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METRIZATION OF QUASI-METRIC SPACES

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0. Introduction.

Let d be a quasi-hemi-metric on a space X. (See Section 2 for definitions!) In this note we prove (Theorem 2.1) that d^p for sufficiently small p is equivalent to a hemi-metric. This gives a kind of generalization of a theorem of Aoki [1] and Rolewicz [10]. (We note that the latter theorem has recently found applications in the theory of interpolation spaces, see Peetre–Sparr [9], Sagher [11].) As an application we give a simplified proof of a metrization theorem of Chittenden [5]. The proof has earlier been simplified by Frink [6] and Aronszajn [2]. Our proof of Theorem 2.1 is based on a more general result (Theorem 1.1) formulated for halfgroupoids. We also give a simple application to capacities.

1. The halfgroupoid case.

DEFINITION 1.1. By a halfgroupoid we mean a set G with a partially defined multiplication which is associative in the following sense: $x(yz)$ is defined iff $(xy)z$ is defined and we have $x(yz)=(xy)z$. By induction $\prod_{j=1}^{n} x_j$ can then unambiguously be defined. (See Bruck [4].)

THEOREM 1.1. *Let f be a realvalued function on a halfgroupoid G and such that*

$$f(xy) \leqq k(f(x)+f(y)), \quad x,y,xy \in G,\ k \text{ constant} > \tfrac{1}{2}\,.$$

For $0<p\leqq \frac{1}{2}\log_{2k}2$ there is a function $\tilde{f}_p$ such that

$$\tilde{f}_p(xy) \leqq \tilde{f}_p(x)+\tilde{f}_p(y), \quad x,y,xy \in G \tag{1.1}$$

and

$$\tilde{f}_p(x) \leqq f^p(x) \leqq 2\tilde{f}_p(x), \quad x \in G\,. \tag{1.2}$$

For the proof we need a lemma.

LEMMA 1.1. *Let f be a function on a halfgroupoid G and such that*

$$f(xy) \leqq C\max(f(x),f(y)), \quad x,y,xy \in G \quad \textit{and} \quad 1 \leqq C \leqq \sqrt{2}\,. \tag{1.3}$$

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If $x=\prod_{j=1}^{n} x_j$, $\sum_{j=1}^{n} 2^{-i_j} \leqq 1$ *and* $i_j \geqq 0$, *we have*

$$f(x) \leqq \max_{1\leqq j\leqq n}(2^{i_j} f(x_j)), \quad x \in G,\ x_j \in G . \tag{1.4}$$

PROOF. The proof is by induction on n. (1.4) is trivially valid for $n=1$. Assume now that (1.4) is valid for $n \leqq m$. Then we want to prove that

$$\prod_{j=1}^{m+1} x_j = x \quad \sum_{j=1}^{m+1} 2^{-i_j} \leqq 1$$

imply

$$f(x) \leqq \max_{1\leqq j\leqq m+1}(2^{i_j} f(x_j)) .$$

There is an integer ν $(1 \leqq \nu \leqq m+1)$ such that

$$\sum_{j=1}^{\nu-1} 2^{-i_j} \leqq \tfrac{1}{2} \quad \text{and} \quad \sum_{j=\nu+1}^{m+1} 2^{-i_j} \leqq \tfrac{1}{2} .$$

Using (1.3) and the induction hypothesis we get

$$\begin{aligned} f(\textstyle\prod_{j=1}^{\nu} x_j) &\leqq C \max(f(\textstyle\prod_{j=1}^{\nu-1} x_j), f(x_\nu)) \\ &\leqq C \max\left(\max_{1\leqq j\leqq \nu-1}(2^{i_j-1} f(x_j)), f(x_\nu)\right) . \end{aligned} \tag{1.5}$$

Using the induction hypothesis again we have

$$f(\textstyle\prod_{j=\nu+1}^{m+1} x_j) \leqq \max_{\nu+1\leqq j\leqq m+1}(2^{i_j-1} f(x_j)) . \tag{1.6}$$

From (1.3), (1.5) and (1.6) we obtain, since $C^2 \leqq 2$,

$$\begin{aligned} f(x) &\leqq C \max(f(\textstyle\prod_{j=1}^{\nu} x_j), f(\textstyle\prod_{j=\nu+1}^{m+1} x_j)) \\ &\leqq C \max\Big(C \max\left(\max_{1\leqq j\leqq \nu-1}(2^{i_j-1} f(x_j)), f(x_\nu)\right), \\ &\qquad\qquad \max_{\nu+1\leqq j\leqq m+1}(2^{i_j-1} f(x_j))\Big) \\ &\leqq \max_{1\leqq j\leqq m+1}(2^{i_j} f(x_j)) . \end{aligned}$$

The proof is complete.

REMARK 1.1. If the multiplication on G is commutative we can prove the lemma for C such that $1 \leqq C \leqq 2$. Indeed, in the commutative case we can find a ν (possibly after a denumeration) such that $\sum_{j=1}^{\nu} 2^{-i_j} \leqq \frac{1}{2}$ and $\sum_{j=\nu+1}^{m+1} 2^{-i_j} \leqq \frac{1}{2}$. (See Peetre–Sparr [9].)

We can now give the proof, which is substantially the one given by Peetre–Sparr [9].

PROOF OF THEOREM 1.1. We have

$$f^p(xy) \leqq k^p(f(x)+f(y))^p \leqq (2k)^p \max(f^p(x), f^p(y)) .$$

$(2k)^p$ will satisfy the restriction on C in Lemma 1.1, if $0 < p \leqq \frac{1}{2}\log_{2k} 2$.

For such p, f^p will thus fulfil (1.3). As $\tilde{f}_p$ we take

$$\tilde{f}_p(x) = \inf\{\textstyle\sum_{j=1}^n f^p(x_j) : x = \prod_{j=1}^n x_j\}.$$

It is clear that $\tilde{f}_p$ satisfies inequality (1.1). Moreover, the left inequality in (1.2) is obvious. For the proof of $f^p(x) \leqq 2\tilde{f}_p(x)$ we choose $i_j \geqq 0$ such that

$$2^{-i_j} \leqq \frac{f^p(x_j)}{\sum_{j=1}^n f^p(x_j)} \leqq 2^{-i_j+1}. \tag{1.7}$$

Using (1.7) and Lemma 1.1 we get

$$f^p(x) \leqq \max_{1\leqq j\leqq n}(2^{i_j} f^p(x_j)) \leqq 2\textstyle\sum_{j=1}^n f^p(x_j).$$

It only remains to take infimum of the right hand side and the proof is complete.

2. The "metric" case.

The purpose of this section is to prove a generalization of a theorem of Aoki [1] and Rolewicz [10]. In the present formulation see also Peetre–Sparr [9].

A function $d: X \times X \to \overline{\mathsf{R}}$, satisfying $d(x,x)=0$, is called *quasi-hemi-metric* if $0 < d(x,y) \leqq \infty$, $x \neq y$, and

(i) $$d(x,z) \leqq k(d(x,y)+d(y,z)), \quad \text{for some } k \geqq 1,$$

and it is called *quasi-metric* if, moreover, $d(x,y)=d(y,x)<\infty$. If we can choose $k=1$, in (i), a quasi-hemi-metric d is called *hemi-metric*, and a quasi-metric d is called *metric*.

Theorem 2.1. *Let X be a set equipped with a quasi-hemi-metric d. For $0<p\leqq \frac{1}{2}\log_{2k} 2$ there is a hemi-metric $\tilde{d}_p$ such that*

$$\tilde{d}_p(x,z) \leqq d^p(x,z) \leqq 2\tilde{d}_p(x,z).$$

Proof. Put $G = X \times X$. For $(x,y) \in G$ and $(y,z) \in G$ we put $(x,y)\circ(y,z) = (x,z)$. With this partial multiplication Theorem 2.1 follows from Theorem 1.1.

Remark 2.1. If d is a quasi-metric, then $\tilde{d}_p$ will be a metric.

Remark 2.2. Since the topology of a locally bounded, topological vector space can be given by a quasi-norm (See Köthe [8, p. 159]), we see that Theorem 2.1 is a generalization of the theorem of Aoki [1] and Rolewicz [10].

3. Application to the metrization theorem.

In this section we give a simplified proof of a metrization theorem of Chittenden [5]. (Cf. Frinck [6], Aronszajn [2], Bourbaki [3, pp. 15–17, 35], Kelley [7, p. 186], Köthe [8, pp. 29–30, 45–47], Weil [12].)

THEOREM 3.1. *A uniform space X is metrizable iff it is Hausdorff and the vicinity filter $\mathscr{F}$ of X has a countable base.*

PROOF. The necessary part is trivial.

For the proof of the sufficient part we take a countable base $(N_i)_{i=1}^{\infty}$ of $\mathscr{F}$. We construct a countable, symmetric base $(U_i)_{i=1}^{\infty}$ of $\mathscr{F}$ such that

$$(3.1)\quad U_{i+1}^2 \subset U_i,\ i=1,2,\dots,\qquad U_{i+1} \subset U_i,\ i=1,2,\dots \text{ and } U_1 \subset N_1 .$$

In fact, let us choose $U_1 = N_1 \cap N_1^{-1}$. Since X is a uniform space there are $M_i \in \mathscr{F}$ such that $M_i^2 \subset U_{i-1}$, $i=2,3,\dots$ (see Köthe [8, p. 29]). Choose inductively

$$U_i = M_i \cap M_i^{-1} \cap N_i \cap N_i^{-1} \cap U_{i-1}, \quad i=2,3,\dots .$$

Then (3.1) is fulfilled and $(U_i)_{i=1}^{\infty}$ is a countable, symmetric base of X. Put

$$d(x,y) = \inf_{(x,y)\in U_i} 2^{-i}$$

and

$$d(x,y) = 1 \quad \text{if } (x,y) \notin \bigcup_{i=1}^{\infty} U_i .$$

Then d will be a quasi-metric on X and $d(x,z) \leqq 2\max(d(x,y),d(y,z))$. Indeed, if say, $d(x,y) = 2^{-k}$ and $d(y,z) = 2^{-l}$ with $k \leqq l$, then $(x,y) \in U_k \subset U_l$ and $(y,z) \in U_l$. Thus $(x,z) \in U_l^2 \subset U_{l-1}$. Consequently

$$d(x,z) \leqq 2^{-l+1} \leqq 2\cdot 2^{-k} \leqq 2\max(d(x,y),d(y,z)) .$$

It is clear that the sets $d^{-1}(0,t)$, $t>0$ give the same uniform structure as the U_i. Nothing will be changed if d is replaced by d^p. But by Theorem 2.1 d^p is equivalent to a metric.

4. Application to capacities.

The following application may be of some interest.

DEFINITION 4.1. A set function f defined on a set U and such that

$$f(A \cup B) \leqq k(f(A)+f(B))$$

is called an (outer) quasi-capacity if $k>1$ and an (outer) capacity if $k \leqq 1$.

THEOREM 4.1. *Let f be an (outer) quasi-capacity on U. For $0<p\leqq \log_{2k}2$ there is an (outer) capacity $\tilde{f}_p$ on U such that*

$$\tilde{f}_p(A) \leqq f^p(A) \leqq 2\tilde{f}_p(A) .$$

PROOF. We define a multiplication $\circ$ on U by $A\circ B=A\cup B$. Then the assertion follows from Theorem 1.1 and Remark 1.1.

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