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ON THE INTEGRABILITY OF THE DERIVATIVE OF A QUASIREGULAR MAPPING

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1. Introduction.

In [3] F. Gehring showed that the derivative of a quasiconformal mapping $f\colon G \to \mathsf{R}^n$, $n \geqq 2$, is integrable to a power $\alpha = \alpha(K(f),n) > n$ over each compact subset of G. Here we prove using the terminology of [4]

1.1. THEOREM. *Let $f\colon G \to \mathsf{R}^n$, $n \geqq 2$, be quasiregular. Then over each compact subset C of G the derivative of f is L^β-integrable with*

$$\beta = \beta(K(f),n,\tilde{N}(f,C)) > n .$$

Here $\tilde{N}(f,C) = \sup_{x \in C} i(x,f)$.

The above theorem follows for $n=2$ directly from Gehring's result, or originally from [1], since if $f\colon G \to \mathsf{R}^2$ is quasiregular then $f = g \circ h$ where h is quasiconformal and g analytic.

The proof of Theorem 1.1 is based on Gehring's method, especially Lemma 3 of [3], and a new linear dilatation for quasiregular mappings.

N. Meyers has reported to the author that he has proved a corresponding result. However, his method is different and based on the theory of elliptic partial differential equations.

2. Capacity estimates for quasiregular mappings.

Suppose that $f\colon G \to \mathsf{R}^n$ is a non-constant quasiregular mapping. We shall use the following capacity inequalities, see [4],

$$\text{(2.1)} \qquad \operatorname{cap} fE \leqq K_I(f) \operatorname{cap} E$$

where E is any condenser in G and

$$\text{(2.2)} \qquad \operatorname{cap} E \leqq K_0(f) N(f,A) \operatorname{cap} fE$$

if $E = (A,C)$ is a normal condenser in G.

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Let $E_n(t)$ denote the Teichmüller condenser

$$(\mathsf{R}^n \setminus I(-\infty, -t], I[0,1]) ,$$

$t > 0$, in R^n,

$$I[a,b] = \{x \in \mathsf{R}^n : x = te_1, \ a \leqq t \leqq b\} .$$

It is well-known, see e.g. [2], that $\varkappa_n(t) = \operatorname{cap} E_n(t)$ is continuous, strictly decreasing, and

$$\lim_{t\to 0+} \varkappa_n(t) = +\infty, \quad \lim_{t\to\infty} \varkappa_n(t) = 0 .$$

The following lemma contains the important symmetrization method in R^n for obtaining significant lower bounds for the capacities of condensers, see [2], [5].

2.3. LEMMA. *Suppose that $E = (A, C)$ is a condenser in R^n such that* (1) *$\complement A$ is connected and meets $S^{n-1}(x,r)$ for some x and $r > 0$ and* (2) *C is connected, $x \in C$, and C meets $S^{n-1}(x,r')$, $r' > 0$. Then*

$$\operatorname{cap} E \geqq \varkappa_n(r/r') > 0 .$$

3. A new dilatation for quasiregular mappings.

Suppose that $f\colon G \to \mathsf{R}^n$ is non-constant and quasiregular. Let $x \in G$. For $r \in (0, d(x, \partial G))$ set

$$L(x,f,r) = \sup_{|x-y|=r} |f(y) - f(x)| ,$$

$$\tilde{l}(x,f,r) = \sup\{s > 0 : U(x,f,s) \subset B^n(x,r)\} ,$$

and for $r > 0$

$$L^*(x,f,r) = \sup_{y \in \partial U(x,f,r)} |y - x| .$$

We recall that $U(x,f,r)$ denotes the x-component of $f^{-1}B^n(f(x),r)$, see [4, p. 9]. Define $\tilde{H}(x,f,r) = L(x,f,r)/\tilde{l}(x,f,r)$.

3.1. REMARK. In the theory of quasiconformal and quasiregular mappings the linear stretching

$$H(x,f) = \limsup_{r\to 0} H(x,f,r)$$

$$= \limsup_{r\to 0} \frac{L(x,f,r)}{\inf_{|y-x|=r} |f(y) - f(x)|}$$

is much used. It can be easily shown that

$$H(x,f) = \limsup_{r\to 0} \tilde{H}(x,f,r) .$$

However, we are interested in the global properties of $\tilde{H}(x,f,r)$ which are better than those of $H(x,f,r)$.

The following lemma is a slightly modified version of [4, Lemma 4.8].

3.2. LEMMA. *Suppose that* $U(x,f,r)$, $0<r<r_0$, *is a normal domain of* f. *Then the mapping* $r \mapsto L^*(x,f,r)$ *is strictly increasing and continuous from the left for* $0<r\leqq r_0$.

3.3. LEMMA. *If* $\bar{B}^n(x,r)\subset G$, *then* $U(x,f,\tilde{l}(x,f,r))\subset B^n(x,r)$.

PROOF. Let $\tilde{l}=\tilde{l}(x,f,r)$ and $\tilde{l}>\varepsilon>0$. Clearly $U(x,f,\tilde{l}-\varepsilon)\subset B^n(x,r)$, hence by [4, Lemma 2.5], $U(x,f,\tilde{l}-\varepsilon)$ is a normal domain of f and by Lemma 3.2,

$$L^*(x,f,\tilde{l}) = \lim_{\varepsilon\to 0} L^*(x,f,\tilde{l}-\varepsilon) \leqq r .$$

3.4. In the following discussion we fix $x_0\in G$ and pick $r_0>0$ so that for $r\in(0,4r_0]$

(a) $U(x_0,f,r)$ is a normal neighborhood of x_0,
(b) $\complement U(x_0,f,r)$ is connected.

This is possible by [4, pp. 9–11]. Let $U_0=U(x_0,f,r_0)$ and $d_0=d(\partial U(x_0,f,2r_0),U_0)>0$.

3.5. LEMMA. $\tilde{H}(x,f,r)\leqq C$ *for all* $x\in U_0$ *and* $r\in(0,d_0]$. *Here* C *depends only on* $K(f)$, n, *and* $i(x_0,f)$.

PROOF. Fix $x\in U_0$ and $r\in(0,d_0]$. Let $L=L(x,f,r)$ and $\tilde{l}=\tilde{l}(x,f,r)$. Denote $U=U(x,f,L)$. Then $U\subset U(x_0,f,4r_0)$. Suppose that $L>\tilde{l}$ and let $0<\varepsilon<L-\tilde{l}$. Now $U(x,f,\tilde{l}+\varepsilon)$ meets $S^{n-1}(x,r)$. The condenser

$$E = (U,\bar{U}(x,f,\tilde{l}+\varepsilon))=(U,C)$$

is a normal condenser. On the other hand $\complement U$ is connected, for if there exists a bounded component F of $\complement U$, then by (b), $F\subset U(x_0,f,4r_0)$. Now $f\partial F\subset f\partial U=\partial fU=S^{n-1}(f(x),L)$, hence there exists $z\in F$ such that $f(z)\in S^{n-1}(f(x),L)$. Pick a line T passing through $f(x_0)$ and $f(z)$ and let T' be the $f(z)$-component of $T\cap\left(\bar{B}^n(f(x_0),4r_0)\setminus B^n(f(x),L)\right)$.

Now $f^{-1}T'\subset\complement U$, thus the z-component of $f^{-1}T'$, say T_1, is contained in F. Because $U(x_0,f,4r_0)$ is a normal domain, T_1 meets $\partial U(x_0,f,4r_0)$ which is impossible since $F\subset U(x_0,f,4r_0)$. Since C is connected and both

$\complement U$ and C meet $S^{n-1}(x,r)$, Lemma 2.3 yields $\operatorname{cap} E \geqq \theta_n > 0$. The inequality (2.2) implies

$$\theta_n \leqq \operatorname{cap} E \leqq K_0(f) N(f,U) \operatorname{cap} fE$$
$$\leqq K(f) i(x_0,f) \omega_{n-1} \ln(L/(\tilde{l}+\varepsilon))^{1-n} .$$

Thus

$$L/(\tilde{l}+\varepsilon) \leqq C = C(n, K(f), i(x_0,f)) .$$

Letting $\varepsilon \to 0$ we deduce the result.

4. Proof of Theorem 1.1.

We may assume that f is non-constant. By [3, Lemma 3] it suffices to show that each $x_0 \in G$ has a neighborhood U such that

$$(4.1) \qquad m_n(Q)^{-1} \int_Q |f'|^n dm_n \leqq b(m_n(Q)^{-1} \int_Q |f'| dm_n)^n$$

for each cube $Q \subset U$ parallel to the coordinate axis with $b = b(n, K(f), i(x_0,f))$. Fix $x_0 \in G$ and let $r_0 > 0$, U_0, and d_0 be as in 3.4. Set $U = U_0 \cap B^n(x_0, d_0)$.

Let $Q \subset U$. We may assume that

$$Q = \{x \in \mathsf{R}^n : \ |x_i| < s, 1 \leqq i \leqq n\}$$

and $f(0) = 0$. Now $s\sqrt{n} \leqq d_0$. At first we shall show that

$$(4.2) \qquad \tilde{l}(0,f,s\sqrt{n}) \leqq C' \tilde{l}(0,f,s)$$

where $C' = C'(n, K(f), i(x_0,f))$.

Let $L = L(0,f,s\sqrt{n})$ and $U' = U(0,f,L)$. The condenser $E = (U', \bar{B}^n(0,s))$ is a normal condenser in $U(x_0,f,4r_0)$. Clearly $\complement U'$ meets $S^{n-1}(s\sqrt{n})$. As in the proof of Lemma 3.5 it can be shown that $\complement U'$ is connected. Lemma 2.3 and (2.2) applied to the condenser E give

$$\varkappa_n(\sqrt{n}) = \varkappa_n(s\sqrt{n}/s) \leqq \operatorname{cap} E \leqq K_0(f) N(f,U') \operatorname{cap} fE$$
$$\leqq K(f) i(x_0,f) \omega_{n-1} \left(\ln \frac{L(0,f,s\sqrt{n})}{L(0,f,s)}\right)^{1-n} .$$

This implies $L(0,f,s\sqrt{n}) \leqq C'' L(0,f,s)$ which, by Lemma 3.5, yields

$$L(0,f,s\sqrt{n}) \leqq C'' C \tilde{l}(0,f,s) = C' \tilde{l}(0,f,s) .$$

Since $\tilde{l}(0,f,s\sqrt{n}) \leqq L(0,f,s\sqrt{n})$, (4.2) follows.

Let $r \in (0, s/\sqrt{n})$ and

$$Q' = \{x \in \mathsf{R}^n : \ |x_i| \leqq r, \ 1 \leqq i \leqq n\} .$$

Then $E=(Q,Q')$ is a condenser in G and

$$(4.3)\qquad \operatorname{cap} E \leqq \omega_{n-1}\left(\ln \frac{s}{r\sqrt{n}}\right)^{1-n} .$$

Define $r'=\sup_{x\in\partial Q'}|f(x)|$, $L=L(0,f,s\sqrt{n})$, and $\tilde{l}=\tilde{l}(0,f,s)$.

$$(4.4)\qquad \begin{aligned}\operatorname{cap} fE &= \operatorname{cap}(fQ,fQ') \geqq \operatorname{cap}(fB^n(s\sqrt{n}),fQ') \\ &\geqq \operatorname{cap}(B^n(L),fQ') \geqq \varkappa_n(L/r')\end{aligned}$$

where Lemma 2.3 is used in the last step. Combining (4.3), (4.4), and (2.1) we get

$$(4.5)\qquad \varkappa_n(L/r') \leqq \operatorname{cap} fE \leqq K_I(f)\operatorname{cap} E \leqq K(f)\omega_{n-1}\left(\ln \frac{s}{r\sqrt{n}}\right)^{1-n} .$$

Let $\alpha=\alpha(n,K(f),i(x_0,f))=\max(C,C')$ where C is given by Lemma 3.5 and C' by (4.2). Choose r so that the right hand side of (4.5) is $=\varkappa_n(2\alpha^2)$. Then $s=\gamma r$ where $\gamma=\gamma(n,K(f),i(x_0,f))$. Since $\varkappa_n$ is decreasing, (4.5) gives $L/\alpha \geqq 2r'\alpha$ and so

$$\tilde{l}(0,f,s\sqrt{n}) \geqq 2r'\alpha \geqq 2r'C' .$$

Now (4.2) yields

$$(4.6)\qquad \tilde{l} \geqq 2r' .$$

Let $P\colon \mathsf{R}^n \to \mathsf{R}^n$ be the projection $P(x)=x-x_ne_n$. For $y\in PQ'$ let J_y be the closed upper segment of $(Q\setminus Q')\cap P^{-1}(y)$. By Fubini's theorem

$$\int_Q |f'|\,dm_n \geqq \int_{PQ'} dm_{n-1}(y) \int_{J_y} |f'|\,dm_1 ,$$

hence there exists $y\in PQ'$ such that

$$(4.7)\qquad \int_{J_y} |f'|\,dm_1 \leqq m_{n-1}(PQ')^{-1}\int_Q |f'|\,dm_n = (2r)^{-(n-1)}\int_Q |f'|\,dm_n$$

and f is absolutely continuous on J_y. Let $l(fJ_y)$ denote the length of the path $f|J_y$. Observe that $f|J_y$ need not be injective. We claim that

$$(4.8)\qquad \tilde{l}-r' \leqq l(fJ_y) .$$

If not, then $l(fJ_y)<\tilde{l}-r'$ and so $fJ_y\subset B^n(\tilde{l})$. By [4, Lemma 2.6] the components of $f^{-1}fJ_y\cap \overline{U}(0,f,\tilde{l})$ are in $U(0,f,\tilde{l})$ and each of them is mapped onto fJ_y. Now J_y is a part of such a component since $U(0,f,\tilde{l})\supset Q'$ by (4.6). On the other hand, by Lemma 3.3, $U(0,f,\tilde{l})\subset B^n(s)$. This implies $J_y\subset B^n(s)$, a contradiction.

The inequalities (4.6) and (4.8) give

$$\tilde{l} = 2\tilde{l}-\tilde{l} \leqq 2(\tilde{l}-r') \leqq 2l(fJ_y) \leqq 2\int_{J_y}|f'|\,dm_1 \leqq 2(2r)^{-(n-1)}\int_Q |f'|\,dm_n ,$$

where (4.7) is used in the last step. Lemma 3.5, (4.2), and the relation $s=\gamma r$ now yield

$$\begin{aligned} m_n(fQ) &\leqq \Omega_n L^n \leqq \Omega_n C\bar{l}(0,f,s\sqrt{n})^n \leqq \Omega_n C C'^n \bar{l}(0,f,s)^n \\ &\leqq \Omega_n \alpha^{n+1} \bar{l}^n \leqq \Omega_n \alpha^{n+1}\big(2(2r)^{-(n-1)} \textstyle\int_Q |f'|\, dm_n\big)^n \\ &= \frac{q}{K(f) i(x_0,f)} m_n(Q) \left(\frac{1}{m_n(Q)} \int_Q |f'|\, dm_n\right)^n . \end{aligned}$$

Here $q=q(n,K(f),i(x_0,f))$. The integration formula [4, 2.15] and the fact $N(y,f,Q) \leqq i(x_0,f)$ for $y \in fQ$ with the above inequality imply

$$\begin{aligned} \frac{1}{m_n(Q)} \int_Q |f'|^n dm_n &\leqq \frac{K(f)}{m_n(Q)} \int_Q J(x,f)\, dm_n(x) \leqq \frac{K(f)}{m_n(Q)} \int_{fQ} N(y,f,Q)\, dm_n(y) \\ &\leqq \frac{K(f) i(x_0,f)}{m_n(Q)}\, m_n(fQ) \leqq q\left(\frac{1}{m_n(Q)} \int_Q |f'|\, dm_n\right)^n . \end{aligned}$$

This is (4.1). The theorem follows.

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