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ON L_p FOURIER MULTIPLIERS ON A COMPACT LIE-GROUP

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0. Introduction.

This note deals with L_p Fourier multipliers on a compact simply connected semi-simple Lie group G. Let $\hat{G}$ be the dual space of equivalence classes of irreducible finite dimensional representations of G. As is well-known such an equivalence class can be identified with the corresponding highest weight Λ. A function φ on $\hat{G}$ is said to be a L_p Fourier multiplier on G if

$$(0.1) \qquad \|\varphi\|_{m_p} = \sup_{0 \neq f \in L_p} \|\check{\varphi} * f\|_{L_p} / \|f\|_{L_p} < \infty .$$

Here $\check{\varphi}$ denotes the inverse Fourier transform of φ, i.e.

$$\check{\varphi}(g) = \sum_{\Lambda \in \hat{G}} d_\Lambda \varphi(\Lambda) \chi_\Lambda(g)$$

where χ_Λ is the character and $d_\Lambda = \chi_\Lambda(e)$ the dimension of the corresponding representation. We use a method similar to the one in [5], [6]. A crucial role is hereby played by a certain recurrence formula for χ_Λ (lemma 1.1).

Previous results on L_p Fourier multipliers have been obtained by Clerc [1] who bases his proofs on the results of Hörmander [4] concerning the spectral function of a general elliptic partial differential operator. E.g. he proves that the Riesz means of order α, corresponding to

$$\varphi(\Lambda) = (1 - N^{-1} \langle \Lambda + \varrho, \Lambda + \varrho \rangle)_+^\alpha ,$$

are uniformly (in N) bounded in m_p, provided $\alpha > (n-1)|p^{-1} - \frac{1}{2}|$, $n = \dim G$. (We remark that this bound on α is presumably not the best one, as can be inferred from the results of Fefferman [2] in the case of the torus T^n). This and other results of Clerc are contained in ours.

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1. Preliminaries on compact Lie groups.

General references for this section are [3], [7] and [8].

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Let $\mathfrak{g}$ be the Lie algebra of G and denote its complexification by $\mathfrak{g}_{\mathbf{C}}$. Pick up a maximal toroidal subgroup H of G. Let $\mathfrak{h}$ be its Lie algebra and set $l = \dim\mathfrak{h} = \text{rank of } G$. We denote by Δ^+ the subset of the root system Δ formed by the positive roots with respect to some compatible ordering in the dual space of $i\mathfrak{h} \subset \mathfrak{g}_{\mathbf{C}}$. Put

$$D = 2^m \prod_{\alpha\in\Delta^+} \sinh \tfrac{1}{2}\alpha = \sum_{S\in W} \det S\, e^{S\varrho}, \qquad \varrho = \tfrac{1}{2} \sum_{\alpha\in\Delta^+} \alpha$$

where W denotes the Weyl group and m is the cardinal number of Δ^+, $n = 2m + l$. Let Q be a fundamental parallelepiped in $\mathfrak{h}$. Then there is a constant C such that for central functions f holds

$$\int_G f(g)\,dg = C \int_Q f(\exp h) |D(h)|^2\, dh .$$

Using the fundamental weights as a basis, we may identify $\hat{G}$ with the following subset of $\mathbf{Z}^l$

$$\mathbf{Z}_+{}^l = \{(\lambda_1, \ldots, \lambda_l) \in \mathbf{Z}^l ;\ \lambda_j \geqq 0,\ j = 1, \ldots, l\} .$$

The character χ_Λ corresponding to the highest weight Λ is given by Weyl's formula

$$(1.1) \qquad \chi_\Lambda = D^{-1} \sum_{S\in W} \det S\, e^{S(\Lambda+\varrho)}$$

from which the dimension d_Λ of the corresponding representation space may be deduced:

$$d_\Lambda = \frac{\prod_{\alpha\in\Delta^+} \langle \Lambda + \varrho, \alpha\rangle}{\prod_{\alpha\in\Delta^+} \langle \varrho, \alpha\rangle} .$$

Here $\langle \cdot, \cdot \rangle$ denotes the Killing form which makes h to a Euclidian space with norm $|h| = (-\langle h, h\rangle)^{\frac{1}{2}}$.

Defining χ_Λ by (1.1) for any linear form Λ on $\mathfrak{h}$ and setting $d_\Lambda = \chi_\Lambda(e)$ (e = the neutral element of G) we have the recurrence formula for the characters.

LEMMA 1.1.

$$\sum_{\alpha\in\Delta} e^{\alpha} \cdot \chi_\Lambda = \sum_{\alpha\in\Delta} \chi_{\Lambda+\alpha} .$$

PROOF. Let $\Delta_1, \Delta_2, \ldots$ be the orbits of W in Δ. Consider Δ_1 and pick up $\alpha_1 \in \Delta_1$. Let W_1 be the isotropy group of α_1 and let $S' \in \Delta$. Then every $\alpha \in \Delta_1$ can be uniquely expressed as $\alpha = S' \dot{S} \alpha_1$ with $\dot{S} \in W/W_1$. It follows in particular that

$$\sum_{\dot{S}\in W/W_1} e^{S' \dot{S} \alpha_1} = \sum_{\alpha\in\Delta_1} e^{\alpha}$$

is independent of S'. Hence by use of Weyl's formula (1.1)

$$\begin{aligned}\sum_{\alpha\in\Delta_1}\chi_{\Lambda+\alpha} &= \sum_{\dot S\in W/W_1}\chi_{\Lambda+\dot S\alpha_1}\\ &= D^{-1}\sum_{\substack{S'\in W\\ \dot S'\in W/W_1}} \det S'\, e^{S'(\Lambda+\dot S\alpha_1+\varrho)}\\ &= D^{-1}\sum_{S'\in W}\det S'\, e^{S'(\Lambda+\varrho)}\sum_{\dot S\in W/W_1} e^{S'\dot S\alpha_1}\\ &= \chi_\Lambda \sum_{\alpha\in\Delta_1} e^{\alpha}\,.\end{aligned}$$

If $\Delta_1=\Delta$ there is nothing more to prove. Otherwise we write down the same expression for $\Delta_2,\ldots$ and form the sum.

Lemma 1.1 will be used later in the following rewritten form, involving the central function

$$\omega(h) = \sum_{\alpha\in\Delta}(e^{\alpha(h)}-1) = 4\sum_{\alpha\in\Delta+}\sinh^2\tfrac{1}{2}\alpha(h),\qquad h\in\mathfrak{h}\,.$$

COROLLARY 1.1.

$$\omega\chi_\Lambda = \sum_{\alpha\in\Delta+}(\chi_{\Lambda+\alpha}-2\chi_\Lambda+\chi_{\Lambda-\alpha})\,.$$

EXAMPLE. $G=\mathrm{SU}(2)$. Since $l=1$ in this case, $\mathfrak{h}$ may be identified with R and the characters are given on $\mathfrak{h}$ by

$$\chi_\Lambda(h) = \frac{\sin(\Lambda+1)h}{\sin h}\qquad \Lambda\in\mathsf{Z}_+\,.$$

The only positive root is $\alpha(h)=2ih$ so lemma 1.1. reduces to the well-known fact that

$$2\cos 2h\cdot\sin(\Lambda+1)h = \sin(\Lambda+3)h+\sin(\Lambda-1)h\,.$$

2. Estimates of the multiplier norm.

Our starting point will be the basic estimate

$$\|\varphi\|_{m_1} \leqq \|\check{\varphi}\|_{L_1} \tag{2.1}$$

which is an immediate consequence of definition (0.1). To proceed further we introduce the difference operator

$$\Delta^\nu = \Delta_1^{\nu_1}\ldots\Delta_l^{\nu_l}\qquad \nu=(\nu_1\ldots\nu_l)$$

where

$$\Delta_j g(\Lambda) = g(\Lambda+e_j)-g(\Lambda)$$

and e_j is the jth unit vector in Z^l.

We also introduce the spaces l_p and $h_p{}^L$ corresponding to the norms

$$\|g\|_{l_p} = (\sum_{\Lambda \in \mathbf{Z}^l} |g(\Lambda)|^p)^{1/p} ,$$
$$\|g\|_{h_p{}^L} = \mathrm{Max}_{|\nu| = L} \|\Delta^\nu g\|_{l_p} ,$$

and put

$$b_p^{s,q} = (l_p, h_p{}^L)_{s/L,q}, \quad L > s .$$

Concerning the interpolation spaces $(\cdot,\cdot)_{\theta,q}$ see [6].

The L_1 norm of $\check{\varphi}$ may now be estimated in terms of these spaces.

LEMMA 2.1. *Assume that*

$$\varphi(S(\Lambda+\varrho)-\varrho) = \varphi(\Lambda) \tag{2.2}$$

for every $S \in W$ and $\Lambda \in \mathbf{Z}^l$. Then

$$\|\varphi\|_{m_1} \leqq C\|d_\Lambda \varphi(\Lambda)\|_{b_2{}^{n/2,1}}$$

PROOF. Let $W_2{}^L$ be the space defined by the norm

$$\|f\|_{W_2{}^L} = \|\omega^L f\|_{L_2}$$

and let $|||\cdot|||$ denote the norm of the interpolation space

$$(L_2, W_2{}^L)_{n/4L,1}$$

We claim that

$$\|f\|_{L_1} \leqq C|||f|||, \quad 4L > n . \tag{2.3}$$

To prove this we put

$$I_k = \{h \in Q \; ; \; 2^{-k-1} \leqq (\omega(h))^{\frac{1}{2}} \leqq 2^{-k}\}$$

and we recall that

$$D = 2^m \prod_{\alpha \in \Delta^+} \sinh \tfrac{1}{2}\alpha, \quad \omega = 4 \sum_{\alpha \in \Delta^+} \sinh^2 \tfrac{1}{2}\alpha, \quad n = 2m + l .$$

Obviously $|D|^2 \leqq C\omega^m$ and $\mathrm{vol}\, I_k \leqq C2^{-kl}$ so using Schwarz' inequality we get

$$\begin{aligned} \int_{I_k} |f|\,|D|^2 dh &\leqq \Big(\int_{I_k} |\omega^M f|^2 |D|^2 dh\Big)^{\frac{1}{2}} \Big(\int_{I_k} \omega^{-2M} |D|^2 dh\Big)^{\frac{1}{2}} \\ &\leqq \|\omega^M f\|_{L_2} (\sup_{I_k} \omega^{-2M} |D|^2 \mathrm{vol}\, I_k)^{\frac{1}{2}} \\ &\leqq C \cdot \|\omega^M f\|_{L_2} \cdot 2^{-k(n/2-2M)} . \end{aligned}$$

For any decomposition $f = f_0 + f_1$ we apply this to f_0 and f_1 with $M = 0$ and L respectively. Hence

$$\int_{I_k} |f|\,|D|^2 dh \leqq C2^{-kn/2} (\|f_0\|_{L_2} + 2^{2kL} \|f_1\|_{W_2{}^L})$$

or after taking inf over all such decompositions

$$\int_{I_k} |f|\,|D|^2 dh \leqq C 2^{-kn/2} K(2^{2kL})$$

where $K(t) = K(t, f, L_2, W_2{}^L)$ is the K-functional of [6]. Summation now yields

$$\|f\|_{L_1} \leqq C \sum_{k=-\infty}^{\infty} 2^{-kn/2} K(2^{kL}) \leqq C \cdot \int_0^\infty t^{-n/2} K(t^{2L}) t^{-1} dt$$

which proves (2.3).

Next for any $\Lambda \in \mathbf{Z}^l$ (see [8, Nachtrag]), either there is a unique $S \in W$ such that $S(\Lambda+\varrho)-\varrho$ is a highest weight and $\chi_\Lambda = \det S \chi_{S(\Lambda+\varrho)-\varrho}$ or χ_Λ and then also d_Λ is identical zero. In any case we see in view of (2.2) that

$$\int f(g)\chi_\Lambda(g)\,dg = d_\Lambda \varphi(\Lambda)\,.$$

By use of Parseval's formula and corollary 1.1 we now get

$$\|f\|_{L_2} = \left(\sum_{\Lambda \in \mathbf{Z}_+{}^l} |d_\Lambda \varphi(\Lambda)|^2\right)^{\frac{1}{2}} \leqq \|d_\Lambda \varphi(\Lambda)\|_{l_2}$$

$$\|\omega f\|_{L_2} \leqq C \|d_\Lambda \varphi(\Lambda)\|_{h_2{}^2}$$

and by iteration

$$\|\omega^L f\|_{L_2} \leqq C \|d_\Lambda \varphi(\Lambda)\|_{h_2{}^{2L}}\,.$$

Hence in view of (2.1) and (2.3) the proof is completed.

Specializing φ to have compact support in the annulus $a \cdot r \leqq |\Lambda| \leqq b \cdot r$ for sufficiently large r and some constants a and b we may replace lemma 2.1 by

LEMMA 2.2. *Let φ be as in lemma 2.1 and have compact support as above. Then*

$$\|\varphi\|_{m_1} \leqq C r^m \|\varphi\|_{b_2{}^{n/2,1}}\,.$$

PROOF. In accordance with the definition of Δ we define the translation operator τ by

$$\tau_j g(\Lambda) = g(\Lambda + e_j)\,.$$

The following rule corresponding to Leibniz' rule for differentiation is valid

$$\Delta^K g_1 g_2 = \sum_{M \leqq K} C_{KM} \Delta^{K-M} \tau^M g_1 \Delta^M g_2\,.$$

Here Δ^K denotes any difference operator of order K.

Now, as a function of Λ, d_Λ is a polynomial of degree m so we must have

$$|\Delta^{K-M} \tau^M d_\Lambda| \leqq C(1+|\Lambda|)^{m-K+M}\,.$$

Thus since φ has compact support we get

$$(2.4) \qquad \|\Delta^K d_\Lambda \varphi(\Lambda)\|_{l_2} \leqq C \textstyle\sum_{M \leqq K} r^{m-K+M} \|\Delta^M \varphi\|_{l_2} .$$

It is however easy to see that

$$\|\varphi\|_{l_2} \leqq Cr\|\varphi\|_{h_2{}^1}$$

and by iteration, which is possible since any difference of φ has support of the same kind as φ, we obtain

$$\|\Delta^M \varphi\|_{l_2} \leqq Cr^{K-M}\|\varphi\|_{h_2{}^K}$$

and hence by (2.4)

$$\|d_\Lambda \varphi(\Lambda)\|_{h_2{}^K} \leqq Cr^m \|\varphi\|_{h_2{}^K} .$$

Taking $K=0$ and $K=2L$ our final estimate of the multiplier norm follows from lemma 2.1 by interpolation.

Remark. Since $m_1 \subset m_p$, $p \geqq 1$, all these estimates trivially also yields m_p results. Sharper m_p results can however be obtained from the corresponding trivial m_2 case by interpolation with m_1 if $1<p<2$ and by duality if $p>2$. This is used e.g. in the proof of theorem 3.2 and theorem 3.3.

3. Formulation of the results.

The multipliers we shall be concerned with are all of the type

$$\varphi(\Lambda) = \Psi(H(\Lambda+\varrho)/N)$$

where $\Psi(t)$ is defined for $t \geqq 0$ and $H(\xi)$, $\xi \in \mathsf{R}^l$ is a homogeneous function of positive degree infinitely differentiable and positive for $\xi \neq 0$. We also assume that $H(\xi)$ is invariant for the Weyl group i.e. $H(S\xi)=H(\xi)$ for every $S \in W$. Then (2.2) is true.

Three theorems are listed below, each of them containing a condition on Ψ which implies that $\|\varphi\|_{m_p}$ is uniformly bounded in N. The proofs as well as the formulations are almost identical with those of theorems 10.1, 10.2 and 10.3 in [5]. There one can find estimates of the $b_2^{n/2,1}$ norm of suitable decompositions of Ψ which by virtue of lemma 2.2 may be applied directly to our case.

Theorem 3.1. *Let $\Psi(t)$ be infinitely differentiable on $0<t<\infty$ and suppose that for some positive α and β*

$$\begin{aligned} |\Psi(t)-\Psi(0)| &\leqq C_0 t^\alpha, && 0 \leqq t \leqq 1 , \\ |\Psi(t)| &\leqq C_0 t^{-\beta}, && 1 \leqq t < \infty , \\ |D^j\Psi(t)| &\leqq C_j t^{-j} \min(t^\alpha, t^{-\beta}), && 0<t<\infty,\ j=1,2,\dots . \end{aligned}$$

Then

$$\|\Psi(H(\Lambda+\varrho)/N)\|_{m_p} \leqq C .$$

THEOREM 3.2. *Let $\Psi(t)$ be infinitely differentiable on $0 \leqq t < \infty$ and suppose that*

$$|D^j \Psi(t)| \leqq C_j t^{-\beta} \quad \text{for } \beta > n|p^{-1}-\tfrac{1}{2}|,\ j=0,1,2,\ldots .$$

Then

$$\|\Psi(H(\Lambda+\varrho)/N)\|_{m_p} \leqq C .$$

THEOREM 3.3. *Let $\Psi(t)$ be infinitely differentiable on $0 \leqq t < \infty$ except for one point $t_0 > 0$ and suppose that*

$$|D^j \Psi(t)| \leqq C_j |t-t_0|^{\alpha-j}, \quad t \neq t_0,\ j=0,1,2,\ldots ,$$

for some α such that $\alpha > (n-1)|p^{-1}-\frac{1}{2}|$. Assume also that the support is compact on $0 \leqq t < \infty$. Then

$$\|\Psi(H(\Lambda+\varrho)/N)\|_{m_p} \leqq C .$$

REMARK. Only the case when $H(\xi)=|\xi|^2$ is treated in [1] and even in this case the "right" bound is not obtained for example if $\Psi(t)=t^{-\beta}e^{it}$ for large t. Clerc's bound for L_1 is $\beta > [\frac{1}{2}n]+1$ instead of the "right" one $\beta > \frac{1}{2}n$ which is obtained from our theorem 3.2. If $\Psi(t)=(1-t)_+^\alpha$ we obtain the result on Riesz means mentioned in the introduction.

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