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A REMARK ON PURE IDEALS AND PROJECTIVE MODULES

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All rings considered in this note will be associative with an identity element and all modules will be unitary.

In the first part of this paper we prove that any twosided ideal which is pure as a left ideal is the trace of a projective left module. It is easily seen that the trace ideal of a projective module over a commutative ring is a pure ideal. We show by an example that there exists a ring and a projective left ideal with a non pure trace ideal.

In the second part of the paper we consider rings in which pure ideals are generated by idempotents.

1. The trace ideal of a projective module.

Let us first recall the definition of the trace ideal of a module. If P is an R-module, P^* denotes its dual module, $\mathrm{Hom}_R(P,R)$, then the trace ideal of P (denoted by $\tau(P)$) is the image of the natural pairing $P^* \otimes_R P \to R$.

It is well-known or readily checked that $\tau(P)$ is a twosided idempotent ideal, when P is a projective module.

Let P be a projective module over a commutative ring R and S a multiplicatively closed subset of R, then $\tau(P_S) = \tau(P)_S$.

Now it follows by the principle of localization that the trace of a projective module over a commutative ring is a pure ideal (i.e. $R/\tau(P)$ is a flat module).

PROPOSITION 1.1. *Let $\mathfrak{a}$ be a twosided ideal which is pure as a left ideal. There exists a projective module P such that $\tau(P) = \mathfrak{a}$.*

PROOF. The proof of the proposition is based on the following result due to D. Lazard [3, Théorème 3.1]:

Let K be a pure submodule of a free module F. Any countably generated

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submodule of K is contained in a countably generated submodule of K which is pure in K and F.

Let us now return to the proof of the proposition. First we note that a twosided ideal, $\mathfrak{a}$, which is pure as a left ideal is its own trace ideal. Clearly $\mathfrak{a} \subseteq \tau(\mathfrak{a})$, since $\mathfrak{a}$ is contained in R. On the other hand if $f(a) \in \tau(\mathfrak{a})$, there exists an $a' \in \mathfrak{a}$ such that $aa' = a$. Hence $f(a) = f(aa') = af(a') \in \mathfrak{a}$. More general the argument above shows that the trace ideal of a pure left ideal is the twosided ideal generated by the pure ideal.

If in proposition 1.1 $\mathfrak{a}$ is countably generated it follows by Jensens lemma [1, lemma 2] that $\mathfrak{a}$ is projective and the proposition is proved.

In the general case let us write $\mathfrak{a}$ as a direct union of countably generated pure left ideals $\mathfrak{a}_i$, $i \in I$ (This can be done by Lazard's Theorem). If $P = \coprod_i \mathfrak{a}_i$, then P is clearly projective as being a direct sum of projective modules. It is readily checked that $\tau(P)$ is the twosided ideal generated by $\tau(\mathfrak{a}_i)$, $i \in I$. It now follows that $\tau(P) = \mathfrak{a}$.

The example given below shows that there exists a ring R and a projective ideal I, such that the trace of I is not pure.

Example 1.2. Let Z_2 denote the field of integers modulo 2. We take R to be the Z_2-algebra on the generators x and y and defining relations

$$y^2 = y \,. \tag{1}$$

$$yx = 0 \text{ and } xy = x \,. \tag{2}$$

(It follows from (1) and (2) that $x^2 = 0$).

$R(1-y)$ is a projective left ideal with trace ideal equal to the twosided ideal generated by $1-y$. The example is completed if we can prove that $R(1-y)R$ is not pure as a left ideal.

Clearly $x = (1-y)x \in R(1-y)R$. It follows immediately from (1) and (2) that $xr = 0$ for all $r \in R(1-y)R$ and consequently $R(1-y)R$ is not pure as a left ideal.

Remark. One might notice that $R(1-y)R$ is pure as a right ideal. A similar example ($xy = 0$, $yx = x$) will however give us a ring and a projective left module with a trace ideal not pure as a right ideal. Taking the direct sum of these rings gives us a projective module with a trace ideal neither pure as a left nor as a right ideal.

One might also notice that the ring R in example 1.2 does satisfy a standard identity of degree 3.

2. Pure ideals.

In [5] W. V. Vasconcelos studied rings in which all pure ideals are generated by idempotents (he called these rings f-rings) and it turned out that a commutative ring R is an f-ring if and only if any projective ideal is a direct sum of finitely generated projective ideals. We will add a few remarks on f-rings.

PROPOSITION 2.1. *Let R be a commutative ring and assume that R/I is an* f*-ring, where I is an ideal contained in the Jacobson radical of R. If idempotents can be lifted from R/I to R, then R is an* f*-ring.*

PROOF. Let $\mathfrak{a}$ be a pure ideal in R. For any $r \in R$, $\bar{r}$ denotes the image of r under the canonical homomorphism from R to R/I.

Given any $a \in \mathfrak{a}$, we have to find an idempotent $f \in \mathfrak{a}$, such that $a = af$.

Since $\mathfrak{a}$ is pure we can choose an element $a_1 \in \mathfrak{a}$, such that $aa_1 = a$. By the hypothesis it follows that there exists an idempotent $f \in \mathfrak{a} + I$, such that $a_1 - a_1 f \in I$. Thus $a - af \in Ia$. Hence $Ra = Raf$ and consequently $Ra \subseteq Rf$.

Since $\bar{f} \in \bar{\mathfrak{a}}$, there exists an element $a_2 \in \mathfrak{a}$, such that $f - fa_2 \in I$. Hence $Rf = Rfa_2 + If$, so $Rf = Rfa_2$. This means that $f \in \mathfrak{a}$.

COROLLARY 1. (cf. [2]). *If R/N is an* f*-ring and N is a nil ideal, then R is an* f*-ring.*

COROLLARY 2. (cf. [2]). *If R is an* f*-ring, then $R[[X]]$ is an* f*-ring.*

EXAMPLE 2.2. In proposition 2.1 the assumption that idempotents can be lifted is essential as shown by the following example.

Let $R = \mathsf{Z}_2[(x_i)]$, $i \in \mathsf{N}$, be the polynomial ring over Z_2 in countably many variables and with the relations

$$x_i x_{i+1} = x_i, \quad i \in \mathsf{N} . \tag{3}$$

Let S be the multiplicatively closed set of polynomials of the form

$$1 + \textstyle\sum_i (x_i - x_i^2) p_i(x_i), \quad p_i(x_i) \in \mathsf{Z}_2[x_i] .$$

The ring R_S will give us the desired example:

It follows from (3) that $x_i - x_i^2 \in J(R_S)$ and it is readily checked that $R_S/J(R_S)$ is isomorphic to the ring of ultimatively constant sequences over Z_2, i.e. a von Neumann regular ring.

Clearly the ideal generated by the x_i's is a pure ideal in R_S. The example is completed if we can prove that the only idempotents in R_S are 0 and 1.

For this let $(\sum_i p_i(x_i))/s$ be an idempotent in R_S. Assume that for some i, $p_i(x_i)$ is a non constant polynomial. Define a homomorphism from R_S to the quotient field of $\mathbf{Z}_2[t]$ as follows:

x_i is mapped to t.
x_j is mapped to 0 for $j<i$.
x_k is mapped to 1 for $j>i$.

This mapping is a well-defined homomorphism from R_S to the field of quotients of $\mathbf{Z}_2[t]$. The idempotent is mapped to

$$a_0+p_i(t)/1+q_i(t)(t-t^2) .$$

Since $p_i(t)$ is non zero this element must be non zero. Consequently $a_0=1$ and $p_i(t)=q_i(t)(t-t^2)$. It is now immediate that the only idempotens in R_S are 0 and 1.

3. Remarks.

It follows from proposition 1.1 that if a ring R satisfies that projective modules are free, then cyclic flat modules are projective (the only pure ideals are (0) and R).

One might also note that if R is a commutative ring with finite Goldie dimension, then R must have pure ideals generated by idempotents (even generated by a single idempotent, i.e. cyclic flat modules are projective). This follows from the existence of the direct sum

$$(x_1-x_2)R\oplus(x_4-x_5)R\oplus\ldots\oplus(x_{3p+1}-x_{3p+2})R\oplus\ \ldots$$

where $x_ix_{i+1}=x_i$ for all $i\in\mathbf{N}$.

In particular $x_1R\oplus(1-x_3)R$ is a direct sum if $x_1x_2=x_1$ and $x_2x_3=x_2$ for some x_2. This shows that a selfinjective commutative ring has pure ideals generated by idempotens (cf. [5]). (The injective hull of Rx_1 is equal to Re for some idempotent e and $Re\oplus R(1-x_3)$ is direct hence $e\geqq ex_3$).

The trace ideal of a projective module has also been studied by E. A. Rutter and he has proved [4, Theorem 4.4] that the trace ideal of a projective left module P is pure (as a right ideal) if and only if any submodule of P is a homomorphic image of a direct sum of copies of P. This last remark is pointed out to us by the referee.

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