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MODULUS OF APPROXIMATE CONTINUITY FOR $R(X)$

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1. Introduction.

Let X be a compact subset of the plane C. We denote by $R_0(X)$ the algebra consisting of the (restrictions to X of) rational functions having no pole on X, and by $R(X)$ the uniform closure of $R_0(X)$. We say that φ is an *admissible function* if (a) φ is a positive, non-decreasing function defined on $(0,\infty)$ and (b) the associated function ψ, defined by $\psi(r) = r/\varphi(r)$, is also non-decreasing, with $\psi(0^+)=0$.

Throughout this paper, Σ will denote the Riemann sphere, $\|\cdot\|$ will denote the supremum norm over the appropriate set,

$$\Delta(x,r) = \{y : |y-x| \leqq r\} ,$$

$$A_n(x) = \{y : 2^{-(n+1)} < |y-x| < 2^{-n}\} ,$$

and m will denote the 2-dimensional Lebesgue measure.

Fix $x \in \mathsf{C}$. We say that a set $E \subset \mathsf{C}$ has *full area density at* x if

$$\lim_{r\to 0} m(E \cap \Delta(x,r))/m(\Delta(x,r)) = 1 .$$

Let F be a function defined on X, $x \in X$. We say that F admits φ as *modulus of approximate continuity at* x if

$$|F(y)-F(x)| \leqq \varphi(|y-x|)$$

for all y in a set having full area density at x; here φ is a positive function on $(0,\infty)$.

In [3], we proved the following theorem: Let φ be an admissible function. Suppose there exists a (complex Borel) measure μ on X representing x for $R(X)$ (i.e. $\int f d\mu = f(x)$ for all $f \in R(X)$) such that $\mu(\{x\})=0$ and $\int \varphi(|z-x|)^{-1} d|\mu| < \infty$. Then the unit ball of $R(X)$ admits $\varepsilon\varphi$ as a modulus of approximate continuity at x for every $\varepsilon > 0$.

The converse is well-known to be true when $\varphi \equiv 1$. One might conjecture that the converse is true, in general. The main result of this paper is to disprove this conjecture. In section 2, we present a special class of compact sets in the plane. In section 3, we give a necessary condition

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for the existence of such representing measures in terms of analytic capacity which was first observed by O'Farrell [2]. We also examine the relations among modulus of approximate continuity, representing measure, and analytic capacity.

2. Construction.

DEFINITION. Let $\overline{D_0}$ be the closed unit disk, and let D_n be the open disk with center a_n and radius ϱ_n. We say that X is a set of type (L) if

(a) $X=\overline{D_0}\setminus\bigcup_1^\infty D_n$, $0\in X$ and
(b) $a_n=\frac{3}{4}2^{-n}$, $\varrho_n\geqq 0$, $\overline{D_n}\subset A_n$ where $A_n=A_n(0)$.

REMARK. Suppose X is a compact set of type (L) with $\sum(\varrho_n/a_n)<\infty$. Let $X_N=\overline{D_0}\setminus\bigcup_1^N D_n$; then 0 lies in the interior of X_N for each N, and

$$f(0)=\frac{1}{2\pi i}\int_{\partial X_N}\frac{f\,dz}{z}=\int f\,d\mu_N \quad \text{for } f\in R_0(X_N),$$

by Cauchy's integral formula. Since $\bigcap X_N=X$, each $f\in R_0(X)$ belongs to $R_0(X_N)$ for N sufficiently large. Now

$$\|\mu_N-\mu_M\|\leqq(2\pi)^{-1}\textstyle\sum_{N+1}^M\int_{\partial D_n}|z|^{-1}dz \quad \text{for } M>N,$$

and

$$(2\pi)^{-1}\textstyle\int_{\partial D_n}|z|^{-1}dz\leqq\varrho_n(a_n-\varrho_n)^{-1}\leqq\frac{3}{2}\varrho_n a_n^{-1}.$$

Hence $\{\mu_N\}$ converges in norm to a measure μ, which represents 0 for $R(X)$ and has no point mass at 0.

Moreover, if φ is an admissible function and $\sum\varrho_n a_n^{-1}\varphi(a_n)^{-1}<\infty$, then

$$\begin{aligned}\int\varphi(|z|)^{-1}d|\mu| &< (2\pi)^{-1}\textstyle\sum_0^\infty\int_{\partial D_n}|z|^{-1}\varphi(|z|)^{-1}|dz|\\ &\leqq\varphi(1)^{-1}+\textstyle\sum_1^\infty\varrho_n(a_n-\varrho_n)^{-1}\varphi(a_n-\varrho_n)^{-1}\\ &\leqq\varphi(1)^{-1}+3\textstyle\sum_1^\infty\varrho_n a_n^{-1}\varphi(a_n)^{-1}<\infty,\end{aligned}$$

since $(a_n-\varrho_n)\geqq\frac{2}{3}a_n>\frac{1}{2}a_n$.

LEMMA 2.1. *Suppose X is a compact set of type* (L) *with $\sum\varrho_n/a_n=\infty$. Then there is no measure μ representing* 0 *for $R(X)$ with $\mu(\{0\})=0$.*

PROOF. Let $f_m=(\sum_1^m\varrho_n/a_n)^{-1}\sum_1^m\varrho_n/(a_n-z)$, then $f_m\in R(X)$ for each m. Suppose $y\in\bar{A}_N\cap X$. Then

$$\begin{aligned}|f_m(y)| &\leqq(\textstyle\sum\varrho_n/a_n)^{-1}(\sum_1^{N-2}\varrho_n/|a_n-y|+3+\sum_{N+2}^\infty\varrho_n/|a_n-y|)\\ &\leqq(\textstyle\sum_1^m\varrho_n/a_n)^{-1}(3\sum_{n<N-1}\varrho_n/a_n+3+2^{N+2}\sum_{n>N+1}\varrho_n)\end{aligned}$$

if $m \geqq N-1$, and

$$|f_m(y)| \leqq (\textstyle\sum_1^m \varrho_n/a_n)^{-1}(3 \sum_{n \leqq m} \varrho_n/a_n)$$

if $m < N-1$. Since $\varrho_n \leqq \frac{1}{3}a_n < 2^{-n}$, we have $\sum_{n>N+1}\varrho_n < 2^{-(N+1)}$, so

$$|f_m(y)| \leqq 3+5(\textstyle\sum_1^m \varrho_n/a_n)^{-1} < C$$

for all m, and for $m \geqq N-1$,

$$|f_m(y)| \leqq \frac{3\sum_{n<N-1}\varrho_n/a_n+5}{\sum_{n\leqq m}\varrho_m/a_n} \to 0$$

as $m \to \infty$. Thus f_m converges boundedly to 0 on $X \setminus \{0\}$. But $f_m(0)=1$ for all m, the lemma is proved.

LEMMA 2.2. *Let φ be an admissible function with $\varphi(0^+)=0$. Suppose X is a compact set of type* (L) *with $\sum \varrho_n a_n^{-1}\varphi(a_n)^{-1}=\infty$. Then there is no measure μ representing* 0 *for $R(X)$ such that $\int \varphi(|z|)^{-1}d|\mu| < \infty$.*

PROOF. We can assume $\sum \varrho_n/a_n = C < \infty$; otherwise we are done by Lemma 2.1. Let

$$f_m = \textstyle\sum_1^m \varrho_n \varphi(a_n)^{-1}(a_n - z)^{-1},$$

then $f_m \in R(X)$ for each m. Suppose $y \in \overline{A_N} \cap X$. Then

$$\begin{aligned} |f_m(y)| &\leqq \textstyle\sum_1^{N-2} \varrho_n \varphi(a_n)^{-1}|a_n-y|^{-1} + \sum_{N-1}^{N+1}\varphi(a_n)^{-1} \\ &\qquad + \textstyle\sum_{N+2}^{\infty} \varrho_n\varphi(a_n)^{-1}|a_n-y|^{-1} \\ &\leqq 3\textstyle\sum_{n<N-1}\varrho_n a_n^{-1}\varphi(a_n)^{-1} + \sum_{N-1}^{N+1}\varphi(a_n)^{-1} + \sum_{n>N+1}\varrho_n\varphi(a_n)^{-1}2^{N+2}. \end{aligned}$$

Now

$$\textstyle\sum_{n<N-1}\varrho_n a_n^{-1}\varphi(a_n)^{-1} \leqq (\sum_{n<N-1}\varrho_n/a_n)\varphi(a_{N-2})^{-1} \leqq C/\varphi(|y|).$$

Also,

$$\begin{aligned} \textstyle\sum_{n>N+1}\varrho_n\varphi(a_n)^{-1}2^{N+2} &= \textstyle\sum_{n>N+1}(\varrho_n/a_n)\psi(a_n)\,2^{N+2} \\ &\leqq (\textstyle\sum_{n>N+1}\varrho_n/a_n)\psi(a_{N+2})2^{N+2} \leqq 4C/\varphi(|y|). \end{aligned}$$

Finally, we have $\sum_{N-1}^{N+1}\varphi(a_n)^{-1} \leqq 7/\varphi(|y|)$ since

$$\varphi(|y|) \leqq \varphi(a_{N-1}), \quad \varphi(|y|) \leqq 2\varphi(a_N) \quad \text{and} \quad \varphi(|y|) \leqq 4\varphi(a_{N+1}).$$

Thus, $|f_m(y)| \leqq 7(C+1)/\varphi(|y|)$ for each m, all $y \in X \setminus \{0\}$. But

$$f_m(0) = \textstyle\sum_1^m \varrho_n a_n^{-1}\varphi(a_n)^{-1} \to \infty \quad \text{as } m \to \infty,$$

the lemma is proved.

MAIN THEOREM. *Let φ be an admissible function with $\varphi(0^+)=0$. Then there is a compact set X and $x\in X$, such that the unit ball of $R(X)$ admits $\varepsilon\varphi$ as modulus of approximate continuity at x for every $\varepsilon>0$, while there is no measure μ representing x for $R(X)$ with $\int\varphi(|z-x|)^{-1}d|\mu|<\infty$.*

PROOF. We can choose an increasing sequence $\{N_k\}$ such that

(i) $\varphi(a_{N_{k+1}}) < \frac{1}{2}\varphi(a_{N_k})$

(ii) $\psi(a_{N_{k+1}})^{-1} > \sum_{j=1}^{k}\psi(a_{N_j})^{-1}$.

We set $\varrho_{N_k}=k^{-1}a_{N_k}\varphi(a_{N_k})$ and $\varrho_n=0$ if $n\notin\{N_k\}$. We form $X=\overline{D_0}\setminus\bigcup D_n$ as above; then X is a compact set of type (L). Immediately we obtain

$$\sum \varrho_n/a_n = \sum k^{-1}\varphi(a_{N_k}) \leqq \sum\varphi(a_{N_k}) < 2\varphi(a_1)\,,$$

and

$$\sum_n \varrho_n a_n{}^{-1}\varphi(a_n)^{-1} = \sum_k k^{-1} = \infty\,,$$

hence there is no measure μ representing 0 for $R(X)$ with $\int\varphi(|z|)^{-1}d|\mu|<\infty$ by Lemma 2.2.

On the other hand, we observe that for every $f\in R(X)$, $y\in\overline{A_N}\cap\mathring{X}$, ($\mathring{X}$ = interior of X), $N\geqq 1$,

$$\begin{aligned}|f(y)-f(0)| &= \left|\frac{1}{2\pi i}\sum_0^\infty\int_{\partial D_n}\frac{fdz}{z-y}-\frac{1}{2\pi i}\sum_0^\infty\int_{\partial D_n}\frac{fdz}{z}\right|\\ &\leqq (2\pi)^{-1}|y|\cdot\|f\|\sum_0^\infty\int_{\partial D_n}|z|^{-1}|z-y|^{-1}|dz|\\ &\leqq |y|\cdot\|f\|[2+\textstyle\sum_1^\infty \varrho_n(a_n-\varrho_n)^{-1}d_n(y)^{-1}]\end{aligned}$$

where $d_n(y)$ denotes the distance between y and D_n. We may assume $\varrho_n/a_n<\frac{1}{4}$ for each n. Then for $n<N$,

$$d_n(y) \geqq (a_n-\varrho_n)-2^{-N} \geqq \tfrac{9}{16}2^{-n}-2^{-N} \geqq \tfrac{1}{16}2^{-n} = \tfrac{1}{12}a_n\,,$$

and for $n>N$,

$$d_n(y) \geqq 2^{-(N+1)}-(a_n+\varrho_n) \geqq 2^{-(N+1)}-\tfrac{15}{16}2^{-n} \geqq \tfrac{1}{16}2^{-(N+1)} = \tfrac{1}{24}a_N\,.$$

Let

$$E = \bigcup_n\{y\in A_n:\ d_n(y)\geqq d_n\}\,,$$

where $d_n=[\varrho_n a_n{}^{-1}\varphi(a_n)^{-1}]^{\frac{1}{2}}a_n$. Then for $y\in\overline{A_N}\cap\mathring{X}\cap E$,

$$\begin{aligned}|f(y)-f(0)| &\leqq 2|y|\cdot\|f\|[1+\textstyle\sum_1^\infty\varrho_n a_n{}^{-1}d_n(y)^{-1}]\\ &\leqq 2|y|\cdot\|f\|[1+12\textstyle\sum_{n<N}\varrho_n a_n{}^{-2}+\varrho_N a_N{}^{-1}d_N{}^{-1}\\ &\qquad +24\textstyle\sum_{n>N}\varrho_n a_n{}^{-1}a_N{}^{-1}]\\ &\leqq 96\varphi(|y|)\|f\|[\psi(a_N)+\textstyle\sum_{n<N}\varrho_n a_n{}^{-2}\psi(a_N)+\varrho_N\varphi(a_N)^{-1}d_N{}^{-1}\\ &\qquad +\textstyle\sum_{n>N}\varrho_n a_n{}^{-1}\varphi(a_N)^{-1}]\,.\end{aligned}$$

We note that E has full area density at 0 for $(d_n+\varrho_n)/a_n \to 0$ as $n \to \infty$. To prove our Main Theorem, it suffices to show that:

(a) $\sum_{n<N} \varrho_n a_n^{-2} = o(\psi(a_N)^{-1})$

(b) $\varrho_N \varphi(a_N)^{-1} d_N^{-1} = o(1)$

(c) $\sum_{n>N} \varrho_n / a_n = o(\varphi(a_N))$.

Clearly (b) is satisfied. If $N_m \leqq N < N_{m+1}$,

$$\begin{aligned}\textstyle\sum_{n<N} \varrho_n a_n^{-2} &\leqq \textstyle\sum_{k=1}^{m} k^{-1}\psi(a_{N_k})^{-1} \\ &= \textstyle\sum_{k=1}^{p-1} k^{-1}\psi(a_{N_k})^{-1} + \sum_{p}^{m} k^{-1}\psi(a_{N_k})^{-1} \\ &\leqq \textstyle\sum_{k=1}^{p-1} \psi(a_{N_k})^{-1} + p^{-1}\sum_{k=p}^{m-1} \psi(a_{N_k})^{-1} + m^{-1}\psi(a_{N_m})^{-1} \\ &< \psi(a_{N_p})^{-1} + p^{-1}\psi(a_{N_m})^{-1} + m^{-1}\psi(a_{N_m})^{-1} \\ &\leqq (\psi(a_{N_m})\psi(a_{N_p})^{-1} + p^{-1} + m^{-1})\psi(a_{N_m})^{-1} \\ &= o(\psi(a_N)^{-1})\end{aligned}$$

by choosing sufficiently large p first, so (a) checks. Also

$$\begin{aligned}\textstyle\sum_{n>N} \varrho_n/a_n &\leqq \textstyle\sum_{k=m+1}^{\infty} k^{-1}\varphi(a_{N_k}) \\ &\leqq (m+1)^{-1}\textstyle\sum_{m+1}^{\infty} \varphi(a_{N_k}) \\ &< (m+1)^{-1}\textstyle\sum_{k=0}^{\infty} \varphi(a_{N_{m+1}})2^{-k} = 2(m+1)^{-1}\varphi(a_{N_{m+1}}) = o(\varphi(a_N)),\end{aligned}$$

so (c) holds.

3. Analytic capacity.

If $U \subset \mathbf{C}$ is a bounded open set, we define the *analytic capacity* of U by

$$\gamma(U) = \sup\{|f'(\infty)| : f \in R(\Sigma \setminus U), \|f\|_{\Sigma \setminus U} \leqq 1, f(\infty)=0\},$$

where $f'(\infty) = \lim_{z\to\infty} zf(z)$.

We remark that, if U is an open disk with radius ϱ, then $\gamma(U)=\varrho$.

THEOREM 3.1. *Let φ be an admissible function and p a non-negative integer. Suppose*

$$\sum 2^{(p+1)n}\varphi(2^{-n})^{-1}\gamma(A_n(x)\setminus X) = \infty .$$

Then there is no measure μ representing x for $R(X)$ such that

$$\mu(\{x\}) = 0 \quad \text{and} \quad \int |z-x|^{-p}\varphi(|z-x|)^{-1} d|\mu| < \infty .$$

PROOF. We may assume $2^{(p+1)n}\varphi(2^{-n})^{-1}\gamma(A_n(x)\setminus X)\leqq 1$ for each n, because if μ is a measure representing x for $R(X)$, then μ is a measure representing x for $R(Y)$ for all compact $Y\supset X$.

We choose $N_1\leqq M_1<N_2\leqq M_2<\dots$ so that

$$1 \leqq \sum_{N_j}^{M_j} 2^{(p+1)n}\varphi(2^{-n})^{-1}\gamma(A_n(x)\setminus X) \leqq 2 .$$

For each n, we choose $f_n\in R\big(X\cup(\Sigma\setminus A_n(x))\big)$ such that $\|f_n\|\leqq 1, f_n(\infty)=0$ and $f_n'(\infty)>\frac{1}{2}\gamma(A_n(x)\setminus X)$. We set

$$g_j(z) = \varphi(|z-x|)(z-x)^{p+1}\sum_{N_j}^{M_j} 2^{(p+1)n}\varphi(2^{-n})^{-1}f_n(z) ,$$

then a familiar type of argument for Melnikov's theorem (cf. [1, p. 206]) shows that $\{g_j\}$ is uniformly bounded on each compact subset of C. Let

$$h_j = \varphi(|z-x|)^{-1}(z-x)g_j \quad \text{and} \quad F_j = (z-x)^{-(p+1)}h_j .$$

We see that h_j and F_j are holomorphic in $\mathsf{C}\setminus\Delta(x,2^{-N_j})$ and $\Sigma\setminus\Delta(x,2^{-N_j})$, respectively,

$$F_j(\infty) = \sum_{N_j}^{M_j} 2^{(p+1)n}\varphi(2^{-n})^{-1}f_n'(\infty)$$

which lies in $[\frac{1}{2},2]$ and $\{h_j\}$, $\{F_j\}$ are uniformly bounded on each compact subset of C and $\mathsf{C}\setminus\{x\}$, respectively. Moreover, $\{F_j\}$ is uniformly bounded on each compact subset of $\Sigma\setminus\{x\}$ by the maximum modulus principle. By passing to a subsequence, we have $\lim_{j\to\infty}F_j(\infty)=\beta$ for some $\beta\in[\frac{1}{2},2]$, and $h_j\to h$, $F_j\to F$ uniformly on each compact subset of $\mathsf{C}\setminus\{x\}$ and $\Sigma\setminus\{x\}$, respectively; whence $F=(z-x)^{-(p+1)}h$ on $\mathsf{C}\setminus\{x\}$. Since h is bounded near x, $\lim_{z\to x}h(z)=0$ and

$$\lim_{z\to\infty}(z-x)^{-(p+1)}h = \lim_{z\to\infty}F(z) = F(\infty) = \lim_{j\to\infty}F_j(\infty) = \beta ,$$

we get that h is entire and

$$h(z) = \beta(z-x)^{p+1}+\sum_1^p \beta_l(z-x)^l$$

where β_l is a constant for each l. Thus,

$$g_j = \varphi(|z-x|)(z-x)^{-1}h_j$$
$$\to \varphi(|z-x|)(z-x)^{-1}h = \beta\varphi(|z-x|)(z-x)^p+\sum_1^p \beta_l\varphi(|z-x|)(z-x)^{l-1}$$

boundedly on each bounded subset of $\mathsf{C}\setminus\{x\}$, so

$$\int g_j d\sigma \to \int \beta\varphi(|z-x|)(z-x)^p d\sigma+\sum_1^p \beta_l \int \varphi(|z-x|)(z-x)^{l-1}d\sigma$$

for every compactly supported measure σ, with $\sigma(\{x\})=0$, by the bounded convergence theorem.

Suppose μ is a measure representing x for $R(X)$ such that $\mu(\{x\})=0$ and

$$\int |z-x|^{-p}\varphi(|z-x|)^{-1}d|\mu| < \infty .$$

Then there is a measure μ_p, which is a linear combination of the measures $(z-x)^{-j}\mu$, $0 \leqq j \leqq p$, so that $\int f d\mu_p = (p!)^{-1} f^{(p)}(x)$ for all $f \in R_0(X)$ (see [3]). Therefore we get a contradiction by taking $\sigma = \varphi(|z-x|)^{-1}\mu_p$.

REMARK. If $\varphi \equiv 1$, then this theorem is only part of Melnikov's theorem: $\sum 2^n \gamma(A_n(x) \setminus X) = \infty$ if and only if there is no measure μ representing x for $R(X)$ such that $\mu(\{x\}) = 0$ (see [1]).

REMARK. For a compact set X of type (L), $\gamma(A_n(0) \setminus X) = \varrho_n$. Also $a_n = \frac{3}{4}2^{-n}$, and $\varphi(a_n) \leqq \varphi(2^{-n}) \leqq 2\varphi(a_n)$. Hence

$$\sum 2^{(p+1)n} \varphi(2^{-n})^{-1} \gamma(A_n(0) \setminus X) = \infty$$

if and only if

$$\sum \varrho_n a_n^{-(p+1)} \varphi(a_n)^{-1} = \infty ,$$

and thus there is a measure μ representing 0 for $R(X)$ such that $\mu(\{0\}) = 0$ and $\int \varphi(|z|)^{-1} d|\mu| < \infty$ if and only if

$$\sum 2^n \varphi(2^{-n})^{-1} \gamma(A_n(0) \setminus X) < \infty .$$

Let φ be an admissible function with $\varphi(0^+) = 0$. The construction in section 2 also demonstrates that there is a compact set X and $x \in X$, such that the unit ball of $R(X)$ admits $\varepsilon\varphi$ as modulus of approximate continuity at x for every $\varepsilon > 0$, while

$$\sum 2^n \varphi(2^{-n})^{-1} \gamma(A_n(x) \setminus X) = \infty .$$

However, it is still unknown whether the following conjectures are true:

CONJECTURE 1. *Suppose* $\sum 2^n \varphi(2^{-n})^{-1} \gamma(A_n(x) \setminus X) < \infty$. *Then the unit ball of* $R(X)$ *admits* $\varepsilon\varphi$ *as modulus of approximate continuity at* x *for every* $\varepsilon > 0$.

CONJECTURE 2. *Suppose* $\sum 2^n \varphi(2^{-n})^{-1} \gamma(A_n(x) \setminus X) < \infty$. *Then there is a measure* μ *representing* x *for* $R(X)$ *with* $\int \varphi(|z-x|)^{-1} d|\mu| < \infty$.

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