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PERFECT CODES IN ANTIPODAL DISTANCE-TRANSITIVE GRAPHS

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Let C be a perfect code in an antipodal distance-transitive graph. In this paper it is shown that if $u \in C$ then any vertex at maximum distance from u also belongs to C. This is a generalisation of a theorem for binary codes of Roos [1].

1.

A graph is a pair $(V(G), E(G))$ where $V(G)$ is a finite and nonempty set of elements called vertices and $E(G)$ is a set of unordered pairs of distinct elements of $V(G)$ called edges.

$(v_0, v_1, \ldots, v_n)$ is a path from v_0 to v_n if v_i, $i=0,1,\ldots,n$ are vertices and $\{v_i, v_{i+1}\}$ are distinct edges. A graph is called *connected* if given any pair of vertices v,w, there is a path from v to w. In this paper we only consider connected graphs.

The number of edges in a path is the length of the path. Let $d(u,v)$, the distance between the vertices u and v, denote the length of the shortest path from u to v. The function $d(u,v)$ defines a metric on the set of vertices.

An *automorphism* φ of a graph is a permutation of $V(G)$ such that for any given pair of vertices u and v it is true that $d(\varphi(u),\varphi(v)) = d(u,v)$.

A graph is called *distance-transitive* if for any given two pairs of vertices u,v and w,z satisfying $d(u,v)=d(w,z)$ there is an automorphism φ for which $\varphi(u)=w$ and $\varphi(v)=z$. All graphs in this paper are distance-transitive.

Let $u \in V(G)$ and

$$\Gamma_i(u) = \{v \in V(G) \mid d(u,v)=i\} .$$

Let d be the maximum possible distance between any two vertices. d is called the *diameter* of G. A graph is called *antipodal* if for all vertices $v,w \in \Gamma_0(u) \cup \Gamma_d(u)$ either $v=w$ or $d(v,w)=d$.

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EXAMPLE. Let Z_n be the integers modulo n. Let Z_n^r be the set of r-tuples of elements of Z_n. Define the distance between r-tuples $\bar{s}=(s_1,\ldots,s_r)$ and $\bar{t}=(t_1,\ldots,t_r)$ to be

$$d(\bar{s},\bar{t}) = |\{i \mid s_i \neq t_i\}| .$$

Z_n^r is a distance-transitive graph where the r-tuples are vertices and $d(\bar{s},\bar{t})$ is the distance-function on the vertices. Z_2^r is an antipodal distance-transitive graph.

A subset C of $V(G)$ is called a *perfect e-error correcting code* if for every vertex v it is true that

$$|\{u \in V(G) \mid d(v,u) \leqq e\} \cap C| = 1 .$$

Let u be a vertex. Define

$$\gamma_i = |\Gamma_i(u) \cap C| \quad i=1,2,\ldots .$$

Call the $d+1$-tuple $(\gamma_0,\gamma_1,\ldots,\gamma_d)$ the *weight-enumerator* of C. The weight enumerator is not independent of the choice of u. But we shall see in section 2 that it only depends on $d(u,C)$, the minimum possible distance between u and any vertex of C. $d(u,C)$ is called the *minimum weight* of C.

Let u and v be two vertices such that $d(u,v)=j$. The numbers

$$\begin{aligned} k_i &= |\Gamma_i(u)| && i=0,1,\ldots,d \\ a_j &= |\Gamma_1(v) \cap \Gamma_j(u)| && \\ b_j &= |\Gamma_1(v) \cap \Gamma_{j+1}(u)| && (\text{defined for } j \leqq d-1) \\ c_j &= |\Gamma_1(v) \cap \Gamma_{j-1}(u)| && (\text{defined for } j \geqq 1) \end{aligned}$$

are independent of the choices of u and v. They satisfy the following relations

(1) $$a_j+b_j+c_j = k_1, \quad j=0,1,\ldots,d ,$$
$$k_ib_i = k_{i+1}c_{i+1}, \quad i=0,1,\ldots,d-1 ,$$

(2) $$k_1 = b_0 > b_1 \geqq \ldots \geqq b_{d-1} \geqq 1, \quad 1=c_1 \leqq c_2 \leqq \ldots \leqq c_d .$$

For a proof of this see [4]. Let

$$\Gamma(G) = \begin{pmatrix} 0 & c_1 & 0 & & & 0 \\ b_0 & a_1 & c_2 & & & \\ 0 & b_1 & a_2 & \ddots & & \\ & & b_2 & \ddots & & \\ & & & \ddots & c_{d-1} & 0 \\ & & & \ddots & a_{d-1} & c_d \\ 0 & & & & b_{d-1} & a_d \end{pmatrix}$$

$\Gamma(G)$ is called the *intersection matrix* of G. If $[1, v_1(\lambda), \ldots, v_d(\lambda)]^t$ is an right eigenvector of $\Gamma(G)$ belonging to the eigenvalue λ, then it must satisfy the relations

$$(3)\qquad \begin{aligned} &v_1(\lambda) = \lambda\,, \\ &c_{i+1}v_{i+1}(\lambda) + (a_i - \lambda)v_i(\lambda) + b_{i-1}v_{i-1}(\lambda) = 0 \quad (v_0(\lambda) = 1) \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad i = 1, 2, \ldots, d-1\,. \end{aligned}$$

$$(4)\qquad b_{d-1}v_{d-1}(\lambda) + (a_d - \lambda)v_d(\lambda) = 0\,.$$

The functions $v_i(\lambda)$, $i = 1, \ldots, d$, are polynomials in λ of degree i.

Biggs has shown [2] and [3] that the $d+1$ eigenvalues of $\Gamma(G)$ are distinct and that they are zeros of the polynomial

$$(\lambda - k_1)(1 + v_1(\lambda) + \ldots + v_d(\lambda))\,.$$

2.

In [3] Biggs shows that if a perfect e-error correcting code exists in the distance-transitive graph G then the polynomial $1 + v_1(\lambda) + \ldots + v_e(\lambda)$ divides the polynomial $1 + v_1(\lambda) + \ldots + v_d(\lambda)$. It is natural to ask which polynomial $f(\lambda)$ satisfy

$$(1 + v_1(\lambda) + \ldots + v_e(\lambda))f(\lambda) = 1 + v_1(\lambda) + \ldots + v_d(\lambda)\,.$$

We shall prove a lemma saying that if $(\gamma_0, \gamma_1, \ldots, \gamma_d)$ is the weight-enumerator of the code then $1 + v_1(\lambda) + \ldots + v_d(\lambda)$ divides

$$(1 + v_1(\lambda) + \ldots + v_e(\lambda))(\gamma_0 + \gamma_1 v_1(\lambda)/k_1 + \ldots + \gamma_d v_d(\lambda)/k_d)\,.$$

Consequently at least $d - e$ eigenvalues of the intersection matrix must be zeros of the polynomial $\gamma_0 + \gamma_1 v_1(\lambda)/k_1 + \ldots + \gamma_d v_d(\lambda)/k_d$. The solution of a system of n such linear equations will only depend on $\gamma_0, \gamma_1, \ldots, \gamma_{d-n}$ as we shall see in lemma 2. Knowing this it will be easy to prove the theorem of Biggs and to prove that the weight-enumerator of the code only depends on the minimum weight for the code.

LEMMA 1. *If C is a perfect code that corrects e errors and $(\gamma_0, \gamma_1, \ldots, \gamma_d)$ is the weight enumerator of C then the polynomial $1 + v_1(\lambda) + \ldots + v_d(\lambda)$ divides the polynomial*

$$(1 + v_1(\lambda) + \ldots + v_e(\lambda))(\gamma_0 + \gamma_1 v_1(\lambda)/k_1 + \ldots + \gamma_d v_d(\lambda)/k_d)\,.$$

PROOF. Let μ be an eigenvalue of the intersection matrix, and u a vertex of G. To every vertex v of G associate the following number

$$v_{d(u,v)}(\mu)/k_{d(u,v)} = f(\mu, v)\,.$$

Using induction over i and the relations (1), (3) and (4) it is straightforward to prove that

$$v_i(\mu)f(\mu,v) = \sum_{w,\, d(v,w)=i} f(\mu,w) \quad \text{for } i=0,1,\ldots,d\,.$$

Consequently if C is a perfect e-error correcting code

$$\Big(\sum_{v\in C} f(\mu,v)\Big)\big(1+v_1(\mu)+\ldots+v_e(\mu)\big) = \sum_{v\in V(G)} f(\mu,v)\,,$$

that is,

$$(\gamma_0+\gamma_1 v_1(\mu)/k_1+\ldots+\gamma_d v_d(\mu)/k_d)(1+v_1(\mu)+\ldots+v_e(\mu))$$
$$= 1+v_1(\mu)+\ldots+v_d(\mu)\,.$$

Since the zeros of $1+v_1(\lambda)+\ldots+v_d(\lambda)$ are eigenvalues of the intersection-matrix, it is necessary that the zeros of $1+v_1(\lambda)+\ldots+v_d(\lambda)$ are zeros of

$$(\gamma_0+\gamma_1 v_1(\lambda)/k_1+\ldots+\gamma_0 v_d(\lambda)/k_d)(1+v_1(\lambda)+\ldots+v_e(\lambda))\,.$$

Consequently the lemma 1 is true.

Lemma 2. *If $\lambda_1,\ldots,\lambda_j$ are distinct eigenvalues of the intersection matrix of G then*

$$\det\begin{pmatrix} \dfrac{v_{d-j+1}(\lambda_1)}{k_{d-j+1}} & \cdots & \dfrac{v_d(\lambda_1)}{k_d} \\ \vdots & & \vdots \\ \dfrac{v_{d-j+1}(\lambda_j)}{k_{d-j+1}} & \cdots & \dfrac{v_d(\lambda_j)}{k_d} \end{pmatrix} \neq 0$$

Proof. Suppose μ is an eigenvalue of $\Gamma(G)$ and $v_d(\mu)=0$. Then we get by recursion using (3) and (4) that $v_0(\mu)=0$. This is impossible since $v_0(\mu)=1$. We conclude that $v_d(\mu)\neq 0$. So by dividing by the nonzero number $v_d(\mu)$ we get an eigenvector

$$(1/v_d(\mu),\ldots,v_{d-1}(\mu)/v_d(\mu),v_d(\mu)/v_d(\mu))^t$$
$$= (v'_0(\mu),\ldots,v'_{d-1}(\mu),1)^t$$

of $\Gamma(G)$ belonging to the eigenvalue μ. Now $v'_i(\mu)$, $i=0,1,\ldots,d-1$ must satisfy the relations

$$b_{d-1}v'_{d-1}(\mu) = \mu-a_d\,,$$

$$c_{i+1}v'_{i+1}(\mu)+(a_i-\mu)v'_i(\mu)+b_{i-1}v'_{i-1}(\mu) = 0 \quad i=1,2,\ldots,d-1\,.$$

Using recursion we see that $v'_i(\mu)$ is a polynomial in μ of degree $d-i$.

So by elementary determinant calculus

$$\det\begin{pmatrix} \frac{v_{d-j+1}(\lambda_1)}{k_{d-j+1}} & \cdots & \frac{v_d(\lambda_1)}{k_d} \\ \vdots & & \vdots \\ \frac{v_{d-j+1}(\lambda_j)}{k_{d-j+1}} & \cdots & \frac{v_d(\lambda_j)}{k_d} \end{pmatrix} = \frac{\prod_{i=1}^{j} v_d(\lambda_i)}{\prod_{i=1}^{j} k_{d-i+1}} \det\begin{pmatrix} v'_{d-j+1}(\lambda_1) & \cdots & 1 \\ \vdots & & \vdots \\ v'_{d-j+1}(\lambda_j) & \cdots & 1 \end{pmatrix}$$

$$= r \det\begin{pmatrix} \lambda_1^{j-1} & \cdots & \lambda_1 & 1 \\ \vdots & & & \vdots \\ \lambda_j^{j-1} & \cdots & \lambda_j & 1 \end{pmatrix} \quad \text{for some } r \neq 0 .$$

Since the λ_i's, $i=1,2,\ldots,j$ are distinct the last determinant is nonzero and the lemma is proved.

THEOREM 1 (Biggs). *If there exists a perfect e-error correcting code C in the distance-transitive graph G then the polynomial* $1+v_1(\lambda)+\ldots+v_e(\lambda)$ *divides the polynomial* $1+v_1(\lambda)+\ldots+v_d(\lambda)$.

PROOF. For every perfect code C with minimum weight less than e there exists an automorphism φ of G such that $\varphi(C)=C'$ is a perfect code with minimum weight equal to e. Suppose that the polynomial $1+v_1(\lambda)+\ldots+v_e(\lambda)$ has less than e zeros among the eigenvalues of $\Gamma(G)$. If $\gamma_0=\ldots=\gamma_{e-1}=0$ there exists a perfect code with such a weight-enumerator, as we saw above. Then by lemma 2 the solutions of the linear system of equations

$$\gamma_0+\gamma_1 v_1(\lambda_i)/k_1+\ldots+\gamma_d v_d(\lambda_i)/k_d = 0, \quad \lambda_i \text{ eigenvalue of } \Gamma(G) \text{ and } i=1,2,\ldots,d-e+1$$

should be $\gamma_j=0$, $j=e,e+1,\ldots,d$. This is impossible.

THEOREM 2. *The weight-enumerator of a perfect code in a distance-transitive graph only depends on the minimum-weight of the code.*

PROOF. Let $(\gamma_0,\gamma_1,\ldots,\gamma_d)$ be the weight enumerator of the perfect e-error correcting code C. From lemma 1 we know that there exist $d-e$ eigenvalues λ_s, $s=1,2,\ldots,d-e$ of $\Gamma(G)$ such that

$$\gamma_0+\gamma_1 v_1(\lambda_s)/k_1+\ldots+\gamma_d v_d(\lambda_s)/k_d = 0 .$$

Suppose that the minimum weight of C is equal to i, that is, $\gamma_0=\ldots=\gamma_{i-1}=\gamma_{i+1}=\ldots=\gamma_e=0$, $\gamma_i=1$. We then get that

$$(*) \quad \gamma_{e+1}v_{e+1}(\lambda_s)/k_{e+1}+\ldots+\gamma_d v_d(\lambda_s)/k_d = v_i(\lambda_s)/k_i \quad s=1,2,\ldots,d-e$$

Since

$$\det \begin{pmatrix} \frac{v_{e+1}(\lambda_1)}{k_{e+1}} & \cdots & \frac{v_d(\lambda_1)}{k_d} \\ \vdots & & \vdots \\ \frac{v_{e+1}(\lambda_{d-e})}{k_{e+1}} & \cdots & \frac{v_d(\lambda_{d-e})}{k_d} \end{pmatrix} \neq 0$$

we get that the solutions of the system of linear equations (*) are unique.

3.

The following relations are easy but useful consequences of the definition of antipodal distance-transitive graph of diameter d.

(5) If $d(u,v) < d$ then $\Gamma_d(u) \cap \Gamma_d(v) = \emptyset$.

(6) If $d(u,v) = d$ and $d(v,w) = i < d/2$ then $d(u,w) = d - i$.

(7) If $d(u,v) = d = 2n+1$ then $\Gamma_n(v) \subseteq \Gamma_{n+1}(u)$.

(8) If $d(u,v) = d = 2n$ then $\Gamma_{n-1}(v) \subseteq \Gamma_{n+1}(u)$.

(9) If $d = 2n+1$ then $\Gamma_{n+1}(u) = \bigcup_{v \in \Gamma_d(u)} \Gamma_n(v)$.

(10) If $d = 2n$ then $\Gamma_{n+1}(u) = \bigcup_{v \in \Gamma_d(u)} \Gamma_{n-1}(v)$.

We need two lemmas for the proof of theorem 3.

LEMMA 3. *If G is an antipodal distance-transitive graph with diameter d then* $1 \leqq k_1 \leqq k_2 \leqq \ldots \leqq k_j > k_{j+1} > \ldots > k_d$ *for some*

$$j \geqq \begin{cases} n+1 & \text{if } d = 2n+1 \\ n & \text{if } d = 2n \,. \end{cases}$$

PROOF. Suppose that $k_j > k_{j+1}$. Then from relation (1) we get that $c_{j+1} > b_j$. So by using relation (2) we see that $c_{s+1} > b_s$ if $s > j$ and consequently $k_s > k_{s+1}$ if $s > j$. By (7) and (8) is $k_n \leqq k_{n+1}$ when $d = 2n+1$ and $k_{n-1} \leqq k_{n+1}$ when $d = 2n$. It follows that $j \geqq n+1$ if $d = 2n+1$ and $j \geqq n$ if $d = 2n$.

LEMMA 4. *If G is an antipodal distance-transitive graph with diameter d then*

$$k_d = \begin{cases} b_n/c_{n+1} & \text{if } d = 2n+1 \\ b_n/c_n & \text{if } d = 2n \end{cases}$$

PROOF. First assume that $d=2n+1$. Let $z \in \Gamma_n(u)$, that is, $d(u,z)=n$. By (9) we have

$$|\Gamma_{n+1}(u) \cap \Gamma_1(z)| = \sum_{v \in \Gamma_d(u)} |\Gamma_n(v) \cap \Gamma_1(z)| ,$$

that is, $b_n = k_d c_{n+1}$, since $d(v,z)=n+1$. When $d=2n=d(u,v)$ choose z such that $d(u,z)=d(v,z)$, and use (10) similarly.

THEOREM 3. *If C is a perfect code in an antipodal distance-transitive graph with diameter d then for any vertex u it is that either $\Gamma_0(u) \cup \Gamma_d(u) \subseteq C$ or $(\Gamma_0(u) \cup \Gamma_d(u)) \cap C = \emptyset$.*

PROOF. Suppose that $u \in C$ and that there exists a vertex $v \in \Gamma_d(u) \setminus C$. Since C is perfect and corrects e errors there must be a vertex v' for which $d(v,v')=i \leqq e$.

Let $w \in \Gamma_i(u)$ and $d(w,v')=d$. It is easy to see that such a vertex must exist. Let φ be an automorphism that satisfy $\varphi(w)=u$ and $\varphi(u)=w$. If $(\gamma_0, \gamma_1, \ldots, \gamma_d)$ is the weight enumerator of $\varphi(C)$ then $\gamma_i = 1$ and $\gamma_d \geqq 1$.

But we get from lemma 3 that $|\Gamma_i(u)| \geqq k_1$ (in the nontrivial cases $e \leqq d/2$) and from lemma 4, since $b_n < k_1$, that $k_d < k_1$. Let $V = \bigcup_{v \in \Gamma_d(u)} \Gamma_i(v)$. Then we find, since C is an e-error correcting code,

$$|C \cap V| \leqq |\Gamma_d(u)| = k_d < k_1 \leqq |\Gamma_i(u)| ,$$

that is, $|C \cap V| < |\Gamma_i(u)|$. Observe that $\Gamma_d(w) \subseteq V$ when $w \in \Gamma_i(u)$, $i \leqq e \leqq d/2$. Hence

$$|C \cap \bigcup_{w \in \Gamma_i(u)} \Gamma_d(w)| \leqq |C \cap V| < |\Gamma_i(u)| .$$

Since $\Gamma_d(w_1) \cap \Gamma_d(w_2) = \emptyset$, when $w_1 \neq w_2 \in \Gamma_i(u)$, we get

$$\sum_{w \in \Gamma_i(u)} |C \cap \Gamma_d(w)| < |\Gamma_i(u)| ,$$

and $C \cap \Gamma_d(w') = \emptyset$ for some $w' \in \Gamma_i(u)$.

Let φ' be an automorphism that satisfy $\varphi'(w')=u$ and $\varphi'(u)=w'$. If $(\gamma_0', \gamma_1', \ldots, \gamma_d')$ is the weight enumerator of $\varphi'(C)$ then $\gamma_i' = 1$ and $\gamma_d' = 0$.

The perfect codes $\varphi(C)$ and $\varphi'(C)$ have the same minimum weight, but their weight enumerators are not equal. Using theorem 2 we see that this is impossible. Consequently $\Gamma_d(u) \setminus C = \emptyset$ if $u \in C$ and the theorem is proved.

In the antipodal distance-transitive graph 2.0_4 (see [5]) it is easy to find a perfect code. 2.0_4 can not be represented as $\mathbf{Z}_2^r$ for any r. So theorem 3 is in fact a generalisation of the theorem of Roos.

In [4] Smith gives an example of an antipodal distance-transitive graph G with intersection-matrix

$$\Gamma(G) = \begin{bmatrix} 0 & 1 & & & & & & & 0 \\ 3 & 0 & 1 & & & & & & \\ & 2 & 0 & 1 & & & & & \\ & & 2 & 0 & 1 & & & & \\ & & & 2 & 0 & 2 & & & \\ & & & & 2 & 0 & 2 & & \\ & & & & & 1 & 0 & 2 & \\ & & & & & & 1 & 0 & 3 \\ 0 & & & & & & & 1 & 0 \end{bmatrix}$$

If $v_0(\lambda), v_1(\lambda), \ldots, v_d(\lambda)$ are defined as in section 1 and $v_0(\lambda) = 1$ it is easy to see that $1 + v_1(\lambda) + v_2(\lambda)$ divides $1 + v_1(\lambda) + \ldots + v_d(\lambda)$ where $d = 8$. This observation was made by Lindström [6].

If there exists a perfect 2-error correcting code C in G then $|C| = 9$. But, using theorem 3 we see that if $u \in C$ then $\Gamma_0(u) \cup \Gamma_8(u) \subseteq C$. The distance between any vertex of G and $\Gamma_0(u) \cup \Gamma_8(u)$ is less or equal to 4 and there can impossibly be any more code vertices of G. Consequently no perfect 2-error correcting code exists in G.

In [4] Smith defines the derived graph G' of the antipodal distance-transitive graph G. The vertices of G' are the sets $\Gamma_0(u) \cup \Gamma_d(u)$, $u \in V(G)$, and there is an edge between the vertices $\Gamma_0(u) \cup \Gamma_d(u)$ and $\Gamma_0(u') \cup \Gamma_d(u')$ of G' iff there are vertices $v \in \Gamma_0(u) \cup \Gamma_d(u)$ and $v' \in \Gamma_0(u') \cup \Gamma_d(u')$ such that $d(v, v') = 1$. Smith then shows that if $d > 2$ for the antipodal distance-transitive graph G, then the derived graph G' is distance-transitive with diameter $[\frac{1}{2}d]$.

We show the following corollary of theorem 3.

Corollary. *If there exists a perfect e-error correcting code in the antipodal distance-transitive graph G then there exists a perfect e-error correcting code in the derived graph G'.*

Proof. Let C be a perfect e-error correcting code of G. Let C' be the vertices of the derived graph G' that satisfy

$$\Gamma_0(u) \cup \Gamma_d(u) \in C' \quad \text{iff} \quad \Gamma_0(u) \cup \Gamma_d(u) \subseteq C .$$

If

$$c_1' = \Gamma_0(c_1) \cup \Gamma_d(c_1) \in C', \quad c_2' = \Gamma_0(c_2) \cup \Gamma_d(c_2) \in C'$$

and $d(c_1', c_2') < 2e+1$ then it is easy to see that there exist vertices $c_1'' \in \Gamma_0(c_1) \cup \Gamma_d(c_1)$, $c_2'' \in \Gamma_0(c_2) \cup \Gamma_d(c_2)$ such that $d(c_1'', c_2'') < 2e+1$. Since C is perfect this is impossible. Using theorem 3 we find that $|C'| = |C|/k_0 + k_d$. Now since $|V(G')| = |V(G)|/k_0 + k_d$ and

$$|\{v \in V(G) \mid d(u,v) \leqq e\}| = |\{v \in V(G') \mid d(u',v) \leqq e\}|$$

for $u \in V(G)$ and $u' \in V(G')$, C' must be a perfect code.

It is well-known that there exists a perfect 3-error correcting code in the antipodal distance-transitive graph $\mathsf{Z}_2^{23} = G$. Consequently there must exist a perfect 3-error correcting code in the derived graph G'. Perhaps this is a code that Biggs [3, p. 296] question for.

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