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ON GROUPS OF AUTOMORPHISMS OF THE TENSOR PRODUCT OF VON NEUMANN ALGEBRAS

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1. Introduction.

Let G be a group of automorphisms of a von Neumann algebra $\mathscr{R}$ (throughout this paper, automorphism will always mean *-automorphism). In [2] and [3], an equivalence relation $\sim_G$ between projections in $\mathscr{R}$ was introduced, which reduces to the usual Murray–von Neumann equivalence on projections when G is the group consisting of the identity automorphism. Størmer, in [3], introduced the concepts of $\sim_G$-abelian projection and $\sim_G$-finite projection. He also defined, following the classical situation, the idea of a $\sim_G$-finite and a $\sim_G$-semifinite von Neumann algebra. In particular he proved that $\mathscr{R}$ is $\sim_G$-semifinite if and only if there is a G-invariant faithful normal semifinite trace on $\mathscr{R}$. We shall say $\mathscr{R}$ is $\sim_G$-type III if $\mathscr{R}$ contains no $\sim_G$-finite projections.

Let $\mathscr{R}$ (respectively $\mathscr{S}$) be a von Neumann algebra and G (respectively H) be a group of automorphisms of $\mathscr{R}$ (respectively $\mathscr{S}$). If $g \in G$, $h \in H$, then $g \otimes h$ is an automorphism of $\mathscr{R} \otimes \mathscr{S}$ (see [1, p. 56. Proposition 2]) and the map $(g,h) \to g \otimes h$ is a group homomorphism identifying the direct product $G \times H$ of G and H as a group of automorphisms of $\mathscr{R} \otimes \mathscr{S}$. We shall show that if either $\mathscr{R}$ is $\sim_G$-type III or $\mathscr{S}$ is $\sim_H$-type III then $\mathscr{R} \otimes \mathscr{S}$ is $\sim_{G \times H}$-type III. Our notation will be as used in [2]. In particular,

$$C^G = \{A \in \mathscr{R} \cap \mathscr{R}' \; ; \; g(A) = A \quad (g \in G)\} .$$

(This set is denoted by $\mathscr{D}$ in [3]).

Remark. If $\mathscr{R}$ has a $\sim_G$-abelian projection, then Lemmas 6 and 9 of [3] show that there is a projection $P \in C^G$ such that $\mathscr{R}P$ has a G-invariant faithful normal semifinite trace, and hence by Theorem 2 of [3], $\mathscr{R}P$ is $\sim_G$-semifinite. If $\mathscr{R}$ has no $\sim_G$-abelian projections, yet contains a $\sim_G$-finite projection E, then by considering a subprojection of E we may assume E is countably decomposable. The proof of Lemma 10 in [3] now shows that there is a projection $Q \in C^G$ such that $\mathscr{R}Q$ is $\sim_G$-semi-

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finite. Conversely, if $\mathscr{R}Q$ is $\sim_G$-semifinite for some projection $Q \in C^G$ let E be a $\sim_G$-finite projection in $\mathscr{R}Q$, then E is a $\sim_G$-finite projection in $\mathscr{R}$. The above comments show that $\mathscr{R}$ is $\sim_G$-type III if and only if $\mathscr{R}Q$ is not $\sim_G$-semifinite for any non zero projection $Q \in C^G$.

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2. The crossed product algebra.

We shall now give a description of the crossed product algebra, as defined in [2], using tensor product notation. Let $\mathscr{R}$ be a von Neumann algebra acting on a Hilbert space $\mathscr{H}$, and G a group of automorphisms of $\mathscr{R}$. Denote by $\{\varepsilon_g; g \in G\}$ the usual orthonormal basis for $l^2(G)$. If $g \in G$, $x \in \mathscr{H}$, define

$$\begin{aligned} U_h(x \otimes \varepsilon_g) &= x \otimes \varepsilon_{gh^{-1}}, \qquad (h \in G)\,, \\ \Phi(A)(x \otimes \varepsilon_g) &= g(A)x \otimes \varepsilon_g, \qquad (A \in \mathscr{R},\ g \in G)\,. \end{aligned}$$

Then U_h extends to a unitary operator on $\mathscr{H} \otimes l^2(G)$, and

$$U_h U_k(x \otimes \varepsilon_g) = U_h(x \otimes \varepsilon_{gk^{-1}}) = x \otimes \varepsilon_{gk^{-1}h^{-1}} = U_{hk}(x \otimes \varepsilon_g)\,.$$

Also, $\Phi(A)$ extends to a bounded linear operator on $\mathscr{H} \otimes l^2(G)$ and

$$\begin{aligned} U_h \Phi(A) U_{h^{-1}}(x \otimes \varepsilon_g) &= U_h \Phi(A)(x \otimes \varepsilon_{gh}) \\ &= U_h(gh(A)x \otimes \varepsilon_{gh}) = gh(A)x \otimes \varepsilon_g \\ &= g(h(A))x \otimes \varepsilon_g = \Phi(h(A))(x \otimes \varepsilon_g)\,. \end{aligned}$$

So $g \to U_g$ is a unitary representation of G on $\mathscr{H} \otimes l^2(G)$ with

$$U_g \Phi(A) U_{g^{-1}} = \Phi(g(A)) \qquad (g \in G,\ A \in \mathscr{R})\,.$$

It is also easy to see that Φ is an ultraweakly continuous *-isomorphism of $\mathscr{R}$. We define $\mathscr{R} \times G$ to be the von Neumann algebra generated by

$$\{\Phi(A), U_g\,;\ A \in \mathscr{R},\ g \in G\}\,.$$

Since

$$(\Phi(A)U_g)^* = U_{g^{-1}}\Phi(A^*) = \Phi(g^{-1}(A^*))U_{g^{-1}}\,,$$

finite sums $\sum_i \Phi(A_i) U_{g_i}$ form a *-algebra weakly dense in $\mathscr{R} \times G$. We call this *-algebra $(\mathscr{R} \times G)_0$. Suppose $\mathscr{S}$ is a von Neumann algebra acting on a Hilbert space $\mathscr{K}$, then denote by V_h (respectively $W_{(g,h)}$) the corresponding group of unitaries in the crossed product algebra $\mathscr{S} \times H$ (respectively $\mathscr{R} \otimes \mathscr{S} \times (G \times H)$).

3. The main result.

LEMMA. *Let $\mathscr{R}$ (respectively $\mathscr{S}$) be a von Neumann algebra acting on a Hilbert space $\mathscr{H}$ (respectively $\mathscr{K}$) and G (respectively H) be a group of automorphisms of $\mathscr{R}$ (respectively $\mathscr{S}$). Then $(\mathscr{R}\times G)\otimes(\mathscr{S}\times H)$ is spatially *-isomorphic to $(\mathscr{R}\otimes\mathscr{S}\times(G\times H))$.*

PROOF. Let (x_α), (y_β) be orthonormal bases of $\mathscr{H}$, $\mathscr{K}$ respectively, and

$$\{\varepsilon_g\,;\; g\in G\},\qquad \{\varepsilon_h\,;\; h\in H\},\qquad \{\varepsilon_{(g,h)}\,;\; (g,h)\in G\times H\}$$

the usual orthonormal bases for $l^2(G)$, $l^2(H)$ and $l^2(G\times H)$ respectively. Define

$$V\big((x_\alpha\otimes y_\beta)\otimes\varepsilon_{(g,h)}\big) = (x_\alpha\otimes\varepsilon_g)\otimes(y_\beta\otimes\varepsilon_h)\qquad (g\in G,\ h\in H)\,.$$

Then V extends to a unitary transformation between $(\mathscr{H}\otimes\mathscr{K})\otimes l^2(G\times H)$ and $(\mathscr{H}\otimes l^2(G))\otimes(\mathscr{K}\otimes l^2(H))$.

If $A\in\mathscr{R}$, $B\in\mathscr{S}$, $x_\alpha\in\mathscr{H}$, $y_\beta\in\mathscr{K}$ and $(g,h),(k,l)\in G\times H$, we have

$$\begin{aligned}
&V^{-1}(\Phi(A)U_g\otimes\Phi(B)V_h)V\big((x_\alpha\otimes y_\beta)\otimes\varepsilon_{(k,l)}\big)\\
&= V^{-1}(\Phi(A)U_g\otimes\Phi(B)V_h)\big((x_\alpha\otimes\varepsilon_k)\otimes(y_\beta\otimes\varepsilon_l)\big)\\
&= V^{-1}\big(\Phi(A)(x_\alpha\otimes\varepsilon_{kg^{-1}})\otimes\Phi(B)(y_\beta\otimes\varepsilon_{lh^{-1}})\big)\\
&= V^{-1}\big((kg^{-1}(A)x_\alpha\otimes\varepsilon_{kg^{-1}})\otimes(lh^{-1}(B)y_\beta\otimes\varepsilon_{lh^{-1}})\big)\\
&= (kg^{-1}(A)x_\alpha\otimes lh^{-1}(B)y_\beta)\otimes\varepsilon_{(kg^{-1},\,lh^{-1})}\\
&= \Phi(A\otimes B)\big((x_\alpha\otimes y_\beta)\otimes\varepsilon_{(kg^{-1},\,lh^{-1})}\big)\\
&= \Phi(A\otimes B)W_{(g,h)}\big((x_\alpha\otimes y_\beta)\otimes\varepsilon_{(k,l)}\big)\,.
\end{aligned}$$

Since linear combinations of elements $(x_\alpha\otimes y_\beta)\otimes\varepsilon_{(k,l)}$ are dense in $\mathscr{H}\otimes\mathscr{K}\otimes l^2(G\times H)$, the following identity gives rise to a mapping from $(\mathscr{R}\times G)_0\otimes(\mathscr{S}\times H)_0$ onto $(\mathscr{R}\otimes\mathscr{S}\times(G\times H))_0$:

$$V^{-1}(\Phi(A)U_g\otimes\Phi(B)V_h)V = \Phi(A\otimes B)W_{(g,h)}\,.$$

This map then extends to the required spatial *-isomorphism.

THEOREM. *Let $\mathscr{R}$ and $\mathscr{S}$ be von Neumann algebras acting on Hilbert spaces $\mathscr{H}$ and $\mathscr{K}$ respectively. Let G (respectively H) be a group of automorphisms of $\mathscr{R}$ (respectively $\mathscr{S}$). If either $\mathscr{R}$ is $\sim_G$-type* III *or $\mathscr{S}$ is $\sim_H$-type* III *then $\mathscr{R}\otimes\mathscr{S}$ is $\sim_{G\times H}$-type* III.

PROOF. Let

$$\{\psi_\varphi\,;\;\varphi\in(\mathscr{S}\times H)_*\}$$

be the projections of Sakai from $(\mathscr{R}\times G)\otimes(\mathscr{S}\times H)$ onto $(\mathscr{R}\times G)$ (see [4, proof of Theorem 2.6.4]), and suppose $\mathscr{R}$ is $\sim_G$-type III. Denote by Ω the *-isomorphism of the previous lemma. Let E be a $\sim_{G\times H}$-finite projection in $\mathscr{R}\otimes\mathscr{S}$. We have to prove $E=0$. Suppose $E\neq 0$ then $0\neq\Phi(E)$ is a finite projection in $(\mathscr{R}\otimes\mathscr{S}\times(G\times H))$ by [2, Theorem 4.1], so $F=\Omega^{-1}\Phi(E)$ is a finite projection in $(\mathscr{R}\times G)\otimes(\mathscr{S}\times H)$. Since $\Phi(E)\in\Phi(\mathscr{R}\otimes\mathscr{S})$, $\Phi(E)$ is the ultraweak limit of elements of the form $\Phi(A_\gamma)$, with

$$A_\gamma = \sum_i \lambda_i C_i\otimes D_i \quad (C_i\in\mathscr{R},\ D_i\in\mathscr{S})\,.$$

Now

$$\begin{aligned}\Omega^{-1}\Phi(A_\gamma) &= \sum_i \lambda_i\Omega^{-1}\Phi(C_i\otimes D_i)\\ &= \sum_i \lambda_i\Phi(C_i)\otimes\Phi(D_i)\in\Phi(\mathscr{R})\otimes\Phi(\mathscr{S})\,,\end{aligned}$$

and $F=$ ultraweak limit of $\Omega^{-1}\Phi(A_\gamma)$ so $F\in\Phi(\mathscr{R})\otimes\Phi(\mathscr{S})$.

If $f\in(\mathscr{R}\times G)_*$, $\varphi\in(\mathscr{S}\times H)_*$, $A\in\Phi(\mathscr{R})$, $B\in\Phi(\mathscr{S})$, then

$$f(\psi_\varphi(A\otimes B)) = (f\otimes\varphi)(A\otimes B) = f(A)\varphi(B) = f(\varphi(B)A)\,.$$

So $\psi_\varphi(A\otimes B)=\varphi(B)A$.

Suppose now $B_\gamma\to F$ ultraweakly, with B_γ of the form $B_\gamma=\sum\lambda_i M_i\otimes N_i$ ($M_i\in\Phi(\mathscr{R})$, $N_i\in\Phi(\mathscr{S})$), then $\psi_\varphi(F)=$ ultraweak limit of $\psi_\varphi(B_\gamma)$ since each ψ_φ is normal, and

$$\psi_\varphi(B_\gamma) = \sum_i \lambda_i\psi_\varphi(M_i\otimes N_i) = \sum_i \lambda_i\varphi(N_i)M_i\,.$$

Thus $\psi_\varphi(B_\gamma)\in\Phi(\mathscr{R})$, and so $\psi_\varphi(F)\in\Phi(\mathscr{R})$ for all $\varphi\in(\mathscr{S}\times H)_*$.

Choose φ_0 with $\psi_{\varphi_0}(F)\neq 0$. The argument now parallels that of [4, Lemma 2.6.5]. Let P be a spectral projection of $\psi_{\varphi_0}(F)$, with $0<\lambda P<\psi_{\varphi_0}(F)$ for some $\lambda>0$, then P is a projection in $\Phi(\mathscr{R})$. Let A_β be a net in $P(R\times G)P$ with $\|A_\beta\|\leqq 1$, $A_\beta\to 0$ ultrastrongly, then $A_\beta F\to 0$ ultrastrongly (we identify $\mathscr{R}\times G$ with

$$(\mathscr{R}\times G)\otimes I\subset(\mathscr{R}\times G)\otimes(\mathscr{S}\times H)\,)\,.$$

Hence, since F is finite, $(A_\beta F)^*=FA_\beta{}^*\to 0$ ultrastrongly ([4, p. 97, Theorem 2.5.6]). Thus since each ψ_φ is ultrastrongly continuous,

$$\psi_{\varphi_0}(FA_\beta{}^*) = \psi_{\varphi_0}(F)A_\beta{}^*\to 0 \quad \text{ultrastrongly}\,.$$

Thus

$$A_\beta{}^* = \{P\psi_{\varphi_0}(F)P+1-P\}^{-1}P\psi_{\varphi_0}(F)A_\beta{}^*\to 0 \quad \text{ultrastrongly}\,.$$

This shows that the *-operation is continuous on bounded spheres of $P(\mathscr{R}\times G)P$, so P is a finite projection in $\mathscr{R}\times G$ (see [4, p. 97, Theorem 2.5.6]). Hence $\Phi^{-1}(P)$ is a $\sim_G$-finite projection in $\mathscr{R}$ (by [2, Theorem 4.1]), a contradiction since $\mathscr{R}$ is $\sim_G$-type III. It follows that $E=0$ and $\mathscr{R}\otimes\mathscr{S}$ is $\sim_{G\times H}$-type III.

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