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INTEGRAL GLOBAL WEIGHTS FOR TORUS ACTIONS ON PROJECTIVE SPACES

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Let a torus T act on a space X such that $H^*(X;\mathsf{Z}_{(p)})$ is a truncated polynomial ring in one variable of degree $2k$. For each component F^i of the fixed point set $F=F(T)$, there is an element $\alpha_i \in H^{2k}(B_T;\mathsf{Z}_{(p)})$ as defined below. Wu-yi Hsiang shows in [1] and [2] how the differences $\alpha_i-\alpha_j$ together with the cohomology groups $H^*(F;\mathsf{Z}_p)$ determine the cohomology ring $H_T{}^*(X;\mathsf{Z}_{(p)})$ and that this ring structure determines the $\alpha_i-\alpha_j$ up to multiplication by a scalar unit. In the present paper we will prove that the content of the polynomials $\alpha_i-\alpha_j$ is determined by the Z_p cohomology groups of the subspaces $F(T_i)$ where T_i is the subgroup of T of elements of order dividing p^i (Theorem 1). Using a result of Hsiang-Su [3], we then completely determine the global weights $\alpha_i-\alpha_j$ from the geometry of the given T-action (Remark 1), up to a scalar unit. In case X has the integral cohomology ring of a complex, quaternionic, or Cayley projective space, the golbal weights are determined up to sign by using the above result at all primes p. In case $k=1$ (complex projective space), Theorem 1 is due to J. C. Su [4].

We define $\boldsymbol{H}^*(X,Y)=\operatorname{proj.lim} H^*(X,Y;\mathsf{Z}_{p^\alpha})$ where Čech cohomology is used on the right hand side. The groups $\boldsymbol{H}^*(X)$ have a technical advantage over the groups $H^*(X,\mathsf{Z}_{(p)})$, set forth in the lemma. Considering only spaces X with $H^q(X;\mathsf{Z}_p)$ finite for each p, $\boldsymbol{H}^*$ is a cohomology functor. The Bockstein sequence

$$\text{(B)} \qquad \to \boldsymbol{H}^q(X) \xrightarrow{p} \boldsymbol{H}^q(X) \longrightarrow H^q(X;\mathsf{Z}_p) \to$$

derived from the coefficient sequences

$$0 \to \mathsf{Z}_{p^\alpha} \to \mathsf{Z}_{p^{\alpha+1}} \to \mathsf{Z}_p \to 0$$

shows that $\boldsymbol{H}^q(X)$ is finitely generated over $A_p=\operatorname{proj.lim} \mathsf{Z}_{p^\alpha}$. Let $\operatorname{cd}_p X$ denote the cohomology dimension of X over Z_p (see [5]). We define

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equivariant cohomology and make topological assumptions about X as in [6].

LEMMA. *Let a compact Lie group G act on X and assume that* $\mathrm{cd}_p X < N$. *For $s \in \boldsymbol{H}^*(B_G)$, set*

$$X^s = \{x \in X \mid s \text{ does not map to zero in } \boldsymbol{H}^*(B_{G_x})\}\,.$$

Then $s^N \boldsymbol{H}_G{}^*(X, X^s) = 0$.

PROOF. It suffices to prove that $s^N H_G{}^*(X, X^s; \mathsf{Z}_{p^\alpha}) = 0$ for all α. This follows from [1] and [5] because the cohomology dimension of X over Z_{p^α} equals $\mathrm{cd}_p X < N$.

Let $S \subset H^*(B_T; \mathsf{Z})$ be the multiplicative system generated by the elements of $H^2(B_T; \mathsf{Z})$ of content 1.

COROLLARY 1. *Assume that T acts on X with finitely many isotropy groups. Then there is an element $s \in S$ such that $s\boldsymbol{H}_T{}^*(X, F(T_1)) = 0$.*

PROOF. For $x \notin F(T_1)$, $T_1 \not\subset G_x$, hence there is a corank one subtorus $C \subset T$ such that $G_x / C \cap G_x \subset T/C$ is a finite group of order prime to p. Let $t(x)$ be the image of a generator by

$$H^2(B_{T/C}; \mathsf{Z}) \to H^2(B_T; \mathsf{Z})\,;$$

then $t(x) \in S$ and $t(x)$ maps to zero in $\boldsymbol{H}^*(B_{G_x})$. Setting $t = t(x_1) \dots t(x_n)$ where each isotropy group G_x for $x \notin F(T_1)$ appears at some $x_i \notin F(T)_1$, we have $X^t = F(T_1)$. Setting $s = t^N$, the conclusion follows from the lemma.

Let $P^n(k)$ denote a space such that

$$H^*(P^n(k); \mathsf{Z}_p) = \mathsf{Z}_p[x]/(x^{n+1}), \quad \deg x = 2k\,,$$

whenever such a space exists. Using the Bockstein sequence (B), we conclude that

$$\boldsymbol{H}^*(P^n(k)) = A_p[x]/(x^{n+1}), \quad \deg x = 2k\,.$$

Let $X \sim_p Y$ mean that X and Y have isomorphic Z_p cohomology rings. Let a torus T act on $X \sim_p P^n(k)$ with fixed point set $F = F^1 + F^2 + \dots$. For each subgroup G of T let $F^i(G)$be the component of $F(G)$ containing F^i. For each finite p-group $G \subset T$ and each i, $F^i(G) \sim_p P^m(d)$ for some $m \leqq n$, $d \leqq k$, according to Bredon [7]. Let $p_i \in F^i$ be a point and let $p_i{}^*$ be the composite homomorphism

$$\boldsymbol{H}_T{}^*(X) \to \boldsymbol{H}_T{}^*(p_i) = \boldsymbol{H}^*(B_T)\,.$$

Let $y \in \boldsymbol{H}_T^{2k}(X)$ map to a generator of $\boldsymbol{H}^{2k}(X)$; then y generates $\boldsymbol{H}_T{}^*(X)$ as an $\boldsymbol{H}^*(B_T)$ algebra. Setting $\alpha_i = p_i{}^*(y)$ the global weights $\alpha_i - \alpha_j$ are uniquely defined up to multiplication by a unit of A_p. Define a sequence of integers $d_1 \geqq d_2 \geqq d_3 \ldots$ by $F^1(T_i) \sim_p P^{n_i}(d_i)$ when $F^2 \subset F^1(T_i)$ and $d_i = 0$ when $F^2 \cap F^1(T_i) = \varnothing$. Our main result is:

THEOREM 1. *Let p^e be the highest power of p dividing $\alpha_1 - \alpha_2$. Then $e = d_1 + d_2 + d_3 + \ldots$.*

COROLLARY 2. *Assume that $k = 1$ or that $F^1 \sim_p P^m(k)$ for some $m \geqq 1$. Then F^1 and F^2 lie in the same component of $F(T_i)$ if and only if $\alpha_1 - \alpha_2$ is divisible by p^{ki}.*

PROOF OF THE COROLLARY. Under the assumption, $F^1(T_i) \sim_p P^{n_i}(k)$ whenever $F^2 \subset F^1(T_i)$. Hence $d_i = k$ or $d_i = 0$ for all i, proving the corollary.

PROOF OF THE THEOREM. Let $Y = \{p_1, p_2\} \subset F^1 + F^2 \subset X$ and let $\varepsilon \in \boldsymbol{H}^0(Y) \subset \boldsymbol{H}_T{}^0(Y)$ be 0 on p_1 and 1 on p_2. Define $I_\varepsilon(T, X) \subset A_p$ to be the ideal of all $m \in A_p$ such that for some $s \in S$, $ms\varepsilon$ lies in the image of $\boldsymbol{H}_T{}^*(X) \to \boldsymbol{H}_T{}^*(Y)$. Similarly, we can define, for every closed subgroup G of T, the ideals $I_\varepsilon(T, F(G))$ and $I_\varepsilon(T/G, F(G))$, the latter because T/G is a torus. It follows from Corollary 1 that

$$I_\varepsilon(T, X) = I_\varepsilon(T, F(T_1)) .$$

After replacing y by $y - \alpha_1$ to obtain $\alpha_1 = p_1{}^*(y) = 0$, it is not difficult to see that $I_\varepsilon(T, X) = (p^e)$ since $\alpha_1 - \alpha_2 = p^e su$ where $s \in S$ and u is a unit of A_p. The reason why $s \in S$ is explained in Remark 1. Since $I_\varepsilon(T, F(T_1)) = (1)$ if and only if $F^2 \cap F^1(T_1) = \varnothing$, $e = 0$ if and only if $d_1 = 0$. Assuming $d_1 > 0$, let $T' = T/T_1$, and let

$$y_1 \in \boldsymbol{H}_{T'}^{2d_1}(F^1(T_1))$$

be a lift of a generator of $\boldsymbol{H}^{2d_1}(F^1(T_1))$, with $p_1{}^*(y_1) = 0$. Let p^{e_1} be the highest power of p dividing $p_2{}^*(y_1)$. Then $I_\varepsilon(T/T_1, F(T_1)) = (p^{e_1})$. There is a commutative diagram:

$$\begin{tikzcd} c^*(y_1) \in \boldsymbol{H}_T{}^*(F^1(T_1)) \arrow[r] & \boldsymbol{H}_T{}^*(Y) \\ y_1 \in \boldsymbol{H}_{T'}{}^*(F^1(T_1)) \arrow[r] \arrow[u, "c^*"] & \boldsymbol{H}_{T'}{}^*(Y) \arrow[u, "c^*"] \end{tikzcd}$$

Since $I_\varepsilon(T, F^1(T_1)) = (p^e)$, it follows that p^e is the highest power of p dividing $p_2{}^*(c^*(y_1)) \in \boldsymbol{H}^*(B_T)$. Because

$$c^* : \boldsymbol{H}^2(B_{T'}) \to \boldsymbol{H}^2(B_T)$$

is "multiplication by p", $\deg y_1 = 2d_1$, and $p_2^*(c^*(y_1)) = c^*(p_2^*(y_1))$, it follows that the highest power of p dividing $p_2^*(c^*(y_1))$ is $p^{d_1+e_1}$, hence $e = d_1 + e_1$. This can be written as

$$I_\varepsilon(T,X) = p^{d_1} I_\varepsilon(T/T_1, F(T_1));$$

by induction this implies

$$I_\varepsilon(T,X) = p^{d_1+d_2+\cdots+d_i} I_\varepsilon(T/T_i, F(T_i))$$

if $d_i > 0$. If $d_{i+1} = 0$, then

$$I_\varepsilon(T/T_i, F(T_1)) = I_\varepsilon(T/T_i, F(T_{i+1})) = (1)$$

by Corollary 1, hence $(p^e) = I_\varepsilon(T,X) = (p^{d_1+d_2+\cdots})$, which proves the theorem.

REMARK 1. Every $w \in H^2(B_T; \mathsf{Q})$ determines a corank one subtorus $C = w^\perp$ such that w generates the kernel of $H^*(B_T; \mathsf{Q}) \to H^*(B_C; \mathsf{Q})$ as an ideal. In Wu-yi Hsiang's paper [1] it is proven that w divides $\alpha_i - \alpha_j$ if and only if $F^i \subset F^j(w^\perp)$. Let $w_1, \ldots, w_r$ ($w_a \neq \pm w_b$) be the elements of $H^2(B_T; \mathsf{Z})$ of content 1 dividing $\alpha_1 - \alpha_2$. Then there is a unit $u \in A_p$ such that

$$\alpha_1 - \alpha_2 = p^e u w_1^{k_1} \ldots w_r^{k_r}$$

where $F^1(w_i^\perp) \sim_p P^{m_i}(k_i)$ for some $m_i \geqq 1$. This formula for the exponents k_i is proven in Hsiang-Su [3]. One can obtain a proof by noting that, in the terminology of Chang-Skjelbred [6],

$$I_\varepsilon = (\alpha_1 - \alpha_2) \subset H^*(B_T) \otimes \mathsf{Q},$$

hence $\alpha_1 - \alpha_2$ is a product of linear terms by Corollary (2.1) ibid. It follows that $I_\varepsilon^{C_i} = (w_i^{k_i})$ for some k_i where $C_i = w_i^\perp$. By calculating $I_\varepsilon^{C_i}$ directly, we find that the above formula for k_i holds.

REMARK 2. In [8] D. Golber determines the powers of p dividing the Euler class of a T-action on a cohomology sphere. In the case of a T-action on a sphere with two fixed points, the result of Golber's paper and this paper coincide. By using the present method of proof, one finds that Golber's result holds without the assumption that the integral cohomology of $F(T_i)$ is finitely generated (ibid., p. 150).

We conclude by marking another application of the method of proof of Theorem 1. Let

$$F^1 \sim_p P^m(d) \quad \text{and} \quad F^1(T_i) \sim_p P^{m(i)}(d(i)).$$

We assume that $m,d \geqq 1$. Let $x \in \boldsymbol{H}^{2d}(F^1)$ be a generator and let $\varphi : F^1 \subset X$. We then have

$$\varphi^*(y) = \alpha_1 + \beta_1 x + \gamma_1 x^2 + \ldots$$

where $\alpha_1, \beta_1, \ldots$ are elements of $\boldsymbol{H}^*(B_T)$. By the localization theorem for the action of T, using the cohomology functor $\boldsymbol{H}^* \otimes \mathrm{Q}$, it follows that $\beta_1 \neq 0$.

THEOREM 2. *The highest power of p dividing β_1 has exponent $\sum_{i \geqq 1} \big(d(i) - d\big)$.*

PROOF. We define the ideal $I(T,X)$ consisting of all $m \in A_p$ such that for some $s \in S$ there is a $z \in \boldsymbol{H}_T^*(X)$ with

$$\varphi^*(z) = msx \; (\operatorname{mod} x^2) \, .$$

Then $I(T,X) = (p^b)$ where p^b is the highest power of p dividing β_1. An induction proof like that of Theorem 1 now applies.

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