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POWER SERIES OVER COHERENT RINGS

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In the first part of this note we give an example of a coherent integral domain R such that the ring of power series $R[[X]]$ is not coherent. An example of a coherent ring R with zerodivisors such that $R[[X]]$ is not coherent is given in [4]. In the second part of this note we prove that if R is a semifir with ACC_n for all n, then $R[[X]]$ is right coherent.

1. Power series over valuation rings.

The main result in this section is the following:

THEOREM 1. *If R is a valuation ring not of rank one, then $R[[X]]$ is not coherent.*

The proof of the theorem is based on a couple of lemmas.

LEMMA 1. *Let R be any ring. If $R[[X]]$ is left coherent, then*

$$\mathrm{w.gl.dim}\,R[[X]] = \mathrm{w.gl.dim}\,R+1 .$$

The lemma might be well-known, but we have not been able to find a proof in the literature.

PROOF OF LEMMA 1. If M is any left R-module, then it is readily checked that

$$\mathrm{w.hd.}_{R[[X]]}M \geqq \mathrm{w.hd.}_R M+1 ,$$

where the structure of M as an $R[[X]]$-module is defined by $Xm=0$ for all $m \in M$. Consequently

$$\mathrm{w.gl.dim}\,R[[X]] \geqq \mathrm{w.gl.dim}\,R+1 .$$

We have now to prove that any left $R[[X]]$-module M has $\mathrm{w.hd.}_{R[[X]]}M \leqq n+1$, where $n=\mathrm{w.gl.dim}\,R$. Since $R[[X]]$ is left coherent it suffices to

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consider left $R[[X]]$-modules of finite presentation. Clearly the result follows if we can prove that

$$\text{w.hd.}_{R[[X]]}M = \text{w.hd}_R M/XM ,$$

where X is assumed to be a non zerodivisor in M and M is of finite presentation. For left modules M of finite presentation over left coherent rings we have that the weak homological dimension is equal to the left projective dimension. Thus we have to prove that

$$\text{l.hd.}_{R[[X]]}M = \text{l.hd.}_R M/XM$$

for any left $R[[X]]$-module having a projective resolution of finitely generated projective modules and X a non zerodivisor in M. This result is essentially proved in [3, Theorem 5.6, p. 49].

LEMMA 2. *Let R be a valuation ring. R is completely integrally closed if and only if R is of rank one.*

PROOF. See [1, chapitre 6, § 4, nº 5, proposition 9].

LEMMA 3. *If R is integrally closed, but not completely integrally closed, then $R[[X]]$ is not integrally closed.*

PROOF. See [1, chapitre 5, § 1, exercise 27].

LEMMA 4. *If S is a coherent integral domain of* $\text{w.gl.dim}\, S = 2$, *then S is integrally closed.*

PROOF. (See also W. V. Vasconcelos [5].) We may assume that S is a local domain with S coherent and $\text{w.gl.dim}\, S = 2$.

The exact sequence

$$0 \to (x) \cap (y) \to S \oplus S \to (x) + (y) \to 0$$

shows that $(x) \cap (y)$ is flat and finitely generated for all x and y. $(x) \cap (y)$ is projective, since S is coherent, thus $(x) \cap (y)$ is free (S local). We have now proved that the intersection of two principal ideals is principal, thus S is integrally closed.

It is obvious that Theorem 1 follows from Lemma 1, 2, 3 and 4.

According to Gruson it is unknown in general whether or not $R[[X]]$ is coherent, if R is a valuation ring of rank one.

We would also like to mention that Raynaud and Gruson have proved that the polynomial ring in any number of indeterminates over a valuation ring is coherent.

2. Power series over semifirs.

For basic results on semifirs we refer to [2]. Let us furthermore recall that a ring is said to have ACC_n, if any ascending chain of n-generated ideals terminates.

The following lemma will be needed in the proof of the main result in this section.

LEMMA 5. *Let* $0 \to K \to L \to M \to 0$ *be an exact sequence of finitely generated modules. If* L *is of finite presentation, then* M *is of finite presentation, too.*

The proof is straightforward.

THEOREM 2. *Let* R *be a semifir with* ACC_n *for all* n. $R[[X]]$ *is right coherent.*

PROOF. If $a = a_0 + \ldots + a_n X^n + \ldots$ is an element in $R[[X]]$, then $v(a)$ denotes the integer n where $a_0 = a_{n-1} = 0$ and $a_n \neq 0$, $l(a)$ denotes the leading term of a, that is, a_n.

Let I be a finitely generated right ideal in $R[[X]]$, generated by n elements, say. We denote by I_0 the right ideal of R generated by leading terms of elements $a \in I$, $v(a) \neq 0$. I_0 is at most n-generated, hence I_0 is free. It follows readily from [2], Theorem 2.6 that we might choose a generating set for I of the form $(a_{01}, \ldots, a_{0j_0}, b_{01}, \ldots, b_{0n-j_0})$, where $v(a_{0t}) = 0$ and $v(b_{0s}) \geqq 1$ and the ideal generated by the leading terms of the a_{0t}'s is I_0.

Put $J_0 = (a_{01}, \ldots, a_{0j_0})$ and $J_0' = (b_{01}, \ldots, b_{0n-j_0})$. We have an exact sequence

$$0 \to XJ_0 \to J_0 \oplus (XJ_0 + J_0') \to I \to 0 .$$

By Lemma 5 it suffices to prove that $XJ_0 + J_0'$ is finitely presented.

I_1 denotes the right ideal in R generated by leading terms of elements $a \in (XJ_0 + J_0')$, where $v(a) = 1$. Clearly $I_0 \subseteqq I_1$ and I_1 is at most n-generated. As before we can choose a generating set for $(XJ_0 + J_0')$ of the following form $(a_{11}, \ldots, a_{1j_1}, b_{11}, \ldots, b_{1n-j_1})$, where $v(a_{1t}) = 1$ and $v(b_{1s}) \geqq 2$ and the ideal generated by the leading terms of the a_{1t}' is I_1.

Put $J_1 = (a_{11}, \ldots, a_{1j_1})$ and $J_1' = (b_{11}, \ldots, b_{1n-j_1})$. We have an exact sequence

$$0 \to XJ_1 \to J_1 \oplus (XJ_1 + J_1') \to XJ_0 + J_0' \to 0 .$$

As before it follows that it suffices to prove that $XJ_1 + J_1'$ is finitely presented.

If we continue this process we get an ascending chain of at most n-generated right ideals $I_0 \subseteqq I_1 \subseteqq \ldots \subseteqq I_m \subseteqq \ldots$. By ACC_n this sequence must terminate. If $I_m = I_{m+1} = \ldots$, then it is readily checked that $XJ_m = XJ_m + J_m'$. (The proof of Theorem 2.2 in [3].) The proof of Theorem 2 is now completed.

If R is an $\aleph_0$-fir (that is R is a semifir and the countably generated ideals are free), then it can be shown that R has ACC_n. Thus we get:

Corollary 1. *If R is a right $\aleph_0$-fir, then $R[[X]]$ is right coherent and* $\text{w.gl.dim}\, R[[X]] = 2$.

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