MATH. SCAND. 34 (1974), 235–240

ON MENSHOFF'S SET OF MULTIPLICITY

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In 1916 Menshoff obtained the first example of a closed set of multiplicity for trigonometric series, of Lebesgue measure 0 [1, 7]. The scope of Menshoff's process was greatly expanded by Bary [1, 2]. Verblunsky then attempted to verify a conjecture of Bary [9], but an error was found by Civen and Chrestenson, who also presented a variant of Menshoff's process [3]. Finally, Pyateckii-Šapiro ingeniously disproved Bary's conjecture with the discovery of a new class of sets of uniqueness [8]. For an exposition of these matters, see [1, pp. 366–387].

In this note we observe that Menshoff's set P carries a probability measure μ with the following property: for any function φ in $C^1(-\infty,\infty)$ with $\varphi' > 0$, we have

$$\lim \int \exp 2\pi i u\varphi(t)\cdot\mu(dt) = 0, \qquad |u| \to \infty .$$

We do not give a detailed proof of this, because we use Menshoff's process to obtain much more subtle examples. Let $C^1{}_+$ be the class of functions defined above, and let $\Lambda^1{}_+$ be the set of increasing functions on R^1, with ψ and ψ^{-1} locally Lipschitzian. The property claimed for Menshoff's set we call $C^1(M)$. In a similar way we can define $\Lambda^1(M)$ sets, but

$$\Lambda^1(M) \text{ sets have positive Lebesgue measure .}$$

Clearly, this assertion merely expresses a property of singular measures; in fact we prove a much stronger property in the last paragraph. The next statement, therefore, cannot be much improved.

THEOREM. *To each Hausdorff measure-function h there is a closed set $P \subseteq [0,1]$ of h-measure* 0, *so that $\psi(P)$ is $C^1(M)$, for each ψ in $\Lambda^1{}_+$.*

Basic facts about Hausdorff measures are presented in [5 II], and related problems are treated in [4, 6]. In most constructions concerning Hausdorff measures and a qualitative property like $C^1(M)$, the argu-

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Received September 10, 1973.

ment for an arbitrary h is the same as that for $h(u)=u^{\frac{1}{2}}$, say. However, Menshoff's set has Hausdorff dimension 1 no matter how the parameters are chosen, and the same appears to be true for its variant [2].

I.1. P is the intersection of a decreasing sequence

$$P_0 \supseteqq P_1 \supseteqq \ldots P_n \supseteqq \ldots ,$$

with $P_0=[0,1]$; P_n is composed of disjoint closed intervals $I_n{}^k$, and in passing from P_n to P_{n+1} we operate only on a single interval $I_n{}^k$, retaining its end-points. Let η_n be the length of the smallest interval occurring in P_n, and r_n the length of the largest open interval removed in passing from P_n to P_{n+1}. We require that $r_n=O(n^{-2}\eta_n)$ and that the largest interval in P_n tends to 0 in length. Each set $\psi(P)$, with ψ in $\Lambda^1{}_+$, differs from some set P^* by at most an affine transformation, whence $\psi(P)$ has all the properties we are going to verify for P itself.

The measure μ is a w^*-limit of measures μ_n carried by P_n, with primitives F_n. As usual, μ_0 is Lebesgue measure on $[0,1]$. To transform μ_n into μ_{n+1} we operate only on the part carried by the dissected interval $I_n{}^k$. Let then $J_1,\ldots,J_r$ be the open intervals removed from $I_n{}^k$, let $y_1,\ldots,y_r$ be their left end-points, while a,b are the end points of $I_n{}^k$. Now F_{n+1} is to be linear on each of the intervals formed from $I_n{}^k$, constant across each J, and

$$F_{n+1}(y_p) = F_n(y_p), \quad 1\leqq p\leqq r ,$$
$$F_{n+1}(a) = F_n(a), \quad F_{n+1}(b) = F_n(b) .$$

By this process F_n will already be linear on $I_n{}^k$, so that F_n-F_{n+1} attains its extreme values at points where its derivative is discontinuous or at a or b. Thus

$$|F_n-F_{n+1}| \leqq r_n/|I_n{}^k|$$

and for the norm in $L^1(0,1)$ we have

$$\|F_n-F_{n+1}\|_1 \leqq r_n .$$

Writing F and μ for the corresponding limits we obtain

$$\|F_n-F\|_1 = o(\eta_n)$$

and of course $\mu(P)=1$. Moreover, the convergence of F_n is uniform, as $r_n/I_n{}^k=O(n^{-2})$. Thus the sequence F_n is equicontinuous, whence

$$\|\mu_n-\mu_{n+1}\| \leqq 2\mu_n(I_n{}^k) = o(1) ,$$

because the length $|I_n{}^k|=o(1)$.

2. Let φ belong to C_+^1, with $\varphi' \geqq c > 0$ on $[0,1]$, and let A be a large number. Henceforth $e(x) \equiv e^{2\pi i x}$. We shall show that

$$|\int e(u\varphi)\,d\mu| < 2(\pi A c)^{-1}$$

for large u, so the property $C^1(M)$ will be proved for P. To do this we take $n = n(u)$ to be the largest solution of the inequality $\eta_n > Au^{-1}$, whence $\eta_{n+1} \leqq Au^{-1}$ for large u. Then an integration by parts leads to

$$|\int e(u\varphi)\,d\mu - \int e(u\varphi)\,d\mu_n| \leqq \|\mu_n - \mu_{n+1}\| + 2\pi u \max|\varphi'| \cdot \|F_{n+1} - F\|_1 .$$

Now we saw that $\|\mu_n - \mu_{n+1}\| = o(1)$, while

$$\|F_{n+1} - F\|_1 = o(\eta_{n+1}) = o(u^{-1}) ,$$

so the bound is $o(1)$ as $u \to \infty$. Now F_n is piecewise linear and the segments J, on which $F_n' > 0$, have length at least Au^{-1}. We shall prove that

$$|\int_J e(u\varphi)\,dx| < 2|J|(\pi A c)^{-1}$$

for all these intervals J, so that

$$|\int e(u\varphi)\,d\mu_n| < 2(\pi A c)^{-1} .$$

Let A_1 be a large number, and suppose that $Au^{-1} \leqq |J| \leqq A_1 A u^{-1}$. The secant line to φ over J, say $\tilde{\varphi}$, fulfills the inequality

$$|u\varphi - u\tilde{\varphi}| \leqq u|J| \sup|\varphi' - \tilde{\varphi}'| \leqq AA_1 o(1) = o(1)$$

as $u \to \infty$, because φ' is uniformly continuous on $[0,1]$. For any number $0 < r < s < 1$ we have

$$|\int_r^s e(u\tilde{\varphi})\,dx| \leqq (\pi c u)^{-1} ,$$

when $\tilde{\varphi}$ is linear on $[r,s]$ with derivative at least c. Since $|J| \geqq Au^{-1}$, we have

$$|\int_J| \leqq (\pi A c)^{-1}|J| .$$

For intervals J of length $|J| > A_1 A u^{-1}$, we divide J into intervals of length exactly Au^{-1} and a remainder J' of length $|J'| \leqq Au^{-1} < A_1^{-1}|J|$. Thus, for large A_1 and large u, we obtain

$$|\int_J e(u\varphi)\,dx| < 2(\pi A c)^{-1}|J|, \quad \text{whenever } |J| \geqq Au^{-1} .$$

3. To complete the proof of this theorem we explain how to construct P so that P has h-measure 0; for definiteness we specify $h(t) > t$ for all $t > 0$, and of course $h(0+) = 0$. To each $\varepsilon > 0$ and $r > 0$, and each interval $[a,b]$, it is easy to remove open, disjoint intervals of length at most r from $[a,b]$, so that the remaining subset of $[a,b]$ is covered by intervals I_m, where $\sum h(|I_m|) < \varepsilon$. In particular, each $|I_m| < \varepsilon$. At a certain stage in

the construction of P, the set P_n consists of q intervals $I_n{}^k$. We apply the dissection just outlined to each $I_n{}^k$ in turn, taking $\varepsilon' = \varepsilon/q$, and using successively smaller values of r. By this procedure, repeated with different values of ε, we construct P of h-measure 0.

The proof that Menshoff's set P has property $C^1(M)$ involves only minor changes in his proof that P is an M_0-set, because there are no exceptional intervals $I_n{}^k$. From the present standpoint, however, this simplification has the limitation that only rather massive sets are obtained.

A set P with property $C^1(M)$ has the property that each transform $\varphi(P)$, φ in $C^1{}_+$, is an M_0-set; we conjecture that the second property is in fact weaker than $C^1(M)$.

II.4. Henceforth λ is a continuous, singular probability measure on $(-\infty, \infty)$; $(v_k)_1^\infty$ is a sequence of positive numbers tending to $+\infty$. Also $c > 1$ is fixed and $\beta = \beta(c) > 0$ is a constant depending only on $c > 1$; a value of β is given below.

THEOREM. *There exists an absolutely continuous function ψ, with $1 \leq \psi' \leq c$ almost everywhere and the following property: the set of w^*-limit points of the sequence $e(v_k \psi)$, in the space $L^\infty(\lambda)$, contains the ball of radius β in $L^\infty(\lambda)$.*

In the proof we apply Baire's Theorem to the set Y of functions named above, with the additional properties $\psi(0) = 0$ and $\int_{-\infty}^{\infty} |\psi' - 1| \, dx < \infty$. The metric in Y is $\|\psi_1' - \psi_2'\|_1$. It is convenient to write S_r for the ball of radius r in $L^\infty(\lambda)$. In the next few sections, some isolated facts are assembled.

a) There is an absolutely continuous function ξ_0, with derivative $1 \leq \xi_0' \leq c_2 < c$, such that $e(\xi_0) = e^{2\pi i \xi_0}$ is periodic and has mean value

$$\beta = \beta(c) = 2(c_2 - 1)/\pi(c_2 + 1) > 0 .$$

In fact $e(\xi_0)$ is periodic with period L if

$$\xi_0(x + L) - \xi_0(x) \equiv 1 ,$$

and the mean value of $e(\xi_0)$ can be made positive by adding a constant to ξ_0, if necessary. If μ is a finite, continuous measure on the line, it is familiar that

$$\left|\int e(\xi_0(vt))\mu(dt) - \beta\mu(\mathbf{R})\right|^2$$

has mean value 0 as a function of the real variable v. Indeed $e(\xi)$ can be approximated uniformly by sums of exponentials; for $b \neq 0$ it is known that $|\int e(bvt)\mu(dt)|^2$ has mean value 0.

b) We describe a collection of open sets Γ in S_1, such that any weak*-closed subset of S_1, intersecting each Γ, contains S_β. Each neighborhood Γ is determined by an $\eta > 0$, disjoint intervals $I_1, \ldots, I_s$, and numbers $c_1, \ldots, c_s$ of modulus β. Then Γ is just

$$\{g \in S_1,\ |\textstyle\int_{I_m} g d\lambda - c_m \lambda(I_m)| < \eta,\ 1 \leqq m \leqq s\} .$$

In choosing the numbers c_m of modulus exactly β, we make use of the continuity of the measure λ; moreover, any complex number of modulus $< \beta$ is the average of two numbers of modulus exactly β. Thus the neighborhoods can be chosen in the special form indicated. Each neighborhood Γ contains a smaller one, Γ', in which the intervals I_m' have total length $\sum |I_m'|$ smaller than any assigned bound, and moreover $0 \notin \bigcup I_m'$. The first assertion is a consequence of the singularity of λ, the second, of its continuity. In the next two sections we show that every neighborhood W in Y contains a function ξ, such that $e(v_k \xi) \in \Gamma$ for some number v_k in the sequence. Since the metric of Y is stronger than the uniform metric, we have $e(v_k \xi^*) \in \Gamma$ for ξ^* in an open subset $W^* \subseteqq W$.

c) Let $\delta > 0$ be so small that $(1+\delta)c_2 < c$ and observe that the set of real numbers v, such that

$$|\textstyle\int_{I_m} e(\xi_0(vt))\lambda(dt) - \beta\lambda(I_m)| < \eta, \quad 1 \leqq m \leqq s ,$$

has density 1; for large k there is such a number v_0 in each interval $v_k < v < (1+\delta)v_k$. Let v_0 be chosen in this way; beginning with a member ψ of $W \subseteqq Y$, we change ψ on $\bigcup I_m$, so that the new function ψ_1 has the property

$$v_k \psi_1(t) - \xi_0(v_0 t) = \text{const.} \quad \text{on each } I_m .$$

This becomes

$$\psi_1'(t) = v_k^{-1} v_0 \xi_0'(v_0 t) ,$$

whence $1 \leqq \psi_1' \leqq c$. Now $\psi_1' = \psi'$ except on $\bigcup I_m$, hence

$$\|\psi_1' - \psi'\|_1 \leqq c \sum |I_m|$$

and this can be made as small as we please.

d) The function ψ_1 has the property that $|\int_{I_m} e(v_k \psi_1)\lambda(dt)|$ differs from $\beta\lambda(I_m)$ by at most η. In order to approximate the value $c_m \lambda(I_m)$, we must define ψ_2 so that

$$v_k \psi_2 - v_k \psi_1 \equiv s_m \ (\text{modulo } 1) \quad \text{on } I_m ,$$

for certain real numbers s_m. Moreover, we have required $\psi_2(0)=0$. This can be accomplished by adjusting ψ_1' between the intervals I_m, while preserving the inequality $1 \leqq \psi' \leqq c$. When v_0 is large, we can attain the estimate

$$\|\psi_2' - \psi_1'\|_1 = O(v_0^{-1}) .$$

Since $\|\psi_1' - \psi'\|_1$ can also be made as small as necessary, we get $\psi_2 \in W$, $e(v_k \psi_2) \in \Gamma$. Thus the set of functions ψ named in the theorem, is a dense G_δ-set in Y.

We conclude that $A^1(M)$ sets — defined after $C^1(M)$ sets — have positive Lebesgue measure, because

$$\int e(u\psi)\,d\lambda \neq o(1)$$

for a certain ψ, if λ contains a discrete component, or if λ is singular and continuous.

ACKNOWLEDGEMENT. We thank the referee for his helpful corrections, and the Universities of Washington and Nantes for their hospitality.

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