operatorname{supp}A_b f \subseteq \operatorname{supp}f$.

(ii) *For all* $f \in D_0$: $\operatorname{supp}A_0 f \subseteq \operatorname{supp}f$.

(iii) *For all* $f \in D_2$: $\operatorname{supp}A_2 f \subseteq \operatorname{supp}f$.

PROOF. (ii) $\Rightarrow$ (iii). Let $f \in D_2$. Since $\mathscr{K} \cap D_0 \cap D_2$ is dense in L^2 (cf. 6) it is enough to prove that $(A_2 f, \varphi) = 0$ for all $\varphi \in \mathscr{K}$ satisfying $\check{\varphi} \in D_0 \cap D_2$ and $\operatorname{supp}\varphi \cap \operatorname{supp}f = \emptyset$. For such a φ we find

$$\begin{aligned} (A_2 f, \varphi) &= \lim_{t\to 0}t^{-1}(\mu_t * f - f, \varphi) \\ &= \lim_{t\to 0}t^{-1}(f, (\mu_t * \check{\varphi} - \check{\varphi})^\vee) \\ &= (f, (A_0\check{\varphi})^\vee) \end{aligned}$$

which is zero, because $\operatorname{supp}(A_0\check{\varphi})^\vee \subseteq \operatorname{supp}\varphi$.

The implication (iii) $\Rightarrow$ (i) can be proved analogously, and (i) $\Rightarrow$ (ii) is trivial.

10. Definition. The semigroup $(\mu_t)_{t>0}$ is said to be of *local type*, if the conditions (i), (ii) and (iii) of Proposition 9 are fulfilled.

11. Lemma. *Let U and V be open, relatively compact subsets of G such that $\overline{U} \subseteq V$. There exists a function $\varphi \in D_0$ satisfying*

$$0 \leqq \varphi \leqq 1, \quad \varphi = 1 \text{ on } U, \quad \varphi = 0 \text{ in } \complement V .$$

Proof. Let $\lambda > 0$ be fixed. The measure ϱ_λ is a "noyau associé" for which the non-negative constants are superharmonic (cf. [2]). Let ω be an open, relatively compact neighbourhood of o such that

$$(\bar{\omega} + \overline{U}) \cap (\bar{\omega} + \complement V) = \emptyset .$$

Using the compactness of $\bar{\omega} + \overline{U}$ it is easy to find a function φ' of the form

$$\varphi' = \varrho_\lambda * f - \varrho_\lambda * g ,$$

where $f, g \in C_0^+$ and the measures $\sigma = f dx$ and $\tau = g dx$ are bounded (cf. the proof of Lemma 5), with the following properties

$$0 \leqq \varphi', \quad \varphi' \geqq 1 \text{ on } \bar{\omega} + U, \quad \varphi' = 0 \text{ in } \bar{\omega} + \complement V .$$

The function

$$\varphi'' = \inf(\varrho_\lambda * g + 1, \varrho_\lambda * f) - \varrho_\lambda * g$$

is then continuous and satisfies

$$0 \leqq \varphi'' \leqq 1, \quad \varphi'' = 1 \text{ on } \bar{\omega} + \overline{U}, \quad \varphi'' = 0 \text{ in } \bar{\omega} + \complement V .$$

Moreover there exists a measure σ' such that

$$\varrho_\lambda * \sigma' = \inf(\varrho_\lambda * g + 1, \varrho_\lambda * f)$$

as measures (cf. [2] p. 79 and 85), and we shall now see that σ' is bounded. The measure $\check{\varrho}_\lambda$ is also a "noyau associé", and there exists consequently (cf. [2] p. 94) for every open, relatively compact set $B \subseteq G$ a positive measure μ_B, supported by $\overline{B}$ and such that

$$0 \leqq \check{\varrho}_\lambda * \mu_B \leqq 1 \quad \text{and} \quad \check{\varrho}_\lambda * \mu_B = 1 \text{ on } B .$$

For a fixed $h \in \mathscr{K}^+$ satisfying $\int h(x) dx = 1$ we find

$$\begin{aligned} \int \check{\varrho}_\lambda * \mu_B * h d\sigma' &= \int \mu_B * h d\varrho_\lambda * \sigma' \\ &\leqq \int \mu_B * h d\varrho_\lambda * \sigma \\ &= \int \check{\varrho}_\lambda * \mu_B * h d\sigma \\ &\leqq \int d\sigma \end{aligned}$$

and it now follows, by taking supremum over sets B as above, that σ' is bounded. Let $h \in \mathscr{K}^+$ satisfy

$$\operatorname{supp} h \subseteq \omega \quad \text{and} \quad \int h(x)dx = 1 .$$

The function

$$\varphi = \varrho_\lambda * (\sigma' * h) - \varrho_\lambda * (\tau * h) = \varphi'' * h$$

belongs to D_0, since $\sigma' * h$, $\tau * h \in C_0$, and it clearly satisfies

$$0 \leqq \varphi \leqq 1, \quad \varphi = 1 \text{ on } U, \quad \varphi = 0 \text{ in } \complement V .$$

12. THEOREM. *The following three conditions are equivalent.*

1) *The semigroup* $(\mu_t)_{t>0}$ *is of local type.*

2) *For all open, relatively compact neighbourhoods* ω *of* o *we have*

$$\lim_{t\to 0} t^{-1}\mu_t(\complement\omega) = 0 .$$

3) *The singular measure* μ *vanishes.*

PROOF. 1) $\Rightarrow$ 2). Let ω be an open, relatively compact neighbourhood of o and let ω_0 be an open, symmetric neighbourhood of o such that $\bar{\omega}_0 \subseteq \omega$. Choose a function $\varphi \in D_0$ satisfying the conditions of Lemma 11 relative to the pair $(\omega_0, \check{\omega})$. Since the total masses of the semigroup measures μ_t are given by

$$\int d\mu_t = e^{-\lambda_0 t}$$

for a suitable $\lambda_0 \geqq 0$, all constant functions belong to D_b, and it follows that the function $\varphi_0 = 1 - \varphi$ belongs to D_b and satisfies

$$0 \leqq \varphi_0 \leqq 1, \quad \varphi_0 = 0 \text{ on } \omega_0, \quad \varphi_0 = 1 \text{ on } \complement\check{\omega} ,$$

and in particular

$$1_{\complement\omega} \leqq \check{\varphi}_0 .$$

Since $(\mu_t)_{t>0}$ is supposed to be of local type we find

$$\begin{aligned} 0 &\leqq \liminf_{t\to 0} t^{-1}\mu_t(\complement\omega) \\ &\leqq \limsup_{t\to 0} t^{-1}\mu_t(\complement\omega) \\ &\leqq \limsup_{t\to 0} t^{-1}\mu_t(\check{\varphi}_0) \\ &= A_b\varphi_0(o) = 0 . \end{aligned}$$

2) $\Rightarrow$ 3). This is clear, since μ is the vague limit on $G \setminus \{o\}$ of the measures $t^{-1}\mu_t | \complement\{o\}$, cf. Remark 8 b).

3) $\Rightarrow$ 1). From the proof of Proposition 9, it is clear that it suffices to show that

$$\operatorname{supp} A_0 f \subseteq \operatorname{supp} f$$

for all $f \in \mathscr{K} \cap D_0$. Let $f \in \mathscr{K} \cap D_0$ and suppose that $f=0$ in the open set ω. Let $x \in \omega$. We shall show that $A_0 f(x)=0$. By the translation invariance of (A_0, D_0) we may (and do) suppose that $x=0$, and Remark 8 b) now gives that

$$\begin{aligned} A_0 f(o) &= \lim_{t\to 0} t^{-1}(\mu_t * f(o) - f(o)) \\ &= \lim_{t\to 0} t^{-1}\mu_t(\check{f}) \\ &= \mu(\check{f}) = 0 . \end{aligned}$$

13. REMARKS. a) It can be shown that a semigroup $(\mu_t)_{t>0}$ consisting of probability measures is of local type if and only if the associated sesquilinear form β has the following local character, cf. Deny [4]: $\beta(f,g)=0$ for all $f,g \in \mathscr{K} \cap D_2$ such that f is constant in a neighbourhood of the support of g.

b) Let X be the Hunt process with state space $(G, \mathscr{B}(G))$ (here $\mathscr{B}(G)$ is the Borel sets in G) and with transition probabilities given by

$$P_t(x,A) = \mu_t(A-x)$$

for $t>0$, $x \in G$ and $A \in \mathscr{B}(G)$. The discussion in [5, § 6] yields that $(\mu_t)_{t>0}$ is of local type if and only if X has continuous trajectories. This is in accordance with the intuitive interpretation of the singular measure μ as determining the "jumps" of X (cf. Remark 8 c)).

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