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REGIONS OF CONVERGENCE FOR HYPERGEOMETRIC SERIES IN THREE VARIABLES

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1.

Horn proved in 1889 a theorem on the region of convergence for a hypergeometric series [3]. For a number of hypergeometric series in three variables studied in recent decades [1], [4]–[8], however, regions of convergence have been incorrectly or incompletely given, even though Horn's results are referred to. In the present paper we shall give an account of Horn's method and thence obtain correct regions of convergence for the above-mentioned series.

2.

To investigate the convergence of a hypergeometric series in three complex variables

$$\sum\nolimits_{m,n,p} A_{m,n,p} x^m y^n z^p ,$$

Horn introduces

$$f(m,n,p) =_{\text{def}} \frac{A_{m+1,n,p}}{A_{m,n,p}}, \qquad g(m,n,p) =_{\text{def}} \frac{A_{m,n+1,p}}{A_{m,n,p}},$$

$$h(m,n,p) =_{\text{def}} \frac{A_{m,n,p+1}}{A_{m,n,p}},$$

which are rational functions, and, discarding possible discontinuities,

$$\Phi(m,n,p) =_{\text{def}} \lim\nolimits_{u\to\infty} f(mu,nu,pu), \qquad m>0,\ n\geqq 0,\ p\geqq 0 ,$$

$$\Psi(m,n,p) =_{\text{def}} \lim\nolimits_{u\to\infty} g(mu,nu,pu), \qquad m\geqq 0,\ n>0,\ p\geqq 0 ,$$

$$\Omega(m,n,p) =_{\text{def}} \lim\nolimits_{u\to\infty} h(mu,nu,pu), \qquad m\geqq 0,\ n\geqq 0,\ p>0 ,$$

which are rational functions, too. From these functions, the following subsets of R_+^3 are constructed,

$$C =_{\text{def}} \{(r,s,t) \mid 0<r<|\Phi(1,0,0)|^{-1} \wedge 0<s<|\Psi(0,1,0)|^{-1} \wedge \wedge\, 0<t<|\Omega(0,0,1)|^{-1}\} ,$$

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$$X =_{\text{def}} \{(r,s,t) \mid \forall (n,p) \in \mathsf{R}_+^2 : 0 < s < |\Psi(0,n,p)|^{-1} \vee$$
$$\vee\ 0 < t < |\Omega(0,n,p)|^{-1}\},$$
$$Y =_{\text{def}} \{(r,s,t) \mid \forall (m,p) \in \mathsf{R}_+^2 : 0 < r < |\Phi(m,0,p)|^{-1} \vee$$
$$\vee\ 0 < t < |\Omega(m,0,p)|^{-1}\},$$
$$Z =_{\text{def}} \{(r,s,t) \mid \forall (m,n) \in \mathsf{R}_+^2 : 0 < r < |\Phi(m,n,0)|^{-1} \vee$$
$$\vee\ 0 < s < |\Psi(m,n,0)|^{-1}\},$$
$$E =_{\text{def}} \{(r,s,t) \mid \forall (m,n,p) \in \mathsf{R}_+^3 : 0 < r < |\Phi(m,n,p)|^{-1} \vee$$
$$\vee\ 0 < s < |\Psi(m,n,p)|^{-1} \vee 0 < t < |\Omega(m,n,p)|^{-1}\},$$
$$D' = E \cap X \cap Y \cap Z \cap C;$$

finally, let $D \subseteq (\mathsf{R}_+ \cup \{0\})^3$ denote the union of D' and its projections upon the coordinate planes.

Horn's theorem can then be stated as follows: The region D is the representation in the absolute octant of the convergence region in C^3.

We shall describe D', and D, by that part S of $\partial D'$ which is not contained in coordinate planes.

3.

Horn's theorem has been applied to the 21 hypergeometric series of Gaussian type in three variables which the author had encountered in the literature. (The number of distinct series however, is about one hundred; a complete discussion would thus not find a proper place here.) The results are compiled in the Table, which gives, for each series considered, the reduced expression for D', the Cartesian equation(s) for S, and, as an identification, a list of Pochhammer symbol subscripts occurring in numerator and denominator of $m!\,n!\,p!\,A_{m,n,p}$. This means that, e.g., F_G is the series

$$\sum_{m,n,p} \frac{(a)_{m+n+p}(b_1)_m(b_2)_n(b_3)_p x^m y^n z^p}{(c_1)_m(c_2)_{n+p} m!\, n!\, p!}$$

and similarly in all other cases; $(h)_r = \Gamma(h+r)/\Gamma(h)$ is the Pochhammer symbol.

For brevity, we do not deduce all results given in the Table; only a few cases that are less simple than the others will be considered in the following sections. For the series F_F, $F_N - F_T$, $G_A - G_D$, H_C, previous results in the literature are corrected; the inclusion of the remaining series was not found out of place.

Table

Function and Pochhammer symbol subscripts		D'	Cartesian equation(s) of S
F_A	$m+n+p, m, n, p$ / m, n, p	E	$r+s+t=1$
F_B	m, m, n, n, p, p / $m+n+p$	C	$r=1,\ s=1,\ t=1$
F_C	$m+n+p, m+n+p$ / m, n, p	E	$\sqrt{r}+\sqrt{s}+\sqrt{t}=1$
F_D	$m+n+p, m, n, p$ / $m+n+p$	C	$r=1,\ s=1,\ t=1$
F_E	$m+n+p, m, n+p$ / m, n, p	E	$r+(\sqrt{s}+\sqrt{t})^2=1$
F_F	$m+n+p, m+p, n$ / $m, n+p$	$E\cap Y$	$\begin{cases} 0<r<(1-s)^2:\ \sqrt{t}+\sqrt{r}=1 \\ (1-s)^2\leqq r<1-s:\ \dfrac{r}{1-s}+\dfrac{t}{s}=1 \end{cases}$
F_G	$m+n+p, m, n, p$ / $m, n+p$	$Y\cap Z$	$r+t=1,\ r+s=1$
F_K	$m, n+p, m+p, n$ / m, n, p	E	$t=(1-r)(1-s)$
F_M	$m, n+p, m+p, n$ / $m, n+p$	$Y\cap C$	$r+t=1,\ s=1$
F_N	$m, n, p, m+p, n$ / $m, n+p$	$Y\cap C$	$r+t=1,\ s=1$
F_P	$m+p, n, m+n, p$ / $m, n+p$	$Y\cap Z$	$r+t=1,\ r+s=1$
F_R	$m+p, n, m+p, n$ / $m, n+p$	$Y\cap C$	$\sqrt{r}+\sqrt{t}=1,\ s=1$
F_S	$m, n+p, m, n, p$ / $m+n+p$	C	$r=1,\ s=1,\ t=1$
F_T	$m, n+p, m+p, n$ / $m+n+p$	C	$r=1,\ s=1,\ t=1$
G_A	$n+p-m, n, m+p$ / $n+p-m$	$Y\cap C$	$r+t=1,\ s=1$
G_B	$n+p-m, m, n, p$ / $n+p-m$	C	$r=1,\ s=1,\ t=1$
G_C	$m+p, n, n, p-m$ / $n+p-m$	$E\cap Y$	$\begin{cases} 0<s<\frac{1}{2}:\ r=1-t \\ \frac{1}{2}\leqq s<1:\ r=\min\left\{1-t,\ \dfrac{1-s}{s}\left(1-\dfrac{1-s}{s}t\right)\right\} \end{cases}$
G_D	$m, n, p, n, p-m$ / $n+p-m$	$Z\cap C$	$s(1+r)=1,\ t=1$
H_A	$p+m, m+n, n+p$ / $m, n+p$	E	$r=(1-s)(1-t)$
H_B	$p+m, m+n, n+p$ / m, n, p	E	$r+s+t+2\sqrt{(rst)}=1$
H_C	$p+m, m+n, n+p$ / $m+n+p$	$E\cap C$	$\begin{cases} r+s+t=2+2\sqrt{((1-r)(1-s)(1-t))}, \\ r=1,\ s=1,\ t=1 \end{cases}$

4.

Throughout this section and the next it will be understood that (m,n,p), $(r,s,t) \in \mathsf{R}_+^3$, and that point sets, in simplified notation, are subsets of R_+^3 consisting of (r,s,t)-triples.

From the definition of the H_C series we obtain

$$\Phi(m,n,p) = \frac{(p+m)(m+n)}{m(m+n+p)}$$

and analogous formulae for Ψ and Ω. It readily follows that C is the open unit cube while

$$X = \{s<1 \vee t<1\} \supset C;$$

similarly, $Y \supset C$ and $Z \supset C$; hence, $D' = E \cap C$. We now show that the algebraic surface in C given by

$$\{P(r,s,t)=0 \wedge r,s,t \in (0;\,1)\},$$

where

$$P(r,s,t) =_{\text{def}} 2-r-s-t+2[(1-r)(1-s)(1-t)]^{\frac{1}{2}},$$

is $C \cap \partial E$ and thus $C \cap S$. (The remaining parts of S are parts of the faces of C.) To prove this we first observe that (r,s,t) belongs to E if and only if for each triple (m,n,p),

$$(1) \qquad 0 < r < \frac{m(m+n+p)}{(p+m)(m+n)} \vee 0 < s < \frac{n(m+n+p)}{(m+n)(n+p)} \vee \vee 0 < t < \frac{p(m+n+p)}{(n+p)(p+m)}.$$

Now, if $(r,s,t) \in C \setminus E$, a triple (m,n,p) exists such that the negation of (1) is true. This means that

$$1-r \leqq \frac{np}{(p+m)(m+n)} \wedge 1-s \leqq \frac{pm}{(m+n)(n+p)} \wedge 1-t \leqq \frac{mn}{(n+p)(p+m)},$$

and these inequalities yield $P(r,s,t) \leqq 0$. Moreover, the points in C for which $P(r,s,t)=0$ do belong to $C \setminus E$ because with $P(r,s,t)=0$ and

$$(m,n,p) = \left([r/(1-r)]^{\frac{1}{2}}, [s/(1-s)]^{\frac{1}{2}}, [t/(1-t)]^{\frac{1}{2}}\right),$$

the negation of (1) can be proved to hold. It follows that $C \cap \partial E$ is the algebraic surface considered.

Finally, consider a point (r,s,t) for which $s+t<1$. For a prescribed triple (m,n,p), the inequality $(1)_2$ may hold; if, not, we must have

$$t < 1-s \leqq 1-\frac{n(m+n+p)}{(m+n)(n+p)} < \frac{p(m+n+p)}{(n+p)(p+m)},$$

which is $(1)_3$. The point thus belongs to E, but for $r \geqq 1$ not to C. We have already seen that $C \setminus E \neq \emptyset$; thus D' cannot be expressed simpler than $E \cap C$.

5.

For the G_C series, the definitions lead to the expressions

$$\Phi(m,n,p) = \frac{(m-n-p)(m+p)}{(m-p)m}, \quad \Psi(m,n,p) = \frac{n}{n+p-m},$$

$$\Omega(m,n,p) = \frac{(p-m)(m+p)}{(n+p-m)p}.$$

It follows that C is the open unit cube and that

$$X = \{s^{-1}+t^{-1} > 1\}, \quad Y = \{r+t < 1\}, \quad Z = \{s < (1+r)^{-1}\}.$$

(Z is related to the region of convergence for the H_2 series; cf. e.g., [2, § 5.7.2].) The inclusions $Z \cap Y \subset C \subset X$ are obvious; hence, $D' = E \cap Y \cap Z$. By definition, a point (r,s,t) belongs to E if and only if for each triple (m,n,p),

$$(2) \qquad 0 < r < \frac{m}{(m+p)|\mu-1|} \vee 0 < s < |1-\mu^{-1}| \vee 0 < t < \frac{p|\mu-1|}{m+p},$$

where for brevity

$$\mu =_{\text{def}} n/(m-p).$$

For a point in E we must have $s < 1$: when $s \geqq 1$ the inequalities (2) all fail if we take a suitable triple (m,n,p) with $n > m > p$. We now prove that

$$(3) \qquad E = \left\{ s < 1 \wedge \left[\left(r \leqq \frac{1-s}{2s} \right) \vee \left(\frac{rs}{1-s} + \frac{(1-s)t}{s} < 1 \right) \right] \right\}.$$

First, suppose that $r \leqq \frac{1}{2}(1-s)/s$, $s < 1$, and that a triple (m,n,p) is given. Then, $(2)_2$ may hold; if not, we must have $|1-\mu^{-1}| \leqq s < 1$; this implies that $\mu > \frac{1}{2}$, and thus in particular $m > p$. Consequently,

$$r \leqq \tfrac{1}{2}(s^{-1}-1) < \frac{m}{m+p}\left(\frac{\mu}{|\mu-1|} - 1\right) \leqq \frac{m}{(m+p)|\mu-1|},$$

which is $(2)_1$. These points thus belong to E.

Next, consider a point (r,s,t) for which $s < 1$ and

$$(4) \qquad r < s^{-1}(1-s)\bigl(1-(1-s)t/s\bigr).$$

If for a prescribed triple (m,n,p) neither $(2)_2$ nor $(2)_3$ holds, then

$$(5) \qquad 1 > s \geqq |1-\mu^{-1}| \wedge t \geqq p|\mu-1|(m+p)^{-1} .$$

By (4) and $(5)_2$ we then obtain

$$r-\frac{m}{(m+p)|\mu-1|} < -\frac{p|\mu-1|}{m+p}\left(\frac{1-s}{s}-\frac{1}{|\mu-1|}\right)\left(\frac{1-s}{s}-\frac{m}{p|\mu-1|}\right) \leqq 0 ,$$

since $(5)_1$ implies that $(1-s)/s \leqq |\mu-1|^{-1}$ and $m>p$. We are thus led to $(2)_1$, and so the point must belong to E.

We finally consider the points for which either

$$\left(t \geqq s/(1-s)-rs^2/(1-s)^2\right) \wedge \left(\tfrac{1}{2}(1-s)/s < r < (1-s)/s\right) \wedge (s < 1)$$

or

$$\left(r \geqq (1-s)/s\right) \wedge (s < 1) .$$

They do not belong to E since for these points the triple

$$(m,n,p) = \left(1, 2/(1-s)-1/(rs), (1-s)/(rs)-1\right)$$

satisfies the negation of (2). This completes the proof of (3).

Comparison of the expressions for E, Y, Z now easily leads to the equations for S given in the Table, and to the fact that D' cannot be written simpler than $E \cap Y$.

The treatment of F_F is quite parallel to that of G_C. We merely notice that in this case the expressions

$$X = \{s<1 \vee t<1\}, \quad Y = \{\sqrt{r}+\sqrt{t}<1\}, \quad Z = \{r+s<1\} ,$$
$$E = \{s<1 \wedge [(r \leqq (1-s)^2) \vee (r<(1-s)(1-t/s))]\}$$

are obtained.

6.

For the series hitherto considered, the region D' reduces to an intersection of two sets at most. It is possible to find a hypergeometric series in three variables, although not of Gaussian type, for which D' cannot be expressed simpler than $E \cap X \cap Y \cap Z$. The series is

$$\sum_{m,n,p} \frac{(a_1)_{m+n+p}(a_2)_{m+n+p}(b_1)_m(b_2)_n(b_3)_p x^m y^n z^p}{(c_1)_{p+m}(c_2)_{m+n}(c_3)_{n+p} m!\, n!\, p!} ;$$

that S is given by the Cartesian equations

$$r+s = 1 \quad s+t = 1, \quad t+r = 1, \quad rs+st+tr = 2\sqrt{(rst)} ,$$

can be proved by methods similar to those applied above.

7.

It does not seem out of place mentioning the possibility of obtaining regions of convergence without application of Horn's theorem. Two examples will be given.

Consider the F_G series explicitly defined in section 3. It follows from Stirling's theorem that the region of convergence of F_G coincides with that of the series

$$S_G =_{\text{def}} \sum\nolimits_{m,n,p} \frac{(m+n+p)!\, m!\, n!\, p!}{m!\,(n+p)!\, m!\, n!\, p!} |x|^m |y|^n |z|^p$$

$$= \sum\nolimits_{m,n,p} \frac{(m+n+p)!}{m!\,(n+p)!} |x|^m |y|^n |z|^p .$$

Convergence being absolute, conditions are obtainable from consideration of a particular order of summation:

$$S_G = \sum\nolimits_{n,p} |y|^n |z|^p \sum\nolimits_m \frac{(m+n+p)!\,|x|^m}{m!\,(n+p)!} = \sum\nolimits_{n,p} \frac{|y|^n |z|^p}{(1-|x|)^{n+p+1}} .$$

Consequently, the region of convergence is determined by the conditions $|y| < 1-|x|$, $|z| < 1-|x|$ (and $|x| < 1$). The boundary S thus has the Cartesian equations $r+s=1$, $r+1=t$, as stated in the Table.

Next, we consider F_P, which, again by Stirling's theorem, has the same region of convergence as the series

$$S_P =_{\text{def}} \sum\nolimits_{m,n,p} \frac{(m+p)!\,(m+n)!}{(m!)^2(n+p)!} |x|^m |y|^n |z|^p$$

$$= \sum\nolimits_{m,n,p} \frac{(m+p)!\,(m+n)!}{m!\,(m+n+p)!} \frac{(m+n+p)!}{m!\,(n+p)!} |x|^m |y|^n |z|^p \leqq S_G ,$$

since $(m+p)!\,(m+n)! \leqq m!\,(m+n+p)!$. The region of convergence of S_P thus contains that of S_G. On the other hand,

$$S_P \geqq \sum\nolimits_{m,n} \frac{(m+n)!}{m!\,n!} |x|^m |y|^n .$$

This series, and so also S_P, is divergent when $|x|+|y| > 1$; similarly, S_P is divergent when $|x|+|z| > 1$. It follows that the region of convergence of F_P equals that of F_G.

By similar methods, regions of convergence can be determined for most functions mentioned in the Table, the possible exceptions being F_F, G_C, G_D, H_C.

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