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REMARKS ON RELATIONS BETWEEN MAXIMAL LATTICES AND RELATIVELY MINIMAL MODELS

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In [4] we have proved that if A is a Dedekind ring of characteristic $\neq 2$ with perfect residue fields (i.e. $A/\mathfrak{p}$ is a perfect field for every maximal ideal $\mathfrak{p}$ in A), E/F a regular extension of transcendence degree 1 and genus 0 of the field of fractions of A then there is a regular quadratic space (V,Q) over F and a lattice L on V such that L defines a relatively minimal Spec(A)-model $M(L)$ of E. The aim of this paper is to characterize those lattices which define relatively minimal models in the way described in [4]. The main result says that every relatively minimal model can be defined by an $\mathfrak{a}$-maximal lattice (see [5, § 82 H]) where $\mathfrak{a}$ is an ideal in A which depends only on the extension E/F and A (Theorem 1).

1. The discriminant of a lattice.

Let (V,Q) be a regular quadratic space over the field of fractions F of a discrete valuation ring A of characteristic $\neq 2$ and $\dim V = 3$. Let π be a generator of the maximal ideal of A. If L is a lattice on V then $L = Ae_0 + Ae_1 + Ae_2$ and this lattice defines a quadratic form

$$f_L = (1/\pi^r) \textstyle\sum_{i,j=0}^{2} B(e_i,e_j)X_iX_j = \sum_{0\leq i\leq j\leq 2} a_{ij}X_iX_j$$

where B is the bilinear form defined by Q, $a_{ii} = (1/\pi^r)B(e_i,e_i)$, $a_{ij} = 2\cdot(1/\pi^r)B(e_i,e_j)$ for $i \neq j$ and $(\pi^r) = \mathfrak{n}L$ is the norm of L (see [5] for the notion of norm, in [4] the form which corresponds to L is

$$(1/\pi^r) \textstyle\sum_{0\leq i\leq j\leq 2} B(e_i,e_j)X_iX_j$$

where $(\pi^r) = \mathfrak{s}L$ is the scale of L. But it will be clear that the present definition of f_L is more convenient). Since $\mathfrak{n}L$ is generated by $B(e_i,e_i)$ and $2B(e_i,e_j)$ the coefficientes of the form f_L belong to A. Now if

$$f = \textstyle\sum_{0\leq i\leq j\leq 2} a_{ij}X_iX_j$$

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then the determinant

$$d(f) = (\tfrac{1}{2})\begin{vmatrix} 2a_{00} & a_{01} & a_{02} \\ a_{01} & 2a_{11} & a_{12} \\ a_{02} & a_{12} & 2a_{22} \end{vmatrix}$$

is called the discriminant of f (see [6, p. 2]). It is easy to check that if $a_{ij} \in A$ then $d(f) \in A$.

If f_L is the form which corresponds to L then

$$d(f_L) = 4(1/\pi^{3r})\begin{vmatrix} B(e_0,e_0) & B(e_0,e_1) & B(e_0,e_2) \\ B(e_0,e_1) & B(e_1,e_1) & B(e_1,e_2) \\ B(e_0,e_2) & B(e_1,e_2) & B(e_2,e_2) \end{vmatrix} = 4(1/\pi^{3r})d(e_0,e_1,e_2)$$

where $d(e_0,e_1,e_2)$ is the discriminant of the base e_0,e_1,e_2 of V over F (see [5, p. 87]). But $d(e_0,e_1,e_2)$ generates the volume $\mathfrak{v}L$ of L (see [5, p. 229]) and the last equality in the global case gives the following result:

LEMMA 1. *Let A be a Dedekind ring and L a lattice on a regular quadratic space (V,Q) over the field of fractions F of A. Then*

$$4\mathfrak{v}L = (\mathfrak{n}L)^3\mathfrak{d}L \tag{1}$$

where $\mathfrak{v}L$ is the volume of L, $\mathfrak{n}L$ the norm of L and $\mathfrak{d}L$ a fractional ideal which is locally defined by $(\mathfrak{d}L)_\mathfrak{p} = (d(f_{L_\mathfrak{p}}))$ where $f_{L_\mathfrak{p}}$ is the quadratic form corresponding to $L_\mathfrak{p}$. Since $d(f_{L_\mathfrak{p}}) \in A_\mathfrak{p}$ for every prime ideal $\mathfrak{p}$ in A the ideal $\mathfrak{d}L$ is integral.

DEFINITION 1. The ideal $\mathfrak{d}L$ will be called the discriminant of L.

REMARK. We have defined $\mathfrak{d}L$ only for lattices on three dimensional spaces. It is clear that this definition can be generalized according to the usual definition of the discriminant of a quadratic form (e.g. [6, p. 2]).

LEMMA 2. *Let $f = \sum_{0 \le i \le j \le 2} a_{ij}X_iX_j$ be a quadratic form with coefficients in a field F of an arbitrary characteristic (it can be equal 2). The form f is reducible in some extension of F if and only if $d(f) = 0$.*

REMARK. If $\operatorname{char}(F) = 2$ then

$$\begin{aligned} d(f) &= 4a_{00}a_{11}a_{22} + a_{01}a_{02}a_{12} - a_{00}a_{12}^2 - a_{11}a_{02}^2 - a_{22}a_{01}^2 \\ &= a_{01}a_{02}a_{12} + a_{00}a_{12}^2 + a_{11}a_{02}^2 + a_{22}a_{01}^2 . \end{aligned}$$

PROOF. If $\operatorname{char}(F) \neq 2$ then the result is well-known. If $\operatorname{char}(F) = 2$ and $f = (a_0x_0 + a_1x_1 + a_2x_2)(b_0x_0 + b_1x_1 + b_2x_2)$ then it is easy to check that

$d(f)=0$. Let $d(f)=0$. Then (a_{12},a_{02},a_{01}) is a zero of f. If $a_{ij}=0$ for $i \neq j$ then the form is reducible in some extension of F. Let $a_{12} \neq 0$. Then the transformation

$$x_0 = a_{12}y_0, \quad x_1 = a_{02}y_0+y_1, \quad x_2 = a_{01}y_0+y_2$$

has determinant not equal to 0 and maps the form f on the form $a_{11}y_1^2 + a_{22}y_2^2+a_{12}y_1y_2$. This form is reducible in some extension of F.

2. Lattices which define models.

We shall assume that A is a Dedekind ring with perfect residue fields such that the characteristic of A is $\neq 2$. E is a regular extension of transcendence degree 1 and genus 0 of F where F is the field of fractions of A.

Definition 2. We shall denote by $\mathfrak{a}_{E/A}$ the ideal of A which is equal to the product of all maximal ideals $\mathfrak{p}$ in A such that the fiber above $\mathfrak{p}$ of a relatively minimal model M of E over $\operatorname{Spec}(A)$ is a form of two intersecting copies of $P^1(A/\mathfrak{p})$ (the projective line over $A/\mathfrak{p}$). This ideal is independent of the relatively minimal model by the Theorem 1 in [2].

Lemma 3. *Let L be a lattice on a regular quadratic space (V,Q) over F.*
a) *If $M(L)$ is a model of E then $\mathfrak{d}L$ is square-free.*
b) *If $M(L)$ is a relatively minimal model of E then $\mathfrak{d}L=\mathfrak{a}_{E|A}$.*

Proof. If $M(L)$ is a model then by Theorem 2 in [3] $v_{\mathfrak{p}}(d(f_{L_{\mathfrak{p}}}))=0$ or 1 where $v_{\mathfrak{p}}$ is the valuation corresponding to $A_{\mathfrak{p}}$. Hence $\mathfrak{d}L$ is square-free. Now if $M(L)$ is a relatively minimal model then the last case takes place if and only if the fiber of this model above $\mathfrak{p}$ is a form of two intersecting copies of $P^1(A/\mathfrak{p})$. In fact, $v_{\mathfrak{p}}(d(f_{L_{\mathfrak{p}}}))=1$ if and only if $d(\bar{f}_{L_{\mathfrak{p}}})=0$ where $\bar{f}_{L_{\mathfrak{p}}}$ is the image of $f_{L_{\mathfrak{p}}}$ under the homomorphism

$$A[x_0,x_1,x_2] \to (A/\mathfrak{p})[x_0,x_1,x_2] .$$

By the Lemma 2, $d(\bar{f}_{L_{\mathfrak{p}}})=0$ if and only if $\bar{f}_{L_{\mathfrak{p}}}$ is reducible in some extension of $A/\mathfrak{p}$, i.e. the fiber of $M(L)$ above $\mathfrak{p}$ is a form of two intersecting copies of $P^1(A/\mathfrak{p})$.

Theorem 1. *Let M be a model of E over* $\operatorname{Spec}(A)$ *such that for every maximal ideal $\mathfrak{p}$ in A the fiber $M_{\mathfrak{p}}$ above $\mathfrak{p}$ is either a form of $P^1(A/\mathfrak{p})$ or a form of two intersecting copies of $P^1(A/\mathfrak{p})$. Then there is a quadratic space (V,Q) and a lattice L on V such that M is* $\operatorname{Spec}(A)$*-isomorphic with $M(L)$ and L is $\mathfrak{d}L$-maximal. Hence if M is a relatively minimal model then L is $\mathfrak{a}_{E/A}$-maximal.*

PROOF. We know that there is a quadratic space (V,Q) and a lattice L on V such that M is $\mathrm{Spec}(A)$-isomorphic to $M(L)$. For relatively minimal M this is proved in [4, Theorem 2]. If M is a model of E then we get L if we apply Theorem 1 in [3] and the same construction as in the proof of Theorem 2 in [4].

Since $\mathfrak{v}L$ defines the neutral element in $\mathrm{Cl}(A)/\mathrm{Cl}(A)^2$ where $\mathrm{Cl}(A)$ denotes the class group of A (by the definition of $\mathfrak{v}L$ — see [5, p. 229]) hence by (1) we get that $\mathfrak{d}L$ and $\mathfrak{n}L$ define the same element in the group $\mathrm{Cl}(A)/\mathrm{Cl}(A)^2$. We know that $\mathfrak{n}(\mathfrak{a}L)=\mathfrak{a}^2(\mathfrak{n}L)$ and $\mathfrak{n}L^\alpha=\alpha(\mathfrak{n}L)$ (see [5, p. 228 and p. 238]). This means that we can choose a lattice $L'=(\mathfrak{a}L)^\alpha$ on a quadratic space V^α such that the model defined by L' is equal to the model defined by L and $\mathfrak{n}L'=\mathfrak{d}L'$. We shall assume that L is such lattice and we shall prove that this lattice is $\mathfrak{d}L$-maximal.

Let K be a lattice such that $\mathfrak{n}K\subseteq\mathfrak{d}L=\mathfrak{n}L$ and $K\supseteq L$. These inclusions give $\mathfrak{n}K=\mathfrak{n}L$ and $\mathfrak{v}L=\mathfrak{a}^2(\mathfrak{v}K)$ where $\mathfrak{a}$ is an ideal in A (see [5, § 82 E, 82:11]). Hence by (1)

$$\mathfrak{d}L = (\mathfrak{n}L)^{-3}4\mathfrak{v}L = (\mathfrak{n}K)^{-3}4\mathfrak{a}^2\mathfrak{v}K = \mathfrak{a}^2\mathfrak{d}K .$$

But $\mathfrak{d}L$ is integral, square-free (Lemma 3) and $\mathfrak{d}K$ is integral. Hence $\mathfrak{a}=A$ and $\mathfrak{v}L=\mathfrak{v}K$. Since $K\supseteq L$ we get $K=L$ (by [5, § 82 E, 82:11a]). This proves that L is $\mathfrak{d}L$-maximal.

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