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BIHARMONIC GREEN'S FUNCTIONS AND QUASIHARMONIC DEGENERACY

C. Y. WANG

On a Riemannian manifold, let γ be the biharmonic Green's function which, roughly speaking, satisfies $\gamma = \Delta\gamma = 0$ on the ideal boundary (Sario [4]). Denote by $O^N{}_\Gamma$ the class of Riemannian N-manifolds on which no γ exists. It is known that the class $O^N{}_G$ of Riemannian N-manifolds which do not admit harmonic Green's functions g satisfies the strict inclusion relation

$$O^N{}_G < O^N{}_\Gamma \tag{1}$$

for every dimension $N \geqq 2$ (Sario [5]). On the other hand, the classes $O^N{}_{QX}$, $X = P,B,D,C$, of Riemannian N-manifolds without quasiharmonic functions q, $\Delta q = 1$, which are positive, bounded, Dirichlet finite, or bounded Dirichlet finite, satisfy

$$O^N{}_G < O^N{}_{QP} < O^N{}_{QB} \cap O^N{}_{QD} < O^N{}_{QB}, O^N{}_{QD} < O^N{}_{QB} \cup O^N{}_{QD} = O^N{}_{QC} \tag{2}$$

for every dimension (Sario [3]). An important question in biharmonic classification theory is to find relations between $O^N{}_\Gamma$ and the $O^N{}_{QX}$. In this paper we shall show that

$$O^N{}_\Gamma < O^N{}_{QP} \tag{3}$$

for every N, so that $O^N{}_\Gamma$ is strictly contained in all classes $O^N{}_{QX}$.

1.

We first establish a useful test for the existence of positive quasiharmonic functions.

Given a Riemannian N-manifold R, take a regular subregion Ω of R. Let $g_\Omega(x,y)$ be the harmonic Green's function on Ω with pole y. The function

$$g(x,y) = \lim_{\Omega \to R} g_\Omega(x,y) ,$$

if it exists, is the harmonic Green's function on R.

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On a Riemannian N-manifold R, fix a regular subregion R_0, and choose regular subregions Ω with $\bar{R}_0 \subset \Omega$. Let ω_Ω be a harmonic function on $\Omega - \bar{R}_0$ and continuous on $\bar{\Omega} - R_0$ such that

$$\omega_\Omega | \partial R_0 = 1 \quad \text{and} \quad \omega_\Omega | \partial \Omega = 0 .$$

The limit $\omega = \lim_{\Omega \to R} \omega_\Omega$ is the harmonic measure of the ideal boundary of R relative to R_0. It is known that $\omega \equiv 1$ if and only if $R \in O^N{}_G$ (e.g. Sario–Nakai [6]).

Consider the equation

$$(4) \qquad \Delta u = f, \quad f \geqq 0, f \not\equiv 0 \text{ on } R$$

with $\Delta = d\delta + \delta d$ the Laplace–Beltrami operator. If

$$(5) \qquad Gf(x) = \int_R g(x,y) f(y)\, dy$$

exists, with dy the volume element, then it is a positive solution of (4). On the other hand, if (4) has a positive solution u, then the Riesz decomposition yields

$$u(x) = h_\Omega(x) + \int_\Omega g_\Omega(x,y) f(y)\, dy$$

on Ω, where $h_\Omega \in H(\Omega) \cap C(\bar{\Omega})$, $h_\Omega | \partial\Omega = u | \partial\Omega$, and H is the family of harmonic functions. Since u is positive superharmonic and $u \geqq h_\Omega$ on Ω, the limit $h = \lim_{\Omega \to R} h_\Omega$ exists and

$$Gf(x) = \lim_{\Omega \to R} \int_\Omega g_\Omega(x,y) f(y)\, dy < \infty .$$

Therefore (4) has a positive solution if and only if $Gf(x) < \infty$ on R.

For a given $x \in R$, choose a regular subregion R_0 containing x. Let

$$(6) \qquad M_x = \max_{y \in \partial R_0} g(x,y), \quad m_x = \min_{y \in \partial R_0} g(x,y) .$$

Then

$$m_x \omega(y) \leqq g(x,y) \leqq M_x \omega(y)$$

for $y \in R - R_0$. Thus

$$(7) \quad m_x \int_{R-R_0} \omega(y) f(y)\, dy \leqq \int_{R-R_0} g(x,y) f(y)\, dy \leqq M_x \int_{R-R_0} \omega(y) f(y)\, dy ,$$

and we have proved for a hyperbolic manifold:

LEMMA. *The equation* (4) *has a positive solution if and only if* $\int_{R-R_0} \omega(y) f(y)\, dy < \infty$ *for some and hence every regular subregion* R_0.

In the special case $f \equiv 1$ on R, we have:

COROLLARY. *The existence of QP-functions on a hyperbolic Riemannian manifold R is equivalent to the finiteness of* $\int_{R-R_0} \omega(y)\, dy$ *for some and hence every regular subregion* R_0.

With the help of the above corollary, the proof of nonexistence of QP-functions on certain Riemannian manifolds is greatly simplified.

2.

Now we are ready to prove our main result:

THEOREM. *The strict inclusion*

$$O^N{}_\Gamma < O^N{}_{QX} \tag{8}$$

is valid for $X=P,B,D,C$, *and every dimension* $N \geqq 2$.

The proof will be given by Nos. 2–3.

Let $\gamma_\Omega(x,y)$ be the biharmonic Green's function on Ω, defined by a biharmonic fundamental singularity $y \in \Omega$ and the boundary conditions $\gamma = \Delta\gamma = 0$ on $\partial\Omega$. In view of

$$\gamma_\Omega(x,y) = \int_\Omega g_\Omega(x,z) g_\Omega(z,y)\, dz ,$$

the biharmonic Green's function, if it exists, on R is

$$\gamma(x,y) = \lim_{\Omega\to R} \gamma_\Omega(x,y) = \int_R g(x,z) g(z,y)\, dz .$$

It is known (loc. cit. [4]) that a hyperbolic $R \in O^N{}_\Gamma$ if and only if $\omega \notin L^2(R-R_0)$ for some R_0. By

$$\int_{R-R_0} g(x,z) g(z,y)\, dz \leqq M \int_{R-R_0} g(x,z)\, dz ,$$

where R_0 is a regular subregion containing y, and $M = \max_{z\in\partial R_0} g(y,z)$, we have $O^N{}_\Gamma \subset O^N{}_{QX}$.

If the volume of R is finite, then

$$\left(\int_{R-R_0} \omega(z)\, dz\right)^2 \leqq \mathrm{Volume}(R-R_0) \int_{R-R_0} \omega^2(z)\, dz .$$

Thus $O^N{}_{QX} \subset O^N{}_\Gamma$ and consequently $O^N{}_{QX} = O^N{}_\Gamma$.

3.

To prove the strictness of $O^N{}_\Gamma \subset O^N{}_{QP}$, let $x, y, \ldots, y_{N-1}$ stand for rectangular coordinates in N-space and consider the N-cylinder

$$T = \{|x| < 1,\ |y_i| \leqq \pi,\ i = 1, 2, \ldots, N-1\} ,$$

where each pair of opposite faces $y_i = \pi$ and $y_i = -\pi$ are identified by a parallel translation orthogonal to the $x-$axis. Endow T with the metric

$$ds = (1-x^2)^{-1}\, ds_E ,$$

where ds_E is the Euclidean metric.

For $h(x) \in H$,

$$\Delta h(x) = -(1-x^2)^N[(1-x^2)^{-(N-2)}h'(x)]' = 0 .$$

If $N=2$, $h(x)=ax+b$. If $N>2$,

$$h'(x) = c(1-x^2)^{N-2} \sim c(1-|x|)^{N-2}$$

as $|x| \to 1$, and

$$h(x) \sim a(1-|x|)^{N-1}+b .$$

This holds, in particular, for the harmonic measure $\omega(x)$ on

$$\{-1<x<-\tfrac{1}{2}\} \cup \{\tfrac{1}{2}<x<1\} ,$$

say. Clearly

$$\omega(x) \sim a(1-|x|)^{N-1}$$

for all $N \geqq 2$. Since

$$\int_{\frac{1}{2}<|x|<1} \omega^2 \, dV \sim c\int_{\frac{1}{2}}^{1} (1-x)^{2(N-1)}(1-x)^{-N}\, dx < \infty ,$$

we have $T \notin O^N{}_\Gamma$. In view of

$$\int_{\frac{1}{2}<|x|<1} \omega \, dV \sim c\int_{\frac{1}{2}}^{1} (1-x)^{N-1}(1-x)^{-N}\, dx = \infty ,$$

we obtain $T \in O^N{}_{QP}$. By (2), the proof is complete.

4.

We like to point out that (8) is not true for all QX-functions, e.g., the class $X=L^p$ of functions whose L^p-norm is finite, $p \geqq 1$. In [1]. it was shown that

$$O^N{}_G \cap O^N{}_{QL^p},\ O^N{}_G \cap \tilde{O}^N{}_{QL^p},\ \tilde{O}^N{}_{QP} \cap O^N{}_{QL^p},\ \tilde{O}^N{}_{QP} \cap \tilde{O}^N{}_{QL^p}$$

are all nonempty, where $\tilde{O}^N$ stands for the complement of O^N with respect to the totality of N-manifolds. By (1) and (8), we see that

(9) $$O^N{}_\Gamma \cap O^N{}_{QL^p},\ O^N{}_\Gamma \cap \tilde{O}^N{}_{QL^p},\ \tilde{O}^N{}_\Gamma \cap O^N{}_{QL^p},\ \tilde{O}^N{}_\Gamma \cap \tilde{O}^N{}_{QL^p}$$

are disjoint and nonempty. Thus, despite the close relationship between the biharmonic Green's function γ and QX-functions, $X=P,B,D,C$, there is no relation whatever between γ and QL^p-functions.

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ARIZONA STATE UNIVERSITY, TEMPE, ARIZONA, U.S.A.
AND
UNIVERSITY OF CALIFORNIA, LOS ANGELES, U.S.A.