MATH. SCAND. 35 (1974), 61–104

GENERAL TAUBERIAN REMAINDER THEOREMS

SONJA LYTTKENS

1. Introduction.

Let Φ be real-valued, bounded and measurable on the real axis and $F \in L^1(-\infty,\infty)$. We introduce the Fourier transform, $\hat{F}$, of F

$$\hat{F}(\xi) = \int_{-\infty}^{\infty} e^{-i\xi x} F(x)\,dx$$

and the convolution

$$\Phi * F(x) = \int_{-\infty}^{\infty} \Phi(x-y)F(y)\,dy\,. \tag{1.1}$$

These notations are used throughout the sequel.

In 1932 N. Wiener [13] proved a general Tauberian theorem. This theorem included most earlier Tauberian theorems, which were stated for special kernels, F. In a form, convenient for our purposes, it can be stated as follows (cf. Achieser [1], p. 393).

Wiener's theorem. *Let Φ be bounded on $(-\infty,\infty)$ and $F \in L^1(-\infty,\infty)$ and suppose that*

$$\Phi * F(x) \to 0, \quad x \to \infty\,. \tag{1.2}$$

If

$$\hat{F}(\xi) \neq 0, \quad -\infty < \xi < \infty \tag{1.3}$$

and if Φ satisfies the Tauberian condition

$$\lim_{h\to\infty} \overline{\lim}_{x\to\infty} \sup_{0<y\leq h} \{\Phi(x) - \Phi(x+y)\} \leqq 0 \tag{1.4}$$

then

$$\Phi(x) \to 0, \quad x \to \infty\,.$$

Wiener's theorem applies to a large class of kernels. Such Tauberian theorems are called *general* and a Tauberian theorem which applies to a specified kernel is called *special*.

Wiener's theorem is a pure Tauberian theorem since it yields the estimate $\Phi(x) \to 0, x \to \infty$ and nothing more.

Received April 17, 1973.

A Tauberian *remainder* theorem on the other hand investigates the order of magnitude of $\Phi(x)$ as $x \to \infty$ provided that the order of magnitude of $\Phi * F(x)$ as $x \to \infty$ is known. Thus we suppose that

$$\Phi * F(x) = O(\varrho(x)), \quad x \to \infty \tag{1.5}$$

where $\varrho(x) \searrow 0, x \to \infty$, and we look for a more refined estimate of Φ than $\Phi(x) \to 0, x \to \infty$. In order to obtain such a result it is necessary to strengthen the Tauberian condition and also to restrict the class of kernels considered.

The classical Tauberian condition used in this connection is the following

$$\Phi(x) + Kx \nearrow, \quad x > x_0, \text{ for some constant } K. \tag{1.6}$$

As regards the kernels considered, the Wiener condition (1.3) is mostly strengthened to the analyticity of $1/\hat{F}$ in a strip around the real axis, together with a restriction of the order of magnitude of $1/\hat{F}$ in this strip.

The first general Tauberian remainder theorem was stated by Beurling [2] in 1938. In this theorem $1/\hat{F}$ is dominated by a polynomial in a strip around the real axis. The theorem can be restated as follows.

Beurling's theorem. A. *Let $F \in L^1(-\infty, \infty)$ and $g(\xi) = 1/\hat{F}(\xi)$, $-\infty < \xi < \infty$. Let $\zeta = \xi + i\eta$ and let a, γ and δ be positive constants. Suppose that*

$$g(\zeta) \text{ is analytic in } -\delta < \eta < \gamma \tag{1.7}$$

and

$$|g(\zeta)| < \text{const}(1 + |\xi|)^a, \quad -\delta < \eta < \gamma . \tag{1.8}$$

Let Φ be bounded and satisfy the Tauberian condition (1.6). *If* (1.5) *holds true with $\varrho(x) = e^{-\alpha x}$ for some $\alpha, 0 < \alpha < \gamma$, then*

$$\Phi(x) = O(\varrho(x)^\lambda), \quad x \to \infty \tag{1.9}$$

where $\lambda = 2/(2a+3)$.

B. *Impose the conditions of part* A *and suppose further that*

$$|g'(\zeta)| < \text{const}(1 + |\xi|)^{a-1}, \quad -\delta < \eta < \gamma .$$

If $a > \frac{1}{2}$ then (1.9) *holds true with $\lambda = 1/(a+1)$.*

The conditions on the kernel F can be weakened in the following way. In part B the strip $-\delta < \eta < \gamma$ can be replaced by $0 < \eta < \gamma$ if we interpret $1/\hat{F}(\xi) = g(\xi), -\infty < \xi < \infty$ as

$$1/\hat{F}(\xi) = \lim_{\eta \to 0+} g(\xi + i\eta), \quad -\infty < \xi < \infty .$$

Also the condition $a > \frac{1}{2}$ can be replaced by $a > 0$ (Lyttkens [8, Theorem 3, p. 581], cf. Ganelius [5, Theorem 1, p. 6]).

The class of kernels to which Beurling's theorem applies is very restricted compared to Wiener's theorem. Yet the conditions on the kernel are relevant in the following sense. If F satisfies a slight additional condition namely $x^{\beta}F(x) \in L^1(-\infty, \infty)$ for some $\beta > \frac{1}{2}$ then the analyticity of $1/\hat{F}$ in a strip above the real axis is necessary in Beurling's theorem (Lyttkens [9, Theorem 6, p. 347]). It was also proved by example that the estimate (1.9) in part B is best possible in the sense that it cannot be replaced by

$$\Phi(x) = O(\delta(x)\varrho(x)^{\lambda}), \quad x \to \infty ,$$

for any function δ such that $\delta(x) \to 0, x \to \infty$ (see [9, p. 348]). Later, Ganelius presented an example which proved that the value $2/(2a+3)$ of the constant λ in part A cannot be improved (see [7, Example 3, p. 45]).

The 'remainder' ϱ in Beurling's theorem is of the form $\varrho(x) = e^{-\alpha x}$, $\alpha < \gamma$. This can be replaced by the weaker condition that $\varrho(x) \geqq e^{-\alpha x}, \alpha < \gamma$ and $\varrho = 1/r$ where r is submultiplicative, i.e.

$$\varrho(x) = 1/r(x), \quad r(x+y) \leqq r(x)r(y), \quad -\infty < x < \infty, \ -\infty < y < \infty \tag{1.10}$$

(Lyttkens [9, Theorem 1 p. 317], cf. Frennemo [3]).

Ganelius studied the corresponding Tauberian remainder problem when $1/\hat{F}$ is dominated by an exponential instead of a polynomial. He proved that if the condition (1.8) in part A of Beurling's theorem is replaced by

$$|g(\zeta)| < \text{const}\, e^{b|\xi|}, \quad -\delta < \eta < \gamma ,$$

then $\Phi * F(x) = O(e^{-\alpha x}), x \to \infty$, for some $\alpha > 0$ implies that

$$\Phi(x) = O(x^{-1}), \quad x \to \infty . \tag{1.11}$$

In [4] Ganelius proved this theorem by using a special Tauberian remainder theorem for the Laplace transform and in [5] he proved it by using a general method for Tauberian remainder theorems. Again the result is best possible in the sense that (1.11) cannot be replaced by $\Phi(x) = O(\delta(x)x^{-1}), x \to \infty$ for any function δ such that $\delta(x) \to 0, x \to \infty$ (see [7, Example 1, p. 39]).

Later Frennemo [3] used Ganelius' method to investigate the case when (1.8) is replaced by

$$|g(\zeta)| \leqq M(|\xi|), \quad -\delta < \eta < \gamma \tag{1.12}$$

and $M(\xi), \xi \geqq 0$ is a submultiplicative increasing function. He also studied the case when it also holds true that $|g'(\zeta)| \leqq M_1(|\xi|)$ in the strip and $M_1(\xi), \xi \geqq 0$, is submultiplicative and increasing.

In his lecture notes of 1971 Ganelius [7] further developed his methods, generalized his results and proved the corresponding theorem when (1.12) is valid and $\xi^{-2}\log M(\xi), \xi \geqq 1$, is bounded. He also proved that the estimates obtained were best possible in many cases.

In her thesis Strube [10] proved a Tauberian remainder theorem with conditions on $1/\hat{F}$ and its first and second derivatives on the real axis only.

Apart from the last mentioned paper of Strube the methods hitherto used lean heavily on the theory of analytic functions and the possibility of moving lines of integration in the plane. The method presented in this paper is quite different and uses no theory of analytic functions but is a pure Fourier method. It is a general method which partly bridges the gap between the Wiener condition (1.3) and the above-mentioned analyticity conditions on $1/\hat{F}$. Thus the theorems obtained can be applied when $1/\hat{F}$ and its derivative are locally in L^2 on the real line as well as when $1/\hat{F}$ is analytic in a strip $-\gamma < \eta < \gamma$.

In spite of their generality the theorems yield sharp results when applied to the above-mentioned theorems of Beurling and Ganelius. They also include the case when (1.12) is valid and $\xi^{-2}\log M(\xi) \to \infty$, $\xi \to \infty$.

I impose conditions on $\hat{F}$ on the real axis only of the following type. Let $g(\xi) = 1/\hat{F}(\xi), -\infty < \xi < \infty$, and suppose

$$(1.13) \qquad |g^{(n)}(\xi)| \leqq P_n M(|\xi|), \qquad -\infty < \xi < \infty, n = 0, 1, 2, \ldots$$

where $M(\xi)$, $\xi \geqq 0$, is an arbitrary increasing function and $P = (P_n)_0^\infty$ is a logarithmically convex sequence, $P_0 = 1$, such that $n!/P_n$ is submultiplicative. If (1.13) holds true with $P_n = n!\gamma^{-n}$, then g is analytic in $-\gamma < \eta < \gamma$ and, in every closed strip inside $-\gamma < \eta < \gamma$, $|g(\zeta)| \leqq \text{const}\, M(|\xi|)$.

In Beurling's theorem the constant λ in (1.9) depends on the majorant of $1/\hat{F}$ in the strip, and the class R_γ, of functions ϱ for which (1.5) implies (1.9), depends on how large the strip is. The class R_γ is, apart from some regularity condition of type (1.10), determined by the condition $\varrho(x) \geqq e^{-\theta\gamma x}$ for some $\theta < 1$.

Under the condition (1.13) the situation is similar. Let Φ satisfy the Tauberian condition (1.6). It will be proved below that it then follows from (1.13) that there exists a function m, depending only on the majorant M in (1.13), and a class R_P, depending on the sequence (P_n) such that (1.5) implies

$$(1.14) \qquad \Phi(x) = O(m(\varrho(x))), \quad x \to \infty$$

for every $\varrho \in R_P$. The function

$$h_P(x) = \sum_{n=0}^{\infty} x^n/P_n \tag{1.15}$$

will play the same role as $e^{\gamma x}$ in Beurling's theorem, i.e. the result (1.14) is, in the general case, valid only if $\varrho(x) \geqq 1/h_P(\theta x)$ for some $\theta < 1$. (In certain special cases it is in fact possible to obtain the result (1.14) for some $\theta > 1$, see Example 5 in Section 5.)

It should be mentioned that the condition (1.13), imposed here for the sake of simplicity, can be considerably weakened (see 2.6).

I use a slightly more general regularity condition on the remainder ϱ introduced in (1.5) than the condition (1.10). Thus I suppose that $\varrho\searrow$ and that, for some constants θ and b, $0<\theta<1$, $b \geqq 1$,

$$\varrho(x-y) \leqq b\varrho(x)h_P(\theta y), \quad y \geqq 0, -\infty < x < \infty . \tag{1.16}$$

If $h_P(x) = e^{\gamma x}$ the use of the condition (1.16) instead of (1.10) has the advantage of including remainders ϱ of the form $\varrho(x) = k(x)e^{-\alpha x}$ where $0<\alpha<\gamma$, $k(x)\nearrow\infty$, $x\to\infty$ and $x^{-1}\log k(x) \to 0, x\to\infty$, and yields sharper results for such remainders in case the function M in (1.13) does not increase too fast, for instance when M is of polynomial type. A function ϱ of the form $\varrho(x) = k(x)e^{-\alpha x}$ cannot satisfy (1.10), since for a submultiplicative r the condition $r(x) = o(e^{\alpha x})$, $x\to\infty$ is equivalent to $r(x) = O(e^{\beta x})$, $x\to\infty$ for some $\beta<\alpha$.

It should be mentioned that if the function h_P, introduced in (1.15), increases very slowly then the condition (1.16) is not quite sufficient but must be replaced by

$$\varrho(x-y) \leqq \varrho(x)v(y), \quad y \geqq 0, -\infty < x < \infty$$

where $v(y)/h_P(\theta y)$ is a bounded function in $L^2(0,\infty)$.

I use the following Tauberian condition

$$\Phi(x) - \Phi(x+y) \leqq \sigma(x), \quad 0<y\leqq t(x), x \geqq x_0 , \tag{1.17}$$

where $\sigma\searrow$, $\sigma \geqq 0$ and $t\searrow, t>0$. Such a condition is quite natural with the methods used below since t and σ will appear in different connections in the estimates. It can easily be adopted to different conditions on Φ. If Φ satisfies the Tauberian condition (1.6) then (1.17) holds true with $\sigma = Kt$ for any positive function t. If $\Phi\nearrow$ then (1.17) holds true with $\sigma \equiv 0$ and $t \equiv t_0$ for any positive constant t_0.

The methods used below are a refinement of the following standard procedure. Let H be an auxiliary function in $L^1(-\infty,\infty)$ such that $\hat{H}$ has compact support and let

$$H_t(x) = t^{-1}H(xt^{-1}), \quad 0<t\leqq 1 .$$

If $\hat{F}(\xi) \neq 0, -\infty < \xi < \infty$, then $\hat{H}_t/\hat{F}$ is the Fourier transform of a function $U_t \in L^1(-\infty, \infty)$. Let $\psi = \Phi * F$. Then $\Phi * H_t = \psi * U_t$, the inversion being justified by absolute convergence. From $|\psi(x)| \leqq \varrho(x)$ and the condition (1.13) I estimate $\psi * U_t(x)$. This is straightforward if (P_n) is non-quasi-analytic, i.e. $\sum_{n=0}^{\infty} P_n{}^{-1/n}$ converges, and H is appropriately chosen. When (P_n) is quasi-analytic, however, complications arise. It is then impossible to get an estimate of the type

$$\psi * U_t(x) = O\big((h_P(\theta x))^{-\lambda}\big), \quad x \to \infty$$

if $\hat{U}_t$ has compact support and t is fixed. I therefore introduce a variable support on the Fourier transform of the auxiliary function H_t in the following way. Let $t(x)\searrow$ and choose $t = t(x)$ in $\psi * U_t(x)$. The support of $\hat{H}_{t(x)}$ and hence of $\hat{U}_{t(x)}$ then increases as x increases. I impose a condition on $t(x)$ of the type

$$t(x) = O\big(1/\varphi(x)\big), \quad x \to \infty, \tag{1.18}$$

where φ is a function determined by the sequence (P_n) such that $\varphi\nearrow$ and $\varphi(x) = o\big((\log x)^{1+\delta}\big), x \to \infty$ for every $\delta > 0$. It is then possible to derive an estimate of $\psi * U_{t(x)}(x)$ which yields sharp results in the Tauberian theorems. In fact, if M does not increase too fast and $\psi(x) = O\big(1/h_P(\theta x)\big)$, $x \to \infty$, then t may be choosen so that

$$\psi * U_{t(x)}(x) = O\big((h_P(\theta x))^{-\lambda}\big), \quad x \to \infty$$

for every λ, $0 < \lambda < 1$.

In Section 2 the method described above is used to estimate $\psi * U_{t(x)}(x)$.

Let

$$I(x) = \int_{-\infty}^{\infty} \Phi(x - ut(x))H(u)\,du\,. \tag{1.19}$$

By choosing the function H in (1.19) appropriately I then, in Section 3, derive an estimate of $\Phi(x)$ as $x \to \infty$ from the order of magnitude of $I(x)$ as $x \to \infty$ and the Tauberian condition (1.17).

The Tauberian theorems are obtained by using the identity

$$I(x) = \psi * U_{t(x)}(x)$$

and the results in Sections 2 and 3. They are collected in Section 4 and in 4.2 they are stated for the Tauberian condition (1.17). In 4.3 they are specialized for the classical Tauberian condition (1.6). The restriction (1.18) on $t(x)$, introduced when (P_n) is quasi-analytic, then transforms to a condition on ϱ. It turns out, however, that this restriction is not a serious one, since it disappears if some slight regularizing conditions are imposed on the sequence (P_n) and the functions M and ϱ.

In Section 5 some applications are given. From Theorem 3 in Section 4, earlier results under analyticity conditions on $1/\hat{F}$ in a strip $-\gamma<\eta<\gamma$ can be deduced and even extended to any order of magnitude of $1/\hat{F}$ in the strip (cf. 5.3). Also, results can be derived when $1/\hat{F}$ is analytic not in a strip but in a domain which tapers off at infinity (see 5.4). Theorem 3_0 yields a sharper form of Strube's Tauberian theorem as a special case (cf. 5.2).

For the sake of formal simplicity I use a convolution of the form (1.1) and suppose Φ real. It is easy to see that the same results are valid if $F(y)dy$ is replaced by $d\mu$, where μ is of bounded variation, or if Φ is complex-valued and the Tauberian condition is imposed on the real and imaginary part of Φ.

A mimeographed version in Swedish of this paper has been in circulation since 1967.

2. Estimation of $I(x)$.

2.1. *Preliminaries.*

All functions are assumed to be measurable. I use the notations

$$M_s\{f;a,b\} = \left(\int_a^b |f(x)|^s dx\right)^{1/s}$$

and

$$\|f\|_s = M_s\{f;-\infty,\infty\}\ .$$

C denotes a constant, not necessarily the same one each time it appears.

Let $P=(P_n)_0^\infty$ be a logarithmically convex sequence, $P_0=1$. I do not exclude the case $P_n=\infty, n>m$, but always suppose P_1 to be finite. Let $n!/P_n$ be submultiplicative, i.e.

$$(2.1.1)\qquad \binom{n}{j}P_jP_{n-j} \leqq P_n,\quad j = 0,1,2,\ldots,n,\ n=1,2,\ldots\ .$$

Note that if $(P_n/n!)^{1/n}\nearrow, n\geqq 1$, then (2.1.1) is satisfied, and that (2.1.1) implies that $P_n^{1/n}\geqq ne^{-1}P_1, n\geqq 2$. Let

$$(2.1.2)\qquad p(x) = \sup_n x^n/P_n,\quad x\geqq 0$$

and

$$(2.1.3)\qquad h_P(x) = \sum_{n=0}^\infty x^n/P_n,\quad x\geqq 0\ .$$

If $r\nearrow$, I introduce the function

$$(2.1.4)\qquad \chi_r(x) = \frac{xD^+r(x)}{r(x)}.$$

The following results are well-known or easily derived from the above definitions. $\log p(x)$ is a convex function of $\log x$ and $\chi_p \nearrow, \chi_p(x) = O(x)$, $x \to \infty$. Furthermore,

$$P_n = \sup_{x \geqq 0} x^n/p(x), \quad n = 0, 1, 2, \ldots . \tag{2.1.5}$$

Let $\alpha_0 = 0$, $\alpha_n = P_n/P_{n-1}, n = 1, 2, \ldots$. Then $\alpha_n \nearrow$,

$$p(x) = x^n/P_n; \quad \alpha_n \leqq x \leqq \alpha_{n+1}, \tag{2.1.6}$$

and

$$\chi_p(x) = n, \quad \alpha_n \leqq x < \alpha_{n+1}. \tag{2.1.7}$$

If $P_n = \infty, n > m$, then χ_p is bounded and h_P is a polynomial of degree $\leqq m$. If P_n is finite for all n, then $p(x)$ grows more rapidly than every power of x and $\chi_p(x) \to \infty, x \to \infty$.
Furthermore,

$$h_P(x) \leqq e, \quad 0 \leqq x \leqq \alpha_1 . \tag{2.1.8}$$

By a well-known formula (see [12, Theorem 11, p. 32]),

$$h_P(x) \leqq p(x)[1 + 2\chi_p(x + x/\chi_p(x))], \quad x \geqq 0, \tag{2.1.9}$$

and it follows that

$$h_P(x) \leqq C(P)(1+x)p(x), \quad x \geqq 0 . \tag{2.1.10}$$

If χ_p is bounded, then $h_P(x) \leqq C(P)p(x), x \geqq 0$, and if $\chi_p(x) \to \infty, x \to \infty$, it follows from (2.1.9) that for every $\delta > 0$

$$h_P(x) = o\big(p((1+\delta)x)\big), \quad x \to \infty . \tag{2.1.11}$$

Hence for every $\delta > 0$ we have

$$h_P(x) \leqq C(P,\delta)p((1+\delta)x), \quad x \geqq 0 . \tag{2.1.12}$$

When (P_n) is quasi-analytic, I need a non-quasi-analytic sequence (Q_n), majorizing (P_n). First, I introduce the function $\varphi(x)$, $x \geqq 0$, as follows. If (P_n) is non-quasi-analytic, let $\varphi \equiv 1$. If (P_n) is quasi-analytic, let us choose φ such that

$$\begin{cases} \varphi(x) = 1, 0 \leqq x \leqq 1, \quad \varphi \nearrow, \quad x \log \varphi(x) \text{ is convex,} \\ \sum_{n=1}^{\infty} \varphi(n)^{-1} P_n^{-1/n} < \infty, \quad \overline{\lim}_{x\to\infty} \chi_\varphi(x) \log x \leqq 1 . \end{cases} \tag{2.1.13}$$

Note that these conditions imply that for every $\delta > 0$

$$\varphi(x) = o\big((\log x)^{1+\delta}\big), \quad x \to \infty \tag{2.1.14}$$

and for every $\delta>0$, $a>1$ and $m \geqq 0$ there exists x_1 such that

$$\sup_{m+1 \leqq y \leqq x^{1+\delta}} \varphi(y)^{y/(y-m)} \leqq a(1+\delta)\varphi(x), \quad x \geqq x_1 . \tag{2.1.15}$$

As an example of a function φ satisfying the above conditions for all sequences (P_n) introduced above, let $x_0 = \exp(e)$ and let us choose

$$\varphi(x) = e^{-1}(\log(x+x_0-1))(\log\log(x+x_0-1))^2, \quad x \geqq 1 . \tag{2.1.16}$$

I introduce the sequence $Q = (Q_n)_0^\infty$ in the following way.

$$Q_n = P_n(\varphi(n))^n, \quad n = 0,1,2,\dots . \tag{2.1.17}$$

Then $Q_n \geqq P_n, n=0,1,2,\dots,(Q_n)$ is non-quasi-analytic and logarithmically convex, and $n!/Q_n$ is submultiplicative. I introduce q and h_Q in the same way as p and h_P, i.e. I put

$$\begin{aligned} q(x) &= \sup_n x^n/Q_n, \quad x \geqq 0 , \\ h_Q(x) &= \textstyle\sum_0^\infty x^n/Q_n, \quad x \geqq 0 . \end{aligned} \tag{2.1.18}$$

Then

$$Q_n = \sup_{x \geqq 0} x^n/q(x) . \tag{2.1.19}$$

The following results are easily verified.

$$p(x) \leqq q\big(x\varphi(\chi_p(x))\big), \quad x \geqq 0 . \tag{2.1.20}$$

If $\chi_p(x) \to \infty, x \to \infty$, then for every $\varepsilon > 0$

$$h_P(x) = o\big(q((1+\varepsilon)x\varphi(x))\big), \quad x \to \infty , \tag{2.1.21}$$

and for every $a \geqq 0$, and every $\alpha, 0<\alpha<1$,

$$\frac{x^a h_P(x^\alpha)}{q(\frac{1}{2}x)} \in L^2(0,\infty) . \tag{2.1.22}$$

Let us further suppose that

$$\log(P_{n+1}/P_n) = o(n), \quad n \to \infty , \tag{2.1.23}$$

which is equivalent to $\chi_p(x)/\log x \to \infty, x \to \infty$. Then for every $a \geqq 0$ and $0<\theta_1<\theta_2$ it holds true that

$$\frac{x^a h_P(\theta_1 x)}{p(\theta_2 x)} \in L^2(0,\infty) . \tag{2.1.24}$$

$S_n(X), X \geqq 0, n=0,1,2,\dots$ are monotonically increasing functions such that, for some positive integer μ,

$$S_{j-1}(X) \leqq \tfrac{3}{2}XS_j(X), \quad X \geqq X_0, j=1,2,\dots \mu . \tag{2.1.25}$$

Introduce the functions

$$\bar{S}_{m,k} = \sup_{m \leqq n \leqq k} S_n, \quad \bar{S}_m = \sup_{n \geqq m} S_n \tag{2.1.26}$$

and

$$S = (S_0 S_1)^{\frac{1}{2}} . \tag{2.1.27}$$

Let f be a function defined on the real axis and let $X \geqq 1$. We shall study a condition of the type

$$M_2\{f^{(n)}; -X, X\} \leqq P_n S_n(X), \quad n = 0, 1, 2, \ldots . \tag{2.1.28}$$

If (2.1.28) is valid for $n = 0, 1, 2, \ldots, k$, only we define $P_n = \infty$, $n > k$ and $S_n = S_k$, $n > k$.

Let

$$t = X^{-1} .$$

These notations are used throughout the rest of the work.

2.2. *A lemma concerning* U_t.

Let us choose the auxiliary function H such that $q(2|x|)H(x) \in L^1(-\infty, \infty)$ and $\hat{H}(\xi) = 0, |\xi| \geqq 1$. Let, for $0 < t \leqq 1, H_t(x) = t^{-1}H(xt^{-1})$. Then $\hat{H}_t(\xi) = \hat{H}(t\xi)$. Let f satisfy (2.1.28) and let

$$\hat{U}_t = \hat{H}_t f . \tag{2.2.1}$$

Then $\hat{U}_t$ is the Fourier transform of a function $U_t \in L^1(-\infty, \infty)$. First, we shall derive some estimates of $\hat{U}_t$ and its derivatives.

By Leibniz's rule

$$\hat{U}_t^{(n)}(\xi) = \sum_{j=0}^{n} \binom{n}{j} t^j \hat{H}^{(j)}(t\xi) f^{(n-j)}(\xi), \quad n = 0, 1, 2, \ldots . \tag{2.2.2}$$

If we use Minkowski's inequality and observe that $\hat{H}^{(j)}(t\xi)$ vanishes for $|\xi| \geqq X = t^{-1}$ and $\|\hat{H}^{(j)}\|_\infty \leqq \|x^j H(x)\|_1$ we obtain from (2.2.2) and (2.1.28) that

$$\|\hat{U}_t^{(n)}\|_2 \leqq \sum_{j=0}^{n} \binom{n}{j} t^j \|x^j H(x)\|_1 P_{n-j} S_{n-j}(X) .$$

Hence, by using (2.1.25) and the notations introduced in (2.1.26),

$$\|\hat{U}_t^{(n)}\|_2 \leqq \bar{S}_{1,n}(X)[\tfrac{3}{2} t^{n-1} \|x^n H(x)\|_1 + \sum_{j=0}^{n-1} \binom{n}{j} t^j \|x^j H(x)\|_1 P_{n-j}], \quad n \geqq 1 . \tag{2.2.3}$$

By (2.1.19) $\sup_x |x|^n / q(2|x|) = 2^{-n} Q_n$, $n = 0, 1, 2, \ldots$, and it follows that

$$\|x^j H(x)\|_1 \leqq 2^{-j} Q_j \|q(2|x|) H(x)\|_1 . \tag{2.2.4}$$

Let

$$M = \|q(2|x|) H(x)\|_1 . \tag{2.2.5}$$

By inserting (2.2.4) in (2.2.3) and using (2.2.5) we get, if $n \geqq 1$,

$$(2.2.6) \quad \|\hat{U}_t^{(n)}\|_2 \leqq M\,\bar{S}_{1,n}(X)[\tfrac{3}{2}Q_n t^{n-1}2^{-n} + \textstyle\sum_{j=0}^{n-1}\binom{n}{j}2^{-j}P_{n-j}t^jQ_j]\,.$$

If $n=1$, we find, since $P_1=Q_1$ and $t \leqq 1$, that

$$(2.2.7) \qquad \|\hat{U}_t'\|_2 \leqq M\,2P_1S_1(X)\,.$$

By the Carlson-Beurling's inequality $\|U_t\|_1 \leqq (\|\hat{U}_t\|_2\|\hat{U}_t'\|_2)^{\frac{1}{2}}$ and (2.2.7) yields

$$(2.2.8) \qquad \|U_t\|_1 \leqq M(2P_1)^{\frac{1}{2}}S(X)\,.$$

In order to estimate the sum in (2.2.6), when $n>1$, we now introduce a condition on t, for the case in which (P_n) is quasi-analytic. We shall prove the following result.

Let μ be the number introduced in (2.1.25) and let us suppose that for some integer $n>\mu$

$$(2.2.9) \qquad t \leqq \varphi(n)^{-n/(n-\mu)}\,.$$

Then

$$(2.2.10) \qquad \|\hat{U}_t^{(n)}\|_2 \leqq 3MP_n\,\bar{S}_{\mu,n}(X)\,.$$

The proof will be given for $\mu=1$. Condition (2.2.9) implies that

$$t^{n-1}Q_n \leqq P_n \quad \text{and} \quad t^jQ_j \leqq P_j, \quad j=0,1,2,\ldots,n-1\,.$$

Therefore (2.2.6) yields

$$(2.2.11) \quad \|\hat{U}_t^{(n)}\|_2 \leqq M\,\bar{S}_{1,n}(X)[\tfrac{3}{2}P_n2^{-n} + \textstyle\sum_{j=0}^{n-1}\binom{n}{j}2^{-j}P_{n-j}P_j]\,.$$

This inequality is derived without Condition (2.1.1). If we use this condition (2.2.10) follows from (2.2.11).

We shall now prove the following lemma, which will be used repeatedly in the rest of the section. It should be noted that the condition (2.2.12) of this lemma is a consequence of $t \leqq 1$, if (P_n) is non-quasi-analytic.

LEMMA 1. *Let f for a fixed $X \geqq 1$ satisfy* (2.1.28). *Let us introduce (P_n), p, φ, q and α_n, as in* 2.1. *Let $t=X^{-1}$ and let us suppose that*

$$(2.2.12) \qquad t \leqq \varphi(n)^{-n/(n-1)}, \quad n=2,3,\ldots,N\,.$$

Let $a>0$ be fixed and $\beta_n=a^{-1}\alpha_n$, $n=1,2,\ldots$. Let $r(x) \geqq 0$, $x \geqq 0$ and $B=\sup_{0\leqq x\leqq\beta_1} r(x)$. Then

$$(2.2.13) \quad \int_{-\infty}^0|U_t(y)|\,dy + \int_0^{\beta_1}r(y)p(ay)|U_t(y)|\,dy \leqq \max(B,1)M(2P_1)^{\frac{1}{2}}S(X)\,.$$

If $\beta_K \leqq x_1 \leqq x_2 \leqq \beta_{N+1}$, where $K \geqq 1$, then

$$(2.2.14) \int_{x_1}^{x_2}r(y)p(ay)|U_t(y)|\,dy \leqq 3(2\pi)^{-\frac{1}{2}}M(\textstyle\sum_{n=K}^N a^{2n})^{\frac{1}{2}}M_2\{r;x_1,x_2\}\bar{S}_{1,N}(X)\,.$$

Let us further suppose that (2.1.25) *is satisfied with* $\mu > 1$ *and*

(2.2.15) $$t \leqq \varphi(n)^{-n/(n-\mu)}, \quad n = \mu+1, \mu+2, \ldots, N .$$

If $K > \mu$ *then* (2.2.14) *holds true with* $\bar{S}_{1,N}$ *replaced by* $\bar{S}_{\mu,N}$.

Proof. The inequality (2.2.13) is a direct consequence of (2.2.8) since $p(ay) = 1$, $0 \leqq y \leqq \beta_1$. In order to prove (2.2.14) let us choose $n, 1 \leqq n \leqq N$. By (2.1.6)

$$\int_{\beta_n}^{\beta_{n+1}} r(y)p(ay)|U_t(y)|\,dy = a^n P_n^{-1} \int_{\beta_n}^{\beta_{n+1}} r(y)y^n |U_t(y)|\,dy .$$

By using Schwarz's inequality, Parseval's relation and (2.2.10), we find after a summation with respect to n, that

$$\int_{\beta_K}^{\beta_{N+1}} r(y)p(ay)|U_t(y)|\,dy \leqq 3(2\pi)^{-\frac{1}{2}} M\left(\sum_{n=K}^{N} a^n M_2\{r;\beta_n,\beta_{n+1}\}\right)\bar{S}_{1,N}(X) .$$

If we apply Schwarz's inequality to the sum in the right-hand member of this inequality, we shall have proved (2.2.14) with $x_1 = \beta_K, x_2 = \beta_{N+1}$. The general result follows in the same way.

In some cases when $f^{(n)}(\xi) = o(1)$, $|\xi| \to \infty, n \geqq 1$, it is better to use L^s-estimates instead of L^2-estimates. In this way the following result is obtained (cf. [8, Theorem 3, p. 582]).

Remark to Lemma 1. Let $1 < s \leqq 2$, $1/s + 1/s' = 1$. Replace condition (2.1.28) in Lemma 1 by

(2.2.16) $$M_s\{f^{(n)}; -X, X\} \leqq P_n S_n^*(X), \quad n = 0, 1, 2, \ldots ,$$

where $S_n^* \nearrow$, $n = 0, 1, 2, \ldots$, and $S_0^*(X) \leqq \frac{1}{2} X S_1^*(X)$. Then (2.2.13) holds true with $(2P_1)^{\frac{1}{2}} S(X)$ replaced by

$$C(s)(X^{-1/s} S_0^*(X) + X^{1/s'} P_1 S_1^*(X)) .$$

2.3. *Estimation of* $\psi * U_t(x)$.

Let $v(x), x \geqq 0$, be a function such that

(2.3.1) $$v \nearrow, \quad v \geqq 1, \quad v(x) \to \infty, \quad x \to \infty ,$$

and introduce the class $R[v]$ of functions $\varrho(x)$, $-\infty < x < \infty$, defined as follows.

Definition. $\varrho \in R[v]$, *if* $\varrho > 0$, $\varrho \searrow$, $\varrho(x) \to 0, x \to \infty$, *and*

(2.3.2) $$\varrho(x-y) \leqq \varrho(x)v(y), \quad y \geqq 0, -\infty < x < \infty .$$

Let $\psi(x), -\infty < x < \infty$, be bounded and

$$(2.3.3) \qquad |\psi(x)| \leqq \varrho(x), \quad x \geqq x_0 ,$$

where $\varrho \in R[v]$. Define $v(x)=1, x<0$. Then (2.3.2) is valid for all y. We shall estimate $\psi * U_t(x)$ when $x \to \infty$. Let

$$(2.3.4) \qquad \int_{-\infty}^{\infty} \psi(x-y)\, U_t(y)\, dy = \int_{-\infty}^{x-x_0} + \int_{x-x_0}^{\infty} = I_1(x) + I_2(x) .$$

From (2.3.2) it follows that $\varrho(x_0) \leqq \varrho(x) v(x-x_0)$. Remembering that $v \nearrow$, we find that

$$(2.3.5) \qquad |I_2(x)| \leqq \|\psi\|_\infty \varrho(x_0)^{-1} \varrho(x) \int_{x-x_0}^{\infty} v(y) |U_t(y)|\, dy .$$

Using (2.3.2) and (2.3.3) we get

$$(2.3.6) \qquad |I_1(x)| \leqq \varrho(x) \int_{-\infty}^{x-x_0} v(y) |U_t(y)|\, dy .$$

Let $v(y)=bw(y), y \geqq 0$, where $b \geqq 1$ is a constant. Let θ be constant, $0<\theta<1$, and let us assume that

$$(2.3.7) \qquad \begin{cases} w(y) \leqq h_Q(\theta y), & y \geqq 0 , \\ M_2\{w(y)/h_Q(\theta y); 0, \infty\} = B_1 < \infty . \end{cases}$$

We shall show that this condition implies

$$(2.3.8) \qquad |\psi * U_t(x)| \leqq M(e(2P_1)^{\frac{1}{2}} S(X) + C B_1 \bar{S}_1(X)) b \varrho(x), \quad x \geqq \xi_1 ,$$

where $C=C(Q,\theta)$, $\xi_1=\xi_1(Q,\theta,\|\psi\|_\infty,\varrho(x_0))$, $X=t^{-1}$ and M is defined by (2.2.5).

Let us choose $a=\theta^{\frac{1}{2}}$ and let $r(y)=bw(y)/q(ay)$. By (2.1.8) and (2.1.12), applied with p replaced by q,

$$h_Q(\theta y) \leqq e, \quad 0 \leqq y \leqq \theta^{-1}\alpha_1 ,$$

and

$$h_Q(\theta y) \leqq Cq(ay), \quad y \geqq 0 .$$

Hence $r(y) \leqq be$, $0 \leqq y \leqq a^{-1}\alpha_1$, and $M_2\{r; 0, \infty\} \leqq CbB_1$. Since $Q_n \geqq P_n$, $n=0,1,2,\ldots$ the function f satisfies (2.1.28) with P_n replaced by Q_n. We may apply Lemma 1 with p replaced by q, in which case Condition (2.2.12) is trivially satisfied. If we choose $x_1=x-x_0$ and let $x_2 \to \infty$ in (2.2.14), we then find from (2.3.5) that

$$(2.3.9) \qquad |I_2(x)| \leqq M B_1 \bar{S}_1(X) b \varrho(x), \quad x \geqq \xi_1 ,$$

where $\xi_1=\xi_1(Q,\theta,\|\psi\|_\infty,\varrho(x_0))$. Lemma 1 and (2.3.6) furthermore yield

$$(2.3.10) \qquad |I_1(x)| \leqq M[e(2P_1)^{\frac{1}{2}} S(X) + C(Q,\theta) B_1 \bar{S}_1(X)] b \varrho(x)$$

and (2.3.8) is proved.

We shall now study the case in which (2.3.7) is satisfied with h_Q replaced by h_P. We first make the following observation, which will be used in other contexts later on. Let $k\nearrow$, $v\nearrow$ and let v^{-1} denote the inverse function of v. Suppose that $s(x)=v^{-1}(k(x)) \to \infty, x \to \infty$. If $x_2 > s(x)$, we find

$$(2.3.11) \qquad \int_{s(x)}^{x_2} |U_t(y)|\,dy \leqq (k(x))^{-1} \int_{s(x)}^{x_2} v(y)|U_t(y)|\,dy\,.$$

Let us suppose further that $v(y)/q(\theta_0 y) \in L^2(0,\infty)$ for some θ_0, $0<\theta_0<1$. As above, we may apply Lemma 1 with p replaced by q and with $r(y)=v(y)/q(\theta_0 y)$ to the integral in the right-hand member of (2.3.11). We then find that for every $\varepsilon>0$ there is $\xi_\varepsilon=\xi_\varepsilon(\theta_0,s,Q,\mu)$ such that

$$(2.3.12) \quad \int_{s(x)}^{\infty} |U_t(y)|\,dy \leqq \varepsilon M(k(x))^{-1}\,\overline{S}_\mu(X) M_2\{r;s(x),\infty\}, \qquad x \geqq \xi_\varepsilon\,,$$

where M is defined by (2.2.5) and $r(y)=v(y)/q(\theta_0 y)$.

We will now prove

LEMMA 2. (1) *Let us introduce* $(P_n), h_P, \varphi, q$ *and* (S_n) *as in* 2.1 *and let* f *satisfy* (2.1.28) *for some* $X=t^{-1}>1$. *Introduce* H_t *as in* 2.2 *and let* $\hat{U}_t=\hat{H}_t f$. *Let* ψ *be bounded on* $(-\infty,\infty)$ *and*

$$(2.3.13) \qquad |\psi(x)| \leqq \varrho(x), \quad x \geqq x_0\,.$$

Let θ, δ *and* b *denote positive constants,* $b \geqq 1$. *Let*

$$(2.3.14) \qquad \varrho \in R[bw]\,.$$

(2) *Suppose further*

$$(2.3.15) \qquad 0 < \theta < 1\,,$$

$$(2.3.16) \qquad w(y) \leqq h_P(\theta y), \quad y \geqq 0\,.$$

(3) *Let*

$$(2.3.17) \qquad w(y)/h_P(\theta y) \in L^2(0,\infty)\,.$$

Then we have the following result:

There is a number $\xi_1=\xi_1(P,\varphi,\delta,\|\psi\|_\infty,\varrho,x_0,\theta)$ *such that if for some* $x \geqq \xi_1$

$$(2.3.18) \qquad 0 < t \leqq \inf_{2\leqq y\leqq x^{1+\delta}} \varphi(y)^{-\nu/(\nu-1)}$$

then

$$(2.3.19) \qquad |\psi * U_t(x)| \leqq M[C_1 S(X)+C_2\overline{S}_1(X)]b\varrho(x)$$

where M *is defined by* (2.2.5), $C_1=C_1(P)$ *and* $C_2=C_2(P,\theta,w)$.

COROLLARY. *Let Conditions* (1) *and* (2) *in Lemma* 2 *hold true and suppose further*

$$\text{(2.3.20)} \qquad \log(P_{n+1}/P_n) = o(n), \quad n \to \infty .$$

Then the same result holds true with $C_2 = C_2(P, \theta)$.

PROOF. If (P_n) is non-quasi-analytic, then $h_P = h_Q$. Thus (2.3.7) is satisfied and the result follows from (2.3.8). Let us suppose that (P_n) is quasi-analytic. Then $\chi_p(x) \to \infty$, $x \to \infty$, and by (2.1.22),

$$\text{(2.3.21)} \qquad h_P(x^\alpha)/q(\tfrac{1}{2}x) \in L^2(0,\infty), \quad 0 < \alpha < 1 .$$

Let $\eta = \delta/2$ and $\alpha = (1+\eta)^{-1}$. Let us write

$$\text{(2.3.22)} \quad \int_{-\infty}^{\infty} \psi(x-y) U_t(y)\,dy = \int_{-\infty}^{x-x_0} + \int_{x-x_0}^{x^{1+\eta}} + \int_{x^{1+\eta}}^{\infty} = I_1(x) + J(x) + K(x) .$$

From $\varrho \in R[bw]$ and (2.3.16) it follows that $\varrho \in R[bh_P(\theta x)]$. According to (2.3.21) we may apply (2.3.12) with $k(x) = h_P(x)$, $v(x) = h_P(x^\alpha)$, $s(x) = x^{1+\eta}$, $\mu = 1$ and $\theta_0 = \frac{1}{2}$. Since $\varrho(0) \leqq bk(x)\varrho(x)$, $x \geqq 0$, we thus find that for every $\varepsilon > 0$ there is $x_1 = x_1(P, \varphi, \delta, \|\psi\|_\infty, \varrho(0), \varepsilon)$ such that

$$\text{(2.3.23)} \qquad |K(x)| \leqq \varepsilon M \bar{S}_1(X) b \varrho(x), \quad x \geqq x_1 .$$

This inequality is derived without the use of Condition (2.3.18). In the rest of the proof I suppose this condition satisfied for every choice of x below.

Using $w \nearrow$ and $\varrho(x_0) \leqq b\varrho(x) w(x-x_0)$, we get

$$\text{(2.3.24)} \qquad |J(x)| \leqq \|\psi\|_\infty \varrho(x_0)^{-1} b \varrho(x) \int_{x-x_0}^{x^{1+\eta}} w(y) |U_t(y)|\,dy .$$

Let $a = \theta^{\frac{1}{2}}$. Then $h_P(\theta y) \leqq C(P,\theta) p(ay)$, $y \geqq 0$, according to (2.1.12). The assumption (2.3.17) thus implies that $r(y) = w(y)/p(ay) \in L^2(0,\infty)$. Furthermore,

$$\chi_p(a x^{1+\eta}) \leqq x^{1+\delta}, \quad x \geqq x_2 = x_2(P,\delta) ,$$

since $\chi_p(x) = O(x)$, $x \to \infty$ and $\eta = \delta/2$. Let us choose $x, x \geqq x_2$. It then follows from (2.3.18) that Condition (2.2.12) in Lemma 1 is satisfied with $N = \chi_p(a x^{1+\eta})$. Thus $x^{1+\eta} < \beta_{N+1} = a^{-1}\alpha_{N+1}$, according to (2.1.7). We may apply (2.2.14) in Lemma 1 to the integral in the right hand member of (2.3.24) and find that there is $x_3 = x_3(P, \theta, \delta, \|\psi\|_\infty, \varrho(x_0))$ such that

$$\text{(2.3.25)} \qquad |J(x)| \leqq M B_2 \bar{S}_1(X) b \varrho(x), \quad x \geqq x_3 ,$$

where $B_2 = M_2\{w(y)/h_P(\theta y); 0, \infty\}$. Finally, the integral $I_1(x)$ may be estimated similarly from (2.3.6) and Lemma 1. Combining the result thus obtained with (2.3.22), (2.3.23) and (2.3.25) we have proved (2.3.19).

To prove the corollary we observe that (2.3.20) implies (2.1.24) and hence $h_P(\theta x)/h_P(\theta_1 x) \in L^2(0,\infty)$ if $\theta < \theta_1$. The result then follows by applying Lemma 2 with θ replaced by $\theta_1 = \theta^{\frac{1}{2}}$.

When (2.3.20) is satisfied we may also derive corresponding results for $\theta = 1$ and for some $\theta > 1$. This is the content of the following lemma.

LEMMA 3. (1) *Let Condition* (1) *in Lemma* 2 *hold true and let* (2.3.20) *be satisfied.*

(2) *Let* $\theta \geqq 1$. *Let* μ *denote the integer introduced in* (2.1.25) *and let* c *and* β *be constants,* $c \geqq 0$, *and*

$$\beta > \overline{\lim}_{y\to\infty} \chi_p(y) y^{-1} . \tag{2.3.26}$$

(3) *Let*

$$w(y) \leqq (1+\theta y)^c \, h_P(\theta y), \quad y \geqq 0 . \tag{2.3.27}$$

Then there is a number $\xi_1 = \xi_1(P, \varphi, \delta, \|\psi\|_\infty, \varrho, x_0, \theta, \mu, c, \beta)$ *such that if, for some* $x \geqq \xi_1$,

$$0 < t \leqq \inf_{\mu+1 \leqq y \leqq x^{1+\delta}} \varphi(y)^{-y/(y-\mu)} \tag{2.3.28}$$

then

$$|\psi * U_t(x)| \leqq \tag{2.3.29}$$

$$\leqq M[C_1 S(X) + C_2 \bar{S}_1(X) + C_3 x^{c+2} \exp(x\beta\theta^2 \log\theta) \bar{S}_\mu(X)] b\varrho(x)$$

where M *is defined by* (2.2.5), $C_1 = C_1(P,c), C_2 = C_2(P,c,\mu,\theta)$ *and* $C_3 = C_3(P)$.

PROOF. Proceeding as in the proof of Lemma 2, let

$$k(x) = (1+\theta x)^c h_P(\theta x), \quad x \geqq 0 .$$

Then $\varrho \in R[bk(x)]$. Let $\eta = \delta/2$ and split up $\psi * U_t(x)$ as in (2.3.22).

In order to estimate $K(x)$ let $\alpha = (1+\eta)^{-1}$ and choose $v(x) = k(x^\alpha)$. Then $v(x)/q(\frac{1}{2}x) \in L^2(0,\infty)$ according to (2.1.22). Using (2.3.12) we find that for every $\varepsilon > 0$ there is $x_1 = x_1(P, \varphi, \delta, \|\psi\|_\infty, \varrho(0), \mu, \varepsilon)$ such that

$$|K(x)| \leqq \varepsilon M \bar{S}_\mu(X) b\varrho(x), \quad x \geqq x_1 .$$

Let $\sigma, 0 < \sigma \leqq 1$, be a number to be fixed later and let us write

$$J(x) = \int_{x-x_0}^{x^{1+\eta}} \psi(x-y) U_t(y)\, dy = \int_{x-x_0}^{\theta(1+\sigma)x} + \int_{\theta(1+\sigma)x}^{x^{1+\eta}} = J_1(x) + J_2(x) .$$

To estimate $J_2(x)$ we choose $v(x) = k(x\theta^{-1}(1+\sigma)^{-1})$. Applying (2.3.11) we get

$$\int_{\theta(1+\sigma)x}^{x^{1+\eta}} |U_t(y)|\, dy \leqq (k(x))^{-1} \int_{\theta(1+\sigma)x}^{x^{1+\eta}} v(y)|U_t(y)|\, dy .$$

Let $a=(1+\sigma)^{-\frac{1}{2}}$. Then $v(y)/p(ay)$ belongs to $L^2(0,\infty)$ according to (2.1.24). Choosing x so large that $\chi_p(ax^{1+\eta})<x^{1+\delta}$ we find from Lemma 1 that for every $\varepsilon>0$ there is $x_2=x_2(P,\theta,\delta,\|\psi\|_\infty,\varrho(0),\sigma,\mu,\varepsilon)$ such that

$$|J_2(x)| \leqq \varepsilon M\bar{S}_\mu(X)b\varrho(x), \quad x\geqq x_2 .$$

To estimate $J_1(x)$ let $r(y)=k(y)/p(\theta y)$. Then $r(y)\leqq C(P)(1+\theta y)^{c+1}$ according to (2.1.10). Let $N=\chi_p(\theta^2(1+\sigma)x)$.

If $\theta=1$ we find from Lemma 1 that for $x\geqq x_3=x_3(P,x_0,\sigma,\mu)$

$$\begin{aligned}\int_{x-x_0}^{\theta(1+\sigma)x} k(y)|U_t(y)|\,dy &\leqq \sigma^{\frac{1}{2}}C(P,c)MN^{\frac{1}{2}}x^{c+3/2}\,\bar{S}_\mu(X)\\ &\leqq \sigma^{\frac{1}{2}}C(P,c)Mx^{c+2}\,\bar{S}_\mu(X)\,.\end{aligned}$$

The last inequality holds since $\chi_p(x)=O(x), x\to\infty$. Now choose $\varepsilon>0$ and

$$\sigma = \varepsilon^2(C(P,c))^{-2}\min\left(1,(\varrho(x_0)/\|\psi\|_\infty)^2\right).$$

It then follows that for $x\geqq x_3'=x_3'(P,x_0,\varrho(x_0),\|\psi\|_\infty,\mu,\varepsilon,c)$

$$|J_1(x)| \leqq \varepsilon Mx^{c+2}\,\bar{S}_\mu(X)b\varrho(x)\,.$$

If $\theta>1$ we find from Lemma 1 that for $x\geqq x_4=x_4(P,\theta,x_0,\sigma,\mu)$

$$\int_{x-x_0}^{\theta(1+\sigma)x} k(y)|U_t(y)|\,dy \leqq MC(P,\theta,c)\theta^N N^{\frac{1}{2}}x^{c+3/2}\,\bar{S}_\mu(X)\,. \tag{2.3.30}$$

Now we choose $\sigma, 0<\sigma<1$, such that

$$\overline{\lim}_{y\to\infty}\chi_p(y)/y < \beta(1+\sigma)^{-2}\,.$$

Then $N=\chi_p(\theta^2(1+\sigma)x)<x\beta\theta^2(1+\sigma)^{-1}, x\geqq x_5=x_5(P,\beta,\sigma)$ and it follows from (2.3.30) that for $x\geqq x_6=x_6(P,\theta,c,x_0,\varrho(x_0),\|\psi\|_\infty,\beta)$

$$|J_1(x)| \leqq M\exp(x\beta\theta^2\log\theta)\bar{S}_\mu(X)b\varrho(x)\,.$$

The integral $I_1(x)$ can be estimated in an analogous way by using Lemma 1. If we combine the inequalities thus obtained, we have proved Lemma 3.

2.4. *Estimation of* $\psi * U_{t(x)}(x)$.

Let $0<t(x)<1, x\geqq x_0$. We shall study the function $\psi * U_{t(x)}(x)$ when $x\to\infty$. To this end we will apply Lemma 2 or Lemma 3 with $t=t(x)$ and let x tend to infinity.

Let us consider Lemma 2. Let the conditions of this lemma hold true and suppose further that (2.1.28) is satisfied for all X belonging to the range of $1/t(x)$. If (P_n) is non-quasi-analytic then $\varphi\equiv 1$ and (2.3.18) holds true for every $\delta>0$ with t replaced by $t(x)$ and for $x\geqq x_0$. Applying Lemma 2 with $t=t(x)$ we find that for $x\geqq\xi_1=\xi_1(P,\varphi,\|\psi\|_\infty,\varrho,x_0,\theta)$

$$|\psi * U_{t(x)}(x)| \leqq M[C_1S(1/t(x))+C_2\bar{S}_1(1/t(x))]b\varrho(x)\,. \tag{2.4.1}$$

To extend this result to quasi-analytic sequences (P_n) we impose the following condition on $t(x)$. Let for some $B>1$

$$t(x)\varphi(x) \leqq B^{-1}, \quad x \geqq x_0 . \tag{2.4.2}$$

Put $B^{\frac{1}{2}} = 1+\delta$. By applying (2.1.15) with $m=1$ and $a=1+\delta$ we get

$$\varphi(x)^{-1} \leqq (1+\delta)^2 \inf_{2\leqq y\leqq x^{1+\delta}} \varphi(y)^{-y/(y-1)}, \quad x \geqq x_1 = x_1(\varphi, B) . \tag{2.4.3}$$

By combining (2.4.2) and (2.4.3) we find

$$t(x) \leqq \inf_{2\leqq y\leqq x^{1+\delta}} \varphi(y)^{-y/(y-1)}, \quad x \geqq x_2 = \max(x_0, x_1) .$$

Therefore (2.3.18) holds true with $t=t(x)$ if $x \geqq x_2$ and we may apply Lemma 2 with $t=t(x)$. Thus we obtain that there is $\xi=\xi(P,\varphi,\|\psi\|_\infty,\varrho, x_0,\theta,B)$ such that (2.4.1) holds true for $x \geqq \xi$ and for all t satisfying (2.4.2).

It is obvious that Lemma 3 can be extended in the same way to hold for $t=t(x)$ if f satisfies (2.1.28) for all X belonging to the range of $1/t(x)$, $x \geqq x_0$, and (2.4.2) is satisfied.

2.5. *Estimation of* $I(x)$.

We shall now apply the above results to the Tauberian problem. Let Φ be bounded on $(-\infty,\infty)$, let $F \in L^1(-\infty,\infty)$ and $\psi = \Phi * F$. Introduce f, H and $\hat{U}_t = \hat{H}_t f$ as in 2.2 and let $1/\hat{F}(\xi) = f(\xi)$, $-\infty<\xi<\infty$. Then, for every fixed t,

$$\psi * U_t(x) = \Phi * H_t(x), \quad -\infty < x < \infty , \tag{2.5.1}$$

the inversion being justified by absolute convergence. Furthermore,

$$\Phi * H_t(x) = \int_{-\infty}^{\infty} \Phi(x-ut)H(u)\,du, \quad -\infty<x<\infty . \tag{2.5.2}$$

Let $0<t(x)<1$, $-\infty<x<\infty$, and introduce the function $I(x)$ as follows

$$I(x) = \int_{-\infty}^{\infty} \Phi(x-ut(x))H(u)\,du . \tag{2.5.3}$$

By choosing $t=t(x)$ in (2.5.1) and (2.5.2) we obtain the identity

$$I(x) = \psi * U_{t(x)}(x), \quad -\infty<x<\infty . \tag{2.5.4}$$

According to (2.5.4) the estimate of $\psi * U_{t(x)}(x)$ obtained in 2.4 can be used to estimate $I(x)$ when $x \to \infty$. In this way the two lemmas below are obtained. They are stated here for the sake of reference and will be used later in the proof of the Tauberian theorems.

The sequence $P=(P_n)$ and the functions φ, h_P, h_Q, S and $\bar{S}_1$ are introduced in 2.1 and the function H in 2.2.

LEMMA 4. (1) *Let* Φ *be bounded on* $(-\infty,\infty)$, *let* $F \in L^1(-\infty,\infty)$ *and*

$$|\Phi * F(x)| \leqq \varrho(x), \quad x \geqq x_0 . \tag{2.5.5}$$

Let $\varrho \in R[bw]$ *for some* $b \geqq 1$. *Let* $0 < t(x) \leqq X_0^{-1}, x \geqq x_0$, *and let* $I(x)$ *be defined by* (2.5.3).

(2) *Let* $1/\hat{F}(\xi) = f(\xi), -\infty < \xi < \infty$, *where* f *satisfies* (2.1.28) *for* $X \geqq X_0 > 1$.

(3) *Let* θ *be constant*, $0 < \theta < 1$.

(4) *Let us suppose either that* (2.3.16), (2.3.17) *and* (2.4.2) *hold true for some* $B > 1$ *or that* (2.3.16) *and* (2.3.17) *hold true with* h_P *replaced by* h_Q.

Then there is ξ *independent of* t *such that*

$$|I(x)| \leqq M\left[C_1 S\left(\frac{1}{t(x)}\right) + C_2 \bar{S}_1\left(\frac{1}{t(x)}\right)\right] b\varrho(x), \quad x \geqq \xi , \tag{2.5.6}$$

where M *is defined by* (2.2.5), $C_1 = C_1(P)$ *and* $C_2 = C_2(P,\varphi,\theta,w)$.

COROLLARY. *Let Conditions* (1), (2) *and* (3) *in Lemma* 4 *hold true and let* (2.3.20) *be satisfied. Suppose either that* (2.3.16) *and* (2.4.2) *hold true or that* (2.3.16) *holds true with* h_P *replaced by* h_Q. *Then* (2.5.6) *holds true with* $C_2 = C_2(P,\varphi,\theta)$.

LEMMA 5. (1) *Let Condition* (1) *in Lemma* 4 *hold true and suppose further that* (2.3.20) *is satisfied.*

(2) *Let Condition* (2) *in Lemma* 4 *hold true.*

(3) *Let* θ *be constant*, $\theta \geqq 1$, *and introduce* μ, c *and* β *as in Lemma* 3.

(4) *Let us suppose either that* (2.3.27) *and* (2.4.2) *are satisfied for some* $B > 1$ *or that* (2.3.27) *is satisfied with* h_P *replaced by* h_Q.

Then there is ξ *independent of* t *such that*

$$|I(x)| \leqq M\left[C_1 S\left(\frac{1}{t(x)}\right) + C_2 \bar{S}_1\left(\frac{1}{t(x)}\right) + \right.$$
$$\left. + C_3 x^{c+2} \exp(x\beta\theta^2 \log\theta)\, \bar{S}_\mu\left(\frac{1}{t(x)}\right)\right] b\varrho(x), \quad x \geqq \xi , \tag{2.5.7}$$

where M *is defined by* (2.2.5), $C_1 = C_1(P,c)$, $C_2 = C_2(P,\varphi,c,\mu,\theta)$ *and* $C_3 = C_3(P,\varphi)$.

PROOF. The results in Lemma 4 and Lemma 5 are immediate consequences of the results in 2.4 and the identity (2.5.4) in case the conditions (2.3.16) and (2.3.17) or the condition (2.3.27) are satisfied. The constants C_2, or C_2 and C_3 respectively are then, in fact, independent of φ. To prove the result when these conditions are satisfied with h_P replaced by h_Q we need only observe that (2.1.28) holds true with (P_n) replaced by (Q_n) and apply the above result with (P_n) replaced by (Q_n).

2.6. *Extension of the class of kernels.*

The class of kernels considered in 2.5, namely those functions $F \in L^1(-\infty,\infty)$ such that $1/\hat{F}(\xi) = f(\xi)$, $-\infty < \xi < \infty$, where f satisfies (2.1.28) for $X \geqq X_0$ is chosen for the sake of simplicity. It is, however, unsatisfactory since it yields a 'symmetric' condition, i.e. a condition of the same strength whether we consider $\Phi * F(x)$ and $\Phi(x)$ as $x \to \infty$ or as $x \to -\infty$. For instance, if $P_n = n!\gamma^{-n}$ then the assumptions imply that $1/\hat{F}$ is analytic in $-\gamma < \eta < \gamma$, a condition which is far from necessary since the same results may be obtained in the Tauberian theorems if $1/\hat{F}$ is supposed to be analytic in $-\delta < \eta < \gamma$ for some $\delta > 0$. In this section we will show how the conditions on F in Lemma 4 and Lemma 5 can be weakened. It will then follow that the conditions in the Tauberian theorems can be weakened in the same way.

The sequence (P_n), the sequence of functions (S_n), and the function S are introduced as in 2.1 and the function H as in 2.2.

Let is untroduce the classes $\mathscr{B}_1 = \mathscr{B}_1((P_n),(S_n))$ and $\mathscr{B}_2 = \mathscr{B}_2((P_n),(S_n))$ of functions $g(\xi), -\infty < \xi < \infty$, as follows.

Definition. $g \in \mathscr{B}_1$ if for every $X \geqq X_0$ there exist functions $f = f_X$ and $k = k_X$ such that

$$g(\xi) = f(\xi) + k(\xi), \quad -X \leqq \xi \leqq X, \tag{2.6.1}$$

where f satisfies (2.1.28) and k is the Fourier transform of a function $K = K_X$ such that $K(x) = 0, x > 0$,

$$\|K\|_\infty \leqq XS(X), \tag{2.6.2}$$

and

$$\|K\|_1 \leqq S(X). \tag{2.6.3}$$

$\mathscr{B}_2$ denotes the class of functions g which satisfy the above conditions but for the fact that k is the Fourier transform in the L^2-sense of K and (2.6.3) is replaced by

$$\|K\|_2 \leqq S(X). \tag{2.6.4}$$

Introduce the notations

$$M_1 = \|(1+|x|)^2 q(2|x|)H(x)\|_1, \quad M_2 = \|(1+\theta|x|)^{c+2} q(2\theta|x|)H(x)\|_1.$$

Let $\psi = \Phi * F$ and

$$\varrho_2(x) = \{\textstyle\int_x^\infty |\psi(y)|^2 dy\}^{\frac{1}{2}}.$$

If $\psi \notin L^2(0,\infty)$ let $\varrho_2 \equiv \infty$.

We will prove the following generalization of Lemma 4 and Lemma 5.

LEMMA 4′. *Let Conditions* (1), (3) *and* (4) *of Lemma* 4 *hold true and let* $1/\hat{F} \in \mathscr{B}_1((P_n),(S_n))$. *Then* (2.5.6) *holds true with* M *replaced by* M_1.

LEMMA 5′. *Let conditions* (1), (3) *and* (4) *of Lemma* 5 *hold true and let* $1/\hat{F} \in \mathscr{B}_1((P_n),(S_n))$. *Then* (2.5.7) *holds true with* M *replaced by* M_2.

REMARK. If the condition $1/\hat{F} \in \mathscr{B}_1((P_n),(S_n))$ in Lemma 4′ or Lemma 5′ is replaced by the condition that $1/\hat{F} \in \mathscr{B}_2((P_n),(S_n))$ and $\hat{F} \neq 0$ on $(-\infty,\infty)$, then (2.5.6) or (2.5.7) respectively hold true if M is exchanged as above and a term $\|H\|_1 S(1/t(x))\varrho_2(x)$ is added to the right hand side of these inequalities.

Let us first prove the following generalization of Lemma 2.

LEMMA 2′. *Replace the definition* $\hat{U}_t = \hat{H}_t f$ *in Lemma* 2 *by*

$$\hat{U}_t = \hat{H}_t(f+k) \tag{2.6.5}$$

where f *and* k *satisfy the conditions in the definition of* $\mathscr{B}_1$ *for some fixed* $X = t^{-1}$. *Then the same result holds true but for the fact that* M *should be replaced by* M_1.

PROOF. Let $\hat{V}_t = \hat{H}_t f$ and $\hat{W}_t = \hat{H}_t k$. The conditions imply that $\hat{U}_t, \hat{V}_t$ and $\hat{W}_t$ are Fourier transforms of U_t, V_t and $W_t = H_t * K$ respectively. Therefore

$$\psi * U_t = \psi * V_t + \psi * W_t\,. \tag{2.6.6}$$

The function V_t now satisfies the same conditions as U_t did in Lemma 2 and hence $\psi * V_t(x)$ can be estimated by use of Lemma 2. It remains to consider $\psi * W_t(x)$. We shall actually estimate $\psi * W_t(x)$ under the weaker condition that $\varrho \in R[bh_P]$ and when $X = t^{-1}$ satisfies

$$\varphi(x) \leqq X\,. \tag{2.6.7}$$

Since $K(y) = 0, y > 0$, we find

$$W_t(y) = \int_y^\infty K(y-u)H_t(u)\,du, \qquad -\infty < y < \infty\,. \tag{2.6.8}$$

Let $v(y) = (1+y)^2 q(2y), y \geqq 0$, $v(y) = 1, y < 0$. Then $v \nearrow$ and $\|vH\|_1 < M_1$. Hence

$$\begin{aligned}\int_y^\infty |H_t(u)|\,du &= \int_{yX}^\infty |H(u)|\,du \leqq (v(yX))^{-1} \int_{yX}^\infty v(u)|H(u)|\,du \\ &\leqq M_1/v(yX), \quad y \geqq 0\,.\end{aligned} \tag{2.6.9}$$

From (2.6.8) and (2.6.9) it follows that

$$(2.6.10)\qquad |W_t(y)| \leqq M_1\|K\|_\infty/v(yX), \quad y \geqq 0\,.$$

We now observe that it follows from (2.1.12), (2.1.15) and (2.1.20) that we can find $x_1 = x_1(P,\varphi) \geqq 0$ such that if (2.6.7) is satisfied for some $x \geqq x_1$, then

$$(2.6.11)\qquad h_P(y) \leqq C(P)q(2yX), \quad 0 \leqq y \leqq x\,.$$

In the sequel we suppose (2.6.7) satisfied for every choice of x below.

By combining (2.6.10) and (2.6.11) and using the definition of v we obtain, if $x \geqq x_1$

$$(2.6.12)\qquad |W_t(y)| \leqq \frac{C(P)M_1\|K\|_\infty}{(1+yX)^2h_P(y)}, \quad 0 \leqq y \leqq x$$

and

$$(2.6.13)\qquad |W_t(y)| \leqq \frac{C(P)M_1\|K\|_\infty}{(1+yX)^2h_P(x)}, \quad x \leqq y\,.$$

Let

$$(2.6.14)\quad \int_{-\infty}^{\infty}\psi(x-y)W_t(y)\,dy = \int_{-\infty}^{0}+\int_0^{x-x_0}+\int_{x-x_0}^{\infty} = I_1(x)+I_2(x)+I_3(x)\,.$$

We may choose $x_0 \geqq 0$. If $x \geqq x_2 = x_1 + x_0$ it follows from (2.6.12) and $\varrho \in R[bh_P]$ that

$$|I_2(x)| \leqq b\varrho(x)\int_0^{x-x_0}h_P(y)|W_t(y)|\,dy \leqq b\varrho(x)C(P)M_1\|K\|_\infty\int_0^{x-x_0}(1+yX)^{-2}dy.$$

Hence, if $x \geqq x_2$

$$(2.6.15)\qquad |I_2(x)| \leqq C(P)M_1X^{-1}\|K\|_\infty b\varrho(x)\,.$$

To estimate $I_3(x)$ we use (2.6.13). If $x \geqq x_2$ we get

$$|I_3(x)| \leqq \|\psi\|_\infty\int_{x-x_0}^{\infty}|W_t(y)|\,dy \leqq \frac{\|\psi\|_\infty C(P)M_1\|K\|_\infty}{h_P(x-x_0)}\int_{x-x_0}^{\infty}\frac{dy}{(1+yX)^2}.$$

The assumption $\varrho \in R[bh_P]$ implies that $1/h_P(x-x_0) \leqq b\varrho(x)/\varrho(x_0)$. Therefore we can find $x_3 = x_3(P,\varphi,\|\psi\|_\infty,\varrho(x_0),x_0) \geqq x_2$ such that if $x \geqq x_3$

$$(2.6.16)\qquad |I_3(x)| \leqq M_1X^{-1}\|K\|_\infty b\varrho(x)\,.$$

It remains to consider $I_1(x)$. Since $|\psi(x)| \leqq \varrho(x)$, $x \geqq x_0$ and $W_t = K * H_t$ we find

$$(2.6.17)\qquad |I_1(x)| \leqq \varrho(x)\|W_t\|_1 \leqq \varrho(x)\|H\|_1\|K\|_1, \quad x \geqq x_0\,.$$

From (2.6.14)-(2.6.17) it follows that if $x \geqq x_3$ then

$$(2.6.18)\qquad |\psi * W_t(x)| \leqq \big(\|H\|_1\|K\|_1 + C(P)M_1X^{-1}\|K\|_\infty b\big)\varrho(x)\,.$$

According to the assumptions (2.6.2) and (2.6.3) we thus have proved that if $x \geqq x_3$ and (2.6.7) is satisfied, then

$$|\psi * W_t(x)| \leqq C(P)M_1 S(X) b_\varrho(x) . \tag{2.6.19}$$

If we combine (2.6.19) with the estimate of $\psi * V_t(x)$ obtained from Lemma 2, and use (2.6.6) we have proved Lemma 2'.

The same generalization of Lemma 3 holds true but for the fact that M should be replaced by M_2 in (2.3.29). This follows by choosing

$$v(y) = (1+\theta y)^{c+2} q(2\theta y), \quad y \geqq 0 ,$$

and repeating the argument.

Let us now suppose that k satisfies the conditions in the definition of $\mathscr{B}_2$ instead of $\mathscr{B}_1$ for some $X = t^{-1}$. If $\hat{U}_t$ and $\hat{W}_t$ denote the Fourier transforms in the L^2-sense of U_t and $W_t = H_t * K$ respectively, then the above argument is still valid but for the estimation of $I_1(x)$. Using Schwarz' inequality we now obtain

$$|I_1(x)| \leqq \varrho_2(x) \|W_t\|_2 \leqq \varrho_2(x) \|H\|_1 \|K\|_2 \tag{2.6.20}$$

and (2.6.4) yields

$$|I_1(x) \leqq \|H\|_1 S(X) \varrho_2(x) . \tag{2.6.21}$$

Proof of Lemma 4'. Let us choose $X = t^{-1} \geqq X_0$ and let $\hat{U}_t = \hat{H}_t / \hat{F}$. Since $\hat{H}_t(\xi) = 0, |\xi| \geqq X$, we find from (2.6.1) that $\hat{U}_t = \hat{H}_t(f+k)$, where f and k satisfy the conditions in the definition of $\mathscr{B}_1$. The assumption $1/\hat{F} \in \mathscr{B}_1$ implies further that $\hat{F} \neq 0$ on $(-\infty, \infty)$ and therefore $\hat{U}_t$ is the Fourier transform of $U_t \in L^1(-\infty, \infty)$. Let $\psi = \Phi * F$. Then $\Phi * H_t = \psi * U_t$, the inversion being justified by absolute convergence. If we use Lemma 2' instead of Lemma 2 for the estimate of $\psi * U_t(x)$, then the result follows in the same way as in 2.4 and 2.5.

Lemma 5' is proved in the same way. In case $1/\hat{F} \in \mathscr{B}_2$ instead of $\mathscr{B}_1$ and $\hat{F} \neq 0$ on $(-\infty, \infty)$ the estimate (2.6.21) of $I_1(x)$ yields the term to be added in the inequalities obtained, and thus the result in the remark follows.

3. Estimation of Φ from the Tauberian condition and $I(x)$.

3.1. *Preliminaries.*

Introduce the class $\mathscr{E}$ of functions $t(x)$, $-\infty < x < \infty$, as follows

Definition. $t \in \mathscr{E}$ if $t \searrow, t > 0$, and for every $\varepsilon > 0$ there exist x_ε and $\delta_\varepsilon > 0$ such that

$$t(x-y) \leqq (1+\varepsilon) t(x), \quad x \geqq x_\varepsilon, 0 \leqq y \leqq \delta_\varepsilon . \tag{3.1.1}$$

I shall use the following Tauberian condition. For some $\lambda > 0$

$$\Phi(x)-\Phi(x+y) \leqq \sigma(x), \quad 0 \leqq y \leqq \lambda t(x),\ x \geqq x_0, \tag{3.1.2}$$

where $t \in \mathscr{E}$ and $0 < t < 1$.

The condition (3.1.2) can be modified in the following way. Let $\Delta_y^n \Phi(x)$ denote the n-th difference of Φ and let δ be a function such that $\delta(x) \to 0, x \to \infty$. It is easy to see that if (3.1.2) is replaced by the condition that for some $n \geqq 1$

$$(-1)^n \Delta_y^n \Phi(x) \leqq \sigma(x) + \delta(x)|\Phi(x)|, \quad 0 \leqq y \leqq \lambda t(x),\ x \geqq x_0, \tag{3.1.3}$$

then the same results will be obtained in Lemma 6 below and in the Tauberian theorems, but for the fact that the constants will depend also on n.

In this section other properties of the auxiliary function H are required than in the preceeding one. Let us suppose that

$$H \geqq 0, \quad \int_{-\infty}^{\infty} H(u)\,du = 1. \tag{3.1.4}$$

Let $0 < \tau(x) < 1$, $-\infty < x < \infty$. Introduce the notation

$$I_\tau(x) = \int_{-\infty}^{\infty} \Phi(x - u\tau(x)) H(u)\,du. \tag{3.1.5}$$

Let $v(x), -\infty < x < \infty$, satisfy

$$v \nearrow, \quad v(x) = 1,\ x < 0, \quad v(x) \to \infty,\ x \to \infty. \tag{3.1.6}$$

The class $R[v]$ is defined in 2.3.

3.2. *Estimation of $\Phi(x)$.*

The main result in this section is contained in the following two lemmas.

LEMMA 6_0.

(1) *Let Φ be bounded and satisfy* (3.1.2), *where $t \in \mathscr{E}$. Let H satisfy* (3.1.4) *and v* (3.1.6) *Let I_τ be defined by* (3.1.5) *and suppose that for some $\varkappa > 1$ and every τ, $\varkappa^{-1} t \leqq \tau \leqq \varkappa t$,*

$$|I_\tau(x)| \leqq \beta m(x),\ x \geqq x_1 \quad \text{and} \quad \sigma(x) \leqq \alpha m(x),\ x \geqq x_1. \tag{3.2.1}$$

(2) *Let $t(x) \to 0, x \to \infty$ and $vH \in L^1(-\infty, \infty)$.*
(3) *Let $m \in R[v]$.*
Then

$$\overline{\lim}_{x\to\infty} \frac{|\Phi(x)|}{m(x)} \leqq C_0\left(\frac{1}{\lambda}+1\right)\alpha + C\beta, \tag{3.2.2}$$

where $C_0 = C_0(v,H)$ and $C = C(v(0+))$.

REMARK. If Condition (2) is replaced by

$$(2') \qquad t(x) < 1,\ x > x_0, \quad v(|x|)v(x)H(x) \in L^1(-\infty,\infty)\,,$$

then (3.2.2) still holds true, with $C = C(v,H)$.

The condition $m \in R[v]$ implies that (3.2.2) cannot yield a better estimate than $\Phi(x) = O(1/v(x)), x \to \infty$. I shall apply the lemma with an auxiliary function H such that $\hat{H}$ has compact support. The assumption $vH \in L^1(0,\infty)$ then implies that v is of the convergence type, i.e.

$$\int^{\infty} x^{-2} \log v(x)\, dx < \infty\,.$$

In order to be able to get an estimate $\Phi(x) = O(1/k_1(x)), x \to \infty$, where k_1 is not of the convergence type, I also give the following formulation of the lemma, in which the condition $m \in R[v]$ is replaced by $m \in R[k]$, together with an additional condition for the function t.

If we choose $k = v$ in Lemma 6 below, Condition (3.2.3) follows from the assumption $t < 1, x > x_0$. Thus Lemma 6_0 is a special case of Lemma 6.

LEMMA 6. *Let Conditions* (1) *and* (2) *of Lemma* 6_0 *hold true. Let* $k\nearrow$, $x \geqq 0, k \geqq 1$, *and let* k^{-1} *and* v^{-1} *denote the inverse functions of* k *and* v *respectively. Let* $m \in R[k]$ *and let us suppose that for some* $u_0 \geqq 0$

$$(3.2.3) \qquad t(x) \leqq k^{-1}(v(u))/u, \quad u_0 \leqq u \leqq v^{-1}\left(\frac{1}{m(x)}\right), \quad x \geqq x_0\,.$$

Then (3.2.2) *holds true with* $C_0 = C_0(v, H, k(0+), u_0)$ *and* $C = C(k(0+))$. *If we further suppose that* m *belongs to* $\mathscr{E}$ *then for every* $\varepsilon > 0$

$$(3.2.4) \qquad \overline{\lim}_{x\to\infty} \frac{|\Phi(x)|}{m(x)} \leqq (1+\varepsilon)[(C_0\lambda^{-1}+1)\alpha + \beta]\,,$$

where $C_0 = C_0(v, H, u_0, \varepsilon)$.

For the sake of clearness, I formulate a detail of the proof as a separate lemma, in which I derive from the Tauberian condition (3.1.2) corresponding results for larger intervals.

LEMMA 7. *Let* Φ *be bounded and satisfy* (3.1.2). *Let* $\sigma\searrow, t\searrow, \sigma \geqq 0, t > 0$ *and let for some constants* a, b *and* B

$$(3.2.5) \qquad \sigma(x-y) \leqq B\sigma(x), \quad 0 \leqq y \leqq 2at(x),\ x \geqq x_0$$

and

$$(3.2.6) \qquad t(x) \leqq bt(x+y), \quad 0 \leqq y \leqq 2at(x),\ x \geqq x_0\,.$$

Then

$$(3.2.7)\qquad \Phi(x)-\Phi(x+y) \leqq \left(\frac{2ab}{\lambda}+1\right)\sigma(x), \quad 0\leqq y\leqq 2at(x),\ x\geqq x_0$$

and

$$(3.2.7)^*\ \Phi(x-y)-\Phi(x) \leqq B\left(\frac{2a}{\lambda}+1\right)\sigma(x), \quad 0\leqq y\leqq 2at(x),\ x\geqq x_0+2at(x_0).$$

PROOF. Let us choose $x=x_1\geqq x_0$ and $y=y_1, 0<y_1\leqq 2at(x_1)$. Let $x_{n+1}=x_n+\lambda t(x_n), n=1,2,\ldots$. Let N be an integer and $x_N<x_1+y_1\leqq x_{N+1}$. From (3.2.6) it follows that

$$N < \frac{by_1}{\lambda t(x_1)}+1 \leqq \frac{2ab}{\lambda}+1 .$$

If we add the inequalities obtained from (3.1.2) by choosing $x=x_n$, $y=\lambda t(x_n), n=1,2,\ldots N-1$ and $x=x_N$, $y=x_1+y_1-x_N\leqq\lambda t(x_N)$, we find

$$\Phi(x_1)-\Phi(x_1+y_1) \leqq \textstyle\sum_{n=1}^{N}\sigma(x_n) \leqq N\sigma(x_1) .$$

Combining this inequality with the above estimate of N we have proved (3.2.7). The inequality (3.2.7)* is proved in an analogous way.

PROOF OF LEMMA 6. I prove the result on the assumption that m and σ belong to $\mathscr{E}$. I choose $\eta, 0<\eta<\frac{1}{2}$ and $a=a(v,H,\eta,u_0)$ such that $a>u_0$ and

$$(3.2.8)\qquad \int_{|u|>a}v(u)H(u)\,du < \tfrac{1}{2}\eta .$$

Since $t(x)\to 0, x\to\infty$ and $t\in\mathscr{E}$ there is ξ_1 such that

$$\varkappa^{-1}t(\xi) \leqq t(\xi+at(x)) \leqq t(\xi-at(x)) \leqq \varkappa t(\xi), \quad \xi\geqq\xi_1, x\geqq\xi_1 .$$

Let $s(x)=v^{-1}(1/m(x))$. Then $s(x)\to\infty, x\to\infty$. Let us choose $\delta, 0<\delta\leqq 1$. Let us also choose $x_\delta'\geqq\max(x_0,x_1,\xi_1)$, such that $m(x-at(x))\leqq(1+\delta)m(x)$, $x\geqq x_\delta'$, such that

$$(3.2.9)\qquad \int_{|u|>s(x)}v(u)H(u)\,du \leqq \delta/\|\Phi\|_\infty, \quad x\geqq x_\delta'$$

and such that Conditions (3.2.5) and (3.2.6) of Lemma 7 are satisfied with $b=B=(1+\delta)^{\frac{1}{2}}$, if $x\geqq x_\delta'$. From Lemma 7 and the Tauberian condition (3.1.2), it follows that (3.2.7) is satisfied. Hence

$$(3.2.10)\qquad \Phi(x)-\Phi(x+y) \leqq (1+\delta)^{\frac{1}{2}}c\sigma(x), \quad 0\leqq y\leqq 2at(x),\ x\geqq x_\delta' ,$$

where

$$(3.2.11)\qquad c = 2a\lambda^{-1}+1 .$$

Let $x_\delta = x_\delta' + 2at(x_\delta')$. Let us choose $x, x \geqq x_\delta$ and let $t = t(x)$. Putting $y = ta - tu$ in (3.2.10), we have

$$\Phi(x) \leqq \Phi(x+ta-tu) + (1+\delta)^{\frac{1}{2}} c\sigma(x), \qquad -a \leqq u \leqq a\,.$$

Let $\eta_1 = \int_{|u|>a} H(u)\,du$. Then $\eta_1 < \eta$ according to (3.2.8). Multiplying the above inequality by $H(u)$ and integrating over the interval $-a \leqq u \leqq a$ we find

$$(3.2.12) \qquad (1-\eta_1)\Phi(x) \leqq \int_{-a}^{a} \Phi(x+ta-tu)H(u)\,du + (1+\delta)^{\frac{1}{2}} c\sigma(x)\,.$$

In an analogous way we obtain, by using (3.2.7)*

$$(3.2.13) \qquad (1-\eta_1)\Phi(x) \geqq \int_{-a}^{a} \Phi(x-ta-tu)H(u)\,du - (1+\delta)^{\frac{1}{2}} c\sigma(x)\,.$$

Let, for $n = 0, 1$,

$$\tau_n(\xi) = t(\xi - (-1)^n at(x)), \qquad \xi \geqq \xi_1\,.$$

Then $\varkappa^{-1} t(\xi) \leqq \tau_n(\xi) \leqq \varkappa t(\xi), \xi \geqq \xi_1$ by our choice of ξ_1. Furthermore

$$I_{\tau_n}(x + (-1)^n at) = \int_{-\infty}^{\infty} \Phi(x + (-1)^n at - ut)H(u)\,du$$
$$= \int_{-a}^{a} + \int_{a<|u|<s(x)} + \int_{|u|>s(x)}\,.$$

Since $v \nearrow$ and $v(s(x)) = 1/m(x)$ it follows that

$$\left| \int_{|u|>s(x)} \Phi(x + (-1)^n at - ut)H(u)\,du \right| \leqq \frac{\|\Phi\|_\infty}{v(s(x))} \int_{|u|>s(x)} v(u)H(u)\,du \leqq \delta m(x)\,,$$

the last inequality by using (3.2.9).

Let

$$J_n(x) = \left| \int_{a<|u|<s(x)} \Phi(x + (-1)^n at(x) - ut(x))H(u)\,du \right|$$

and

$$J = \max(J_0, J_1)\,.$$

Then

$$(3.2.14) \qquad \left|\int_{-a}^{a} \Phi(x + (-1)^n at - ut)H(u)\,du\right| \leqq$$
$$\leqq |I_{\tau_n}(x + (-1)^n at)| + J(x) + \delta m(x)\,.$$

Combining the inequalities (3.2.12), (3.2.13) and (3.2.14) and using the assumption (3.2.1) and the choice of x_δ we have proved

$$(3.2.15) \qquad (1-\eta)|\Phi(x)| \leqq m(x)[(1+\delta)(c\alpha+\beta)+\delta] + J(x), \qquad x \geqq x_\delta\,.$$

I shall use this inequality for the estimation of $\overline{\lim}_{x\to\infty}(|\Phi(x)|/m(x))$. To this end, I introduce a monotonous majorant μ of $|\Phi|$ in the following way. Let

$$\nu(x) = \sup_{y \geqq x} |\Phi(y)|, \qquad \mu(x) = m(x) \sup_{y \leqq x} (\nu(y)/m(y))\,.$$

It is easy to verify that $\mu \geqq |\Phi|$ and

$$\mu \searrow , \tag{3.2.16}$$

$$\mu/m \nearrow , \tag{3.2.17}$$

and that either μ/m is bounded or

$$\overline{\lim}_{x\to\infty}(|\Phi(x)|/\mu(x)) = 1 . \tag{3.2.18}$$

Now, by (3.2.16) and (3.2.17),

$$\mu(x \pm at(x) - ut(x))m(x) \leqq \begin{cases} \mu(x)m(x-at(x)-ut(x)), & u \geqq 0 , \\ \mu(x)m(x-at(x)) , & u \leqq 0 . \end{cases}$$

Using $m \in R[k]$ and the choice of x_δ, we find that, if $x \geqq x_\delta$,

$$\mu(x \pm at(x) - ut(x)) \leqq \begin{cases} (1+\delta)\mu(x)k(ut(x)) , & u \geqq 0 , \\ (1+\delta)\mu(x) , & u \leqq 0 . \end{cases}$$

By the assumption (3.2.3), $k(ut(x)) \leqq v(u)$, $a \leqq u \leqq s(x)$. Hence

$$J(x) \leqq \mu(x)(1+\delta) \int_{a<|u|<s(x)} v(u)H(u)\,du \leqq \eta\mu(x), \quad x \geqq x_\delta , \tag{3.2.19}$$

the last inequality by the choice of a. Combining (3.2.15) and (3.2.19) we get

$$(1-\eta)|\Phi(x)| \leqq [(1+\delta)(c\alpha+\beta)+\delta]m(x)+\eta\mu(x), \quad x \geqq x_\delta . \tag{3.2.20}$$

If (3.2.18) holds true we find, by dividing both sides of the above inequality by $\mu(x)$ and taking the upper limit when $x \to \infty$, that

$$\overline{\lim}_{x\to\infty} \frac{m(x)}{\mu(x)} \geqq \frac{1-2\eta}{(1+\delta)(c\alpha+\beta)+\delta}.$$

Since δ was an arbitrary positive number, we have proved

$$\underline{\lim}_{x\to\infty} \frac{\mu(x)}{m(x)} \leqq \frac{c\alpha+\beta}{1-2\eta}. \tag{3.2.21}$$

Now μ/m is non-decreasing according to (3.2.17). Therefore (3.2.21) implies that $\lim_{x\to\infty}(\mu(x)/m(x))$ exists and is $\leqq (c\alpha+\beta)(1-2\eta)^{-1}$. Since μ was a majorant of $|\Phi|$, we have proved that

$$\overline{\lim}_{x\to\infty} \frac{|\Phi(x)|}{m(x)} \leqq \frac{c\alpha+\beta}{1-2\eta}. \tag{3.2.22}$$

By (3.2.11), $c = 2a\lambda^{-1}+1$. The result (3.2.4) therefore follows from (3.2.22), through the choice $(1-2\eta)^{-1} = 1+\varepsilon$.

If μ/m is bounded then $\overline{\lim}_{x\to\infty}\Phi(x)/m(x) = K < \infty$. It is easy to see that this implies that

$$\overline{\lim}_{x\to\infty} J(x)/m(x) \leqq \eta K .$$

The result then follows directly from (3.2.15).

The other results in Lemma 6_0 and Lemma 6 follow in an analogous way. I omit the details.

4. Tauberian theorems.

4.1. *Preliminaries.*

I introduce the sequence (P_n), the functions h_P and h_Q and the sequence of functions (S_n), the integer μ and the functions S and $\bar{S}_m$ as in 2.1. Thus $S=(S_0S_1)^{\frac{1}{2}}$, $\bar{S}_m=\sup_{n\geqq m}S_n$ and

$$(4.1.1) \qquad S_{j-1}(X) \leqq \tfrac{3}{2}XS_j(X), \quad j=1,2,\ldots\mu\,.$$

The class $R[v]$ is defined in 2.3 and the class $\mathscr{E}$ in 3.1.

For the sake of reference the theorems are stated for the condition that $1/\hat{F}\in\mathscr{B}_1((P_n),(S_n))$, where $\mathscr{B}_1$ denotes the class introduced in 2.6. In the applications in this paper, however, I shall only use the fact that the theorems hold true when $1/\hat{F}(\xi)=f(\xi), -\infty<\xi<\infty$, where

$$(4.1.2) \qquad M_2\{f^{(n)};\ -X,X\} \leqq P_nS_n(X), \quad n=0,1,2,\ldots,\ X\geqq X_0.$$

In 4.2 I use the Tauberian condition introduced in 3.1. Thus I suppose that for some positive constant λ,

$$(4.1.3) \qquad \Phi(x)-\Phi(x+y) \leqq \sigma(x), \quad 0\leqq y\leqq \lambda t(x),\ x\geqq x_0\,,$$

where $t\in\mathscr{E}$ and $0<t<1$. In 4.3 I specialize the results for the Tauberian condition, which corresponds to the 'classical' condition (1.6). Finally, in 4.4, I state the results obtained when $1/\hat{F}\in\mathscr{B}_2$.

4.2. *Tauberian theorems with a general Tauberian condition.*

I shall prove the following theorems.

THEOREM 1. (1) *Let* $F\in L^1(-\infty,\infty)$ *and* $1/\hat{F}(\xi)\in\mathscr{B}_1((P_n),(S_n))$. *Let* $\bar{S}_1\leqq S$. *Let* Φ *be bounded on* $(-\infty,\infty)$ *and*

$$(4.2.1) \qquad |\Phi*F(x)| \leqq \varrho(x), \quad x\geqq x_0\,.$$

(2) *Let* Φ *satisfy* (4.1.3) *where* $t\in\mathscr{E}$.

(3) *Let* ϱ *and* m *belong to* $R[bw]$ *for some* $b\geqq 1$.

(4) *Let* a *and* α *be constants, and*

$$(4.2.2) \qquad \varrho(x)S(1/t(x)) \leqq am(x), \quad \sigma(x) \leqq \alpha m(x), \quad x\geqq x_0\,.$$

(5) *Let* θ *be constant,* $0<\theta<1$. *Let os suppose either that*

$$(4.2.3) \qquad w(x) \leqq h_P(\theta x), \quad x\geqq 0\,,$$

$$(4.2.4)\qquad w(x)/h_P(\theta x) \in L^2(0,\infty)$$

and

$$(4.2.5)\qquad \overline{\lim}_{x\to\infty} t(x)\varphi(x) < 1$$

or that (4.2.3) *and* (4.2.4) *hold true with* h_P *replaced by* h_Q.

Then

$$(4.2.6)\qquad \overline{\lim}_{x\to\infty} \frac{|\Phi(x)|}{m(x)} \leqq C_0(\lambda^{-1}+1)\alpha + Ca ,$$

where $C_0 = C_0(P,\varphi,b)$ *and* $C = C(P,\varphi,b,\theta,w)$.

Corollary. (1) *Let Conditions* (1)-(4) *of Theorem* 1 *hold true and suppose further*

$$(4.2.7)\qquad \log(P_{n+1}/P_n) = o(n), \quad n\to\infty .$$

(2) *Let us suppose either that* (4.2.3) *and* (4.2.5) *hold true or that* (4.2.3) *holds true with* h_P *replaced by* h_Q.

Then (4.2.6) *is valid with* $C = C(P,\varphi,b,\theta)$.

Theorem 2. (1) *Let Conditions* (1)-(4) *of Theorem* 1 *hold true.*

(2) *Let* (4.2.7) *be satisfied. Let* μ *denote the integer introduced in* (4.1.1) *and let* c, β *and* θ *be constants,* $c \geqq 0, \beta > \overline{\lim}_{x\to\infty} x^{-1}\chi_P(x)$ *and* $\theta \geqq 1$.

(3) *Let*

$$(4.2.8)\qquad \bar{S}_\mu\left(\frac{1}{t(x)}\right) \leqq x^{-c-2}\exp(-x\beta\theta^2\log\theta) S\left(\frac{1}{t(x)}\right), \quad x \geqq x_0 .$$

(4) *Let*

$$(4.2.9)\qquad w(x) \leqq (1+\theta x)^c h_P(\theta x), \quad x \geqq 0 .$$

Then (4.2.6) *is valid with* $C_0 = C_0(P,b,c)$ *and* $C = C(P,b,c,\mu,\theta)$.

If $\lim_{x\to\infty} t(x) = t_1 > 0$ it is sufficient to impose the conditions in the definition of $\mathscr{B}_1$ for $X_0 \leqq X \leqq t_1^{-1}$. If $t(x)\to 0$, $x\to\infty$, and $\bar{S}_1(X) = o(S(X))$, $X\to\infty$, then C is independent of θ, μ and w. If $t(x)\to 0$, $x\to\infty$ and $\chi_S(X)\to\infty, X\to\infty$, we may choose $C=0$ in Theorem 1 and its Corollary. If $t(x)\varphi(x)\to 0, x\to\infty$, $\chi_S\nearrow$, and for some $c\geqq 0$

$$(4.2.10)\qquad \chi_S\left(\frac{1}{e^{c+2}t(x)}\right) \geqq \log x, \quad x \geqq x_0,$$

then Condition (2) of the Corollary can be replaced by $w(x) \leqq (1+x)^c h_P(x)$, $x \geqq 0$, and (4.2.6) is valid with $C_0 = C_0(P,\varphi,b,c)$ and $C = 0$.

Theorem 2 can be applied only if $t(x)$ is sufficiently small. In fact, the condition (4.2.8) implies, according to (4.1.1), that

$$t(x)^{\mu-\frac{1}{2}} \leqq C(\mu)x^{-c-2}\exp(-x\beta\theta^2\log\theta), \quad x \geqq x_0 , \tag{4.2.11}$$

where $C(\mu)=3^{\mu-\frac{1}{2}}2^{-\mu+\frac{1}{2}}$.

PROOF OF THEOREM 1. The theorem will be proved on the assumption that $t(x) \to 0, x \to \infty$. Let H denote a function such that

$$(1+|x|^2)h_Q(2|x|)H(x) \in L^1(-\infty,\infty) ,$$

$\hat{H}(\xi)=0, |\xi| \geqq 1$, and such that $H \geqq 0$ and $\int_{-\infty}^{\infty} H(u)\,du=1$. Such a function certainly exists since

$$\int^{\infty} x^{-2}\log h_Q(x)\,dx < \infty .$$

Choose $\varkappa, 1<\varkappa\leqq 2$, such that $\overline{\lim}_{x\to\infty} t(x)\varphi(x)<\varkappa^{-2}$. Let I_τ be defined by (3.1.5). From Lemma 4′ and the assumption (4.2.2) it follows that there is ξ such that for every τ, $t\leqq\tau\leqq\varkappa^2 t$,

$$|I_\tau(x)| \leqq C_3abm(x), \quad x \geqq \xi , \tag{4.2.12}$$

where $C_3=C_3(P,\varphi,\theta,w,H)$. Let us now apply Lemma 6 with t replaced by $t_1=\varkappa t$ and λ replaced by $\lambda_1=\lambda\varkappa^{-1}$. The condition (3.2.1) of Lemma 6_0 is satisfied with $\beta=C_3ab$ according to (4.2.12). If (P_n) is non-quasi-analytic, (4.2.6) follows from Lemma 6_0 applied with $v(x)=bh_P(x), x\geqq 0$. If (P_n) is quasi-analytic let $v(x)=bh_Q(2x), x\geqq 0$, and $k(x)=bh_P(x), x\geqq 0$. Using (2.1.15), (2.1.21) and the assumption (4.2.5) it is easy to see that Condition (3.2.3) of Lemma 6 is satisfied with $u_0=u_0(P,\varphi)$. The result (4.2.6) then follows from Lemma 6.

The corollary follows in the same way as the corollary of Lemma 2 by applying Theorem 1 with θ replaced by $\theta^{\frac{1}{2}}$. The remaining results of Theorems 1 and 2 are proved similarly by an appropriate choice of the functions v and k in Lemma 6 and by using the remark following Lemma 6_0. Theorem 2 may be proved by using the function φ defined in (2.1.16) since (4.2.11) implies that (4.2.5) is satisfied.

The above remarks are obvious consequences of the corresponding results in Lemma 4′ and Lemma 5′, except for the statements involving χ_S, which I shall now prove.

Let us suppose that $\chi_S(\xi) \to \infty, \xi \to \infty$. Choose $\delta, 0<\delta\leqq 1$, such that

$$\overline{\lim}_{x\to\infty} t(x)\varphi(x) < (1+\delta)^{-1} .$$

Let $\tau(x)=(1+\delta)t(x)$. From the assumption $\varrho(x)S(1/t(x))=O(m(x)), x \to \infty$ and $\chi_S(\xi) \to \infty, \xi \to \infty$, it follows that

$$\varrho(x)S(1/\tau(x)) = o(m(x)), \quad x \to \infty .$$

If we apply Theorem 1 with t replaced by τ and λ replaced by $\lambda/(1+\delta)$, we find that (4.2.6) is valid with $C=0$.

The result under assumption (4.2.10) is proved similarly by an application of Lemma 5′ and Lemma 6 with t replaced by $2e^{c+2}t$, and λ replaced by $2^{-1}e^{-c-2}\lambda$.

4.3. *Tauberian theorems with the classical Tauberian condition.*

I shall now derive the results obtained for the Tauberian condition (1.6). Let

$$T(X) = XS(X) \tag{4.3.1}$$

and let T^{-1} denote the inverse function of T. Introduce the function $\tau=\tau_{S,\varrho}$ as follows

$$\tau(x) = \frac{1}{T^{-1}(1/\varrho(x))}. \tag{4.3.2}$$

Then $\tau\searrow$, $\tau(x)\to 0$, $x\to\infty$, and

$$\varrho(x)S(1/\tau(x)) = \varrho(x)\tau(x)T(1/\tau(x)) = \tau(x) . \tag{4.3.3}$$

It is easy to see that $\varrho \in \mathscr{E}$ implies that $\tau \in \mathscr{E}$ and that $\varrho \in R[k]$ implies that $\tau \in R[k]$. We further observe that if, for some $\delta>0$,

$$\overline{\lim}_{x\to\infty}\varrho(x)T((1+\delta)\varphi(x)) < 1 \tag{4.3.4}$$

then (4.2.5) is satisfied with $t=\tau$.

Let Φ satisfy the following Tauberian condition. Let for some positive constants K and λ

$$\Phi(x)-\Phi(x+y) \leqq K\lambda\tau(x), \quad 0\leqq y\leqq \lambda\tau(x),\ x\geqq x_0 . \tag{4.3.5}$$

From (4.3.5) it follows that the Tauberian condition (4.1.3) of Theorem 1 is satisfied with $t=\tau$ and $\sigma=K\lambda\tau$. If we apply Theorem 1 with $t=\tau=m$ and $\sigma=K\lambda\tau$ and observe that the first inequality in (4.2.2) is satisfied with $a=1$ according to (4.3.3) and the second one is satisfied with $\alpha=K\lambda$ we thus get the following result.

THEOREM 3_0. (1) *Let Condition* (1) *of Theorem* 1 *hold true and let* $\varrho \in \mathscr{E}$. *Let* τ *defined by* (4.3.2) *and let* (4.3.5) *be satisfied.*

(2) *Let* θ *be constant,* $0<\theta<1$, *and let* $\varrho \in R[bw]$ *for some* $b\geqq 1$.

(3) *Let us suppose either that* (4.2.3), (4.2.4) *and* (4.3.4) *are satisfied or that* (4.2.3) *and* (4.2.4) *are satisfied with* h_P *replaced by* h_Q.

Then

(4.3.6) $$\overline{\lim}_{x\to\infty} \frac{|\Phi(x)|}{\tau(x)} \leqq C_0 K(1+\lambda)+C\,,$$

where $C_0 = C_0(P, \varphi, b)$ *and* $C = C(P, \varphi, b, \theta, w)$.

COROLLARY. *Impose the conditions of Theorem* 3_0 *except those involving* (4.2.4). *Let* (4.2.7) *hold true. Then* (4.3.6) *is valid with* $C = C(P, \varphi, b, \theta)$.

We shall now prove that the condition (4.3.4) in Theorem 3_0 and its corollary can be omitted if (P_n), ϱ and S are sufficiently regular. For the sake of simplicity we introduce the following conditions. It is easy to see that they may be considerably weakened.

Let us call (P_n), ϱ and S *regular* if the following conditions are satisfied.

(4.3.7) $$\lim_{n\to\infty} \frac{\log n}{\log (P_{n+1}/P_n)} = \nu\,.$$

Let $r = 1/\varrho$ and

(4.3.8) $$\lim_{x\to\infty} \frac{\log \chi_r(x)}{\log x} = \omega$$

(4.3.9) $$\lim_{x\to\infty} \frac{\chi_S(x)}{(\log S(x))(\log\log S(x))} = \varkappa \leqq \infty$$

Condition (4.3.7) implies that p is of regular growth and of order v, i.e.

$$\lim_{x\to\infty} \frac{\log\log p(x)}{\log x} = \nu$$

Condition (4.3.8) implies that r is of regular growth and of order ω.

By using these assumptions Theorems 3_0 and its corollary may be modified as follows.

THEOREM 3′. *Let* (P_n), ϱ *and* S *be regular. Let Conditions* (1) *and* (2) *of Theorem* 3_0 *hold true and let* (4.2.3) *and* (4.2.4) *be satisfied. Then* (4.3.6) *is valid with* $C_0 = C_0(P, b)$ *and* $C = C(P, b, \theta, w)$.

THEOREM 3. (1) *Let* (P_n), ϱ *and* S *be regular, let Condition* (1) *of Theorem* 3_0 *hold true and let* (4.2.7) *be satisfied.*

(2) *Let* θ *be constant,* $0 < \theta < 1$, *and* $\varrho \in R[bh_P(\theta x)]$ *for some* $b \geqq 1$.

Then (4.3.6) *is valid with* $C_0 = C_0(P, b,)$ *and* $C = C(P, b, \theta)$.

PROOF. If (P_n) is non-quasi-analytic then $\varphi \equiv 1$ and the results follow from Theorem 3_0. If (P_n) is quasi-analytic then $\nu = 1$ in (4.3.7) and

(4.2.7) is satisfied. Thus it suffices to prove Theorem 3 and suppose (P_n) quasi-analytic. Let φ be defined by (2.1.16). Then for every $\eta < 1$, $h_Q(x)\exp(-x^\eta) \to \infty$, $x \to \infty$.

Introduce ω as in (4.3.8) and $\varkappa$ as in (4.3.9). The proof will be different for different values of ω and $\varkappa$ and we consider the cases a), b) and c) below. It should be observed that in a) and b) the result (4.3.6) is derived without use of Condition (2).

a) $$\omega \leqq \tfrac{1}{2}\,.$$

Let $v(x) = \exp(x^{\frac{2}{3}})$. Then $v(x) \leqq b h_Q(\frac{1}{2}x), x \geqq 0$, for some $b = b(P)$ and $v(x)/h_Q(\frac{1}{2}x) \in L^2(0,\infty)$. Since $\omega \leqq \frac{1}{2}$ there exists x_1 such that $\chi_r(x) \leqq \frac{2}{3}x^{\frac{2}{3}}$, $x \geqq x_1$. Let $\varrho(x) = \varrho(x_1), x \leqq x_1$. The function ϱ, thus redefined, belongs to $R[v]$. It then follows from Theorem 3_0, applied with $\theta = \frac{1}{2}$, $w = v/b$, that (4.3.6) holds true with $C_0 = C_0(P)$ and $C = C(P)$.

b) $$\omega > \tfrac{1}{2}, \quad \varkappa \geqq 1\,.$$

Let $\varrho_0(x) = \exp(-x^{\frac{1}{2}})$ and let τ_0 be defined by (4.3.2) with ϱ replaced by ϱ_0. Then $\varrho(x) \leqq \varrho_0(x), x \geqq x_1$ and $\tau(x) \leqq \tau_0(x), x \geqq x_1$. Therefore, by the argument used in Lemma 7, Condition (1) is satisfied with ϱ replaced by ϱ_0 and K replaced by $3K$. By using (4.3.9) and $\varkappa \geqq 1$ it is easy to see that

$$\overline{\lim}_{x\to\infty} \tau_0(x)/\tau(x) \leqq 2\,.$$

From the result proved in a) it follows that (4.3.6) holds true with τ replaced by τ_0. Thus (4.3.6) holds true with $C_0 = C_0(P)$ and $C = C(P)$.

c) $$\omega > \tfrac{1}{2}, \quad \varkappa < 1\,.$$

From $\omega > \frac{1}{2}$ it follows that $\varrho(x) = o(\exp(-x^{\frac{1}{2}})), x \to \infty$, and from $\varkappa < 1$ it follows that

$$T(X)\exp(-e^{X^\alpha}) \to 0, \qquad X \to \infty$$

for every α, $\varkappa < \alpha < 1$. By combining these relations with (2.1.14) we find that (4.3.4) is satisfied for every $\delta > 0$. The result (4.3.6) then follows from the corollary of Theorem 3_0.

The remarks following Theorem 2 are valid for the above theorems as well. Thus, if $\chi_S(X) \to \infty$ as $X \to \infty$ then (4.3.6) holds true with $C = 0$. If $\chi_S \nearrow$,

$$\tau(x) = o((\log x)(\log\log x)^{1+\delta}), \qquad x \to \infty$$

for some $\delta > 0$ and (4.2.10) is satisfied with $t = \tau$ then Condition (2) of Theorem 3 can be replaced by $\varrho \in R[b(1+x)^c h_P(x)]$ and (4.3.6) is valid with $C_0 = C_0(P,b,c,\delta)$ and $C = 0$.

The restriction $\theta<1$ imposed in the above theorems has no real importance if S increases fast enough. This is a consequence of the following result which we shall now prove.

Let for some $A>0$

$$\log S(X) \leqq A\chi_S(X), \quad X \geqq X_0 . \tag{4.3.10}$$

Then the condition $0<\theta<1$ in Theorems 3_0, $3'$ and 3 can be replaced by $\theta \geqq 1$ and the same result holds true but for the fact that C_0 will depend also on θ and A.

Let n be a positive integer, $n \geqq 2\theta > n-1$. Let us first observe that $\varrho \in R[v(x)]$ implies that $\varrho^{1/n} \in R[v(x/n)]$ and that (4.3.10) implies that, for every $\delta>0$,

$$T^{-1}(y) \leqq n^{A+\delta} T^{-1}(y^{1/n}), \quad y \geqq y_\delta .$$

The result then follows in the same way as in b) by applying the theorems with ϱ replaced by $\varrho^{1/n}$.

The following theorem is derived from Theorem 2 in the same way as Theorem 3_0 was derived from Theorem 1.

THEOREM 4. *Let Condition* (1) *of Theorem* 3_0 *and Condition* (2) *of Theorem* 2 *hold true and suppose further*

$$\bar{S}_\mu\left(\frac{1}{\tau(x)}\right) \leqq x^{-c-2} \exp(-x\beta\theta^2 \log\theta) S\left(\frac{1}{\tau(x)}\right), \quad x \geqq x_0 . \tag{4.3.11}$$

Let $\varrho \in R[b(1+\theta x)^c h_P(\theta x)]$ *for some* $b \geqq 1$. *Then* (4.3.6) *is valid with* $C_0 = C_0(P,b,c)$ *and* $C = C(P,b,c,\mu,\theta)$.

Theorem 4 can be applied only if S and ϱ are sufficiently small. In fact, if (4.3.11) is satisfied for some $\theta>1$ then S is dominated by a polynomial and ϱ is exponentially decreasing. If (4.3.11) is satisfied with $\theta=1$ then

$$\log\log S(X) = O(\log X), \quad X \to \infty ,$$

and

$$\overline{\lim}_{x\to\infty} (\log x)^{-1} \log \varrho(x) < 0 .$$

To see this we observe that (4.3.11) implies that (4.2.11) is satisfied with $t=\tau$. Therefore, by the definition of τ,

$$x^{c+2} \exp(x\beta\theta^2 \log\theta) \leqq C(\mu) \left[T^{-1}\left(\frac{1}{\varrho(x)}\right)\right]^{\mu-\frac{1}{2}}, \quad x \geqq x_0 . \tag{4.3.12}$$

Since $T(X)=XS(X)$ the result stated follows from (4.3.12) and the inequality $1/\varrho(x) \leqq e^{\beta x}, x \geqq x_1$.

Let us finally note the following concerning the theorems in 4.3. If $\chi_S(X) \to \infty, X \to \infty$ then the condition $\varrho \in \mathscr{E}$ can be omitted. If $\bar{S}_1(X) = o(S(X)), X \to \infty$ then C is independent of θ, μ and w. If $\chi_S(X) \to \infty$, $X \to \infty$, and (4.2.8) holds true for $\tau \leqq t \leqq B\tau$ for some $B > 1$ then $C=0$ in Theorem 4.

4.4. *Tauberian theorems when* $1/\hat{F}$ *belongs to* $\mathscr{B}_2$.

The result below is stated here for the sake of reference. If is easily proved by using the remark to Lemma 4′ and Lemma 5′ in 2.6.

Remark. Impose the conditions of any of the theorems in 4.2 or 4.3 but for the fact that the condition $1/\hat{F} \in \mathscr{B}_1$ is replaced by the condition that $1/\hat{F} \in \mathscr{B}_2$ and $\hat{F} \neq 0$ on $(-\infty, \infty)$. Let $\psi = \Phi * F$ and suppose further

$$M_2\{\psi; x, \infty\} \leqq \varrho(x), \quad x \geqq x_0 . \tag{4.4.1}$$

Then the same result is valid.

5. Examples and applications.

5.1. *Assumptions.*

Let $F \in L^1(-\infty, \infty)$ and

$$1/\hat{F}(\xi) = g(\xi), \quad -\infty < \xi < \infty . \tag{5.1.1}$$

Let Φ be bounded on $(-\infty, \infty)$ and

$$|\Phi * F(x)| < \varrho(x), \quad x \geqq x_0 , \tag{5.1.2}$$

where $\varrho \in \mathscr{E}$ and ϱ is regular in the sense of (4.3.8). These assumptions are maintained throughout this section.

For the sake of brevity I use the following Tauberian condition

$$\Phi(x) + Kx \nearrow, \quad x \geqq x_0 \quad \text{for some } K \geqq 0 . \tag{5.1.3}$$

It may, of course, always be relaxed in the sense of (4.3.5).

5.2. *Conditions on* $1/\hat{F}$ *on the real axis.*

Two examples will be considered in which conditions are imposed on the first m derivatives of g.

EXAMPLE 1. Let a be a positive number and m a positive integer. Let $g^{(m)}$ exists on the real axis and $g'(\xi)=O(|\xi|^{a-1}), |\xi| \to \infty$. If $m>1$ we also suppose that

$$g^{(n)}(\xi) = O(|\xi|^{a-\frac{1}{2}}), \quad |\xi| \to \infty,\ n=2,3,\ldots,m .$$

Let Φ satisfy (5.1.3) and let $\varrho \in R[(1+x)^c]$. If $0<c<m-1+b$, where $b=\min(\frac{1}{2},a)$, then

$$\Phi(x) = O(\varrho(x)^{1/(a+1)}), \quad x \to \infty . \tag{5.2.1}$$

PROOF. Let $v(x)=(1+x)^c$. Since $h_P=h_Q$ is a polynomial of degree m the function $v(x)/h_Q(\frac{1}{2}x), x \geqq 0$, is bounded and belongs to $L^2(0,\infty)$. If $a>\frac{1}{2}$, $b=\frac{1}{2}$ the result follows directly from Theorem 3_0 by choosing, apart from constant factors, $S_0(X)=X^{a+\frac{1}{2}}$, $S_1(X)=X^{a-\frac{1}{2}}$, $\bar{S}_1(X)=X^a$, $S(X)=X^a, T(X)=X^{a+1}$ and $\tau(x)=\varrho(x)^{1/(a+1)}$. If $0<a\leqq \frac{1}{2}$ the result follows similarly by using the L^s-estimate in the remark to Lemma 1 and choosing $1<s<(1-a)^{-1}$.

It follows from a theorem of Ganelius [7, Theorem 4.2.1, p. 34] that the estimate (5.2.1) is best possible in the sense that it cannot be replaced by

$$\Phi(x) = O(\delta(x)\varrho(x)^{1/(a+1)}), \quad x \to \infty ,$$

for any function δ such that $\delta(x) \to 0, x \to \infty$.

Another application of Theorem 3_0 yields the following result.

EXAMPLE 2. Let m be a positive integer. Let $g^{(m)}$ exist on the real axis and

$$g^{(n)}(\xi) = O(M_0(|\xi|)), \quad |\xi| \to \infty,\ n=0,1,2,\ldots,m , \tag{5.2.2}$$

where $M_0(\xi)\nearrow, \xi \geqq 0$. Let $T(X)=X^{\frac{3}{2}}M_0(X)$ and let T^{-1} denote the inverse function of T. Let Φ satisfy (5.1.3). If $\varrho \in R[(1+x)^c]$ for some c, $0<c<m-\frac{1}{2}$, then

$$\Phi(x) = O\big(1/T^{-1}(1/\varrho(x))\big), \quad x \to \infty . \tag{5.2.3}$$

If (5.2.2) is valid with $M_0(\xi)=\xi^a$ for some $a>0$ we thus get

$$\Phi(x) = O(\varrho(x)^{2/(2a+3)}), \quad x \to \infty \tag{5.2.4}$$

and if $M_0(\xi)=\exp(\xi^a)$ for some $a>0$, and $\varrho(x)=e^{-V(x)}$ we get

$$\Phi(x) = O(V(x)^{-1/a}), \quad x \to \infty . \tag{5.2.5}$$

By using the above-mentioned theorem of Ganelius it is easy to see that if

(5.2.6) $$\overline{\lim}_{\xi\to\infty} \xi^{-2} \log T(\xi) < \infty$$
and
(5.2.7) $$\underline{\lim}_{\xi\to\infty} \chi_T(\xi)(\log \xi)^{-1} > 0$$

then the estimate (5.2.3) is best possible in the sense that it cannot be replaced by

$$\Phi(x) = O\big(\delta(x)/T^{-1}(1/\varrho(x))\big), \quad x \to \infty$$

for any function δ such that $\delta(x) \to 0, x \to \infty$.

5.3. *The function* $1/\hat{F}$ *analytic in a strip.*

Let us now consider the case in which g is analytic in a strip around the real axis in the ζ-plane, $\zeta = \xi + i\eta$.

Let $M_0(\xi)\nearrow, \xi \geqq 0$, and let M_0 be regular in the sense of (4.3.9). Introduce the condition

(5.3.1) $$\begin{cases} g \text{ is analytic in the strip } -\gamma < \eta < \gamma\,, \\ |g(\xi + i\eta)| \leqq M_0(|\xi|), \quad -\gamma < \eta < \gamma\,. \end{cases}$$

From (5.3.1) it follows, by Cauchy's formula, that

$$|g^{(n)}(\xi)| \leqq M_0(|\xi| + \gamma)\gamma^{-n} n!, \quad n = 0, 1, 2, \ldots .$$

Hence, for $X \geqq 0$,

(5.3.2) $$M_2\{g^{(n)}; -X, X\} \leqq P_n S_0(X), \quad n = 0, 1, 2, \ldots,$$

where $S_0(X) = 2^{1/2} X^{1/2} M_0(X+\gamma)$ and $P_n = n!\, \gamma^{-n}$. The theorems in Section 4 can be applied with $h_P(x) = e^{\gamma x}$ and $S_n = S_0, n = 0, 1, 2, \ldots$. Applying Theorem 3 we get the following result.

Example 3. Let g satisfy (5.3.1) and Φ (5.1.3). Introduce the function T^{-1} as in Example 2. If $\varrho \in R[e^{\theta\gamma x}]$ for some θ, $0 < \theta < 1$, then (5.2.3) is valid.

This result was proved by Frennemo [3, Theorem 1, p. 80] in the case when M_0 is submultiplicative and thus of at most exponential growth and $1/\varrho$ is submultiplicative.

Let $M_1(\xi), \xi \geqq 0$, be regular in the sense of (4.3.9), $M_1 \leqq M_0$ and $\xi^{\frac{1}{2}-\delta} M_1(\xi)\nearrow$ for some $\delta > 0$. Let g be analytic in $-\gamma < \eta < \gamma$ and

(5.3.3) $$|g'(\xi + i\eta)| \leqq M_1(|\xi|), \quad -\gamma < \eta < \gamma\,.$$

From (5.3.3) it follows, by Cauchy's formula that for $X \geqq 0$

$$M_2\{g^{(n)};-X,X\} \leqq P_n S_1(X), \quad n=1,2,\dots,$$

where $P_n=(n-1)!\gamma^{-n+1}$ and $S_1(X)=2^{\frac{1}{2}}X^{\frac{1}{2}}M_1(X+\gamma)$.

If (5.3.1) and (5.3.3) both hold true then the theorems of section 4 can be applied with $\bar{S}_1(X)=S_1(X)$ and $h_P(x)=1+xe^{\gamma x}$.

An application of Theorem 3 yields the following result.

Example 4. Let (5.3.1), (5.3.3) and (5.1.3) be satisfied. Let $T(X)=X^{3/2}(M_0(X)M_1(X))^{\frac{1}{2}}$ and let T^{-1} denote the inverse function of T. If $\varrho \in R[e^{\theta\gamma x}]$ for some θ, $0<\theta<1$, then

$$\Phi(x) = O\big(1/T^{-1}(1/\varrho(x))\big), \quad x \to \infty . \tag{5.3.4}$$

This result was proved by Frennemo [3, Theorem 2, p. 84] in the case when M_0, M_1 and $1/\varrho$ are submultiplicative and $M_1\nearrow$.

In the same way as before it follows that the estimate in Example 3 is best possible in the sense described above if (5.2.6) and (5.2.7) are satisfied and that the estimate in Example 4 is best possible in the same sense if (5.2.6) is satisfied and either (5.2.7) is satisfied or $XM_1(X)=O(M_0(X))$, $X\to\infty$.

The following example has been chosen to show that, with appropriate conditions on g' in the relevant strip, results can be derived for $\varrho(x)=e^{-\theta\gamma x}$ for some $\theta>1$. To this end I impose the following condition on ϱ

$$\lim_{x\to\infty} x^{-1}\log\varrho(x) = -\alpha . \tag{5.3.5}$$

Example 5. Let a be a positive constant and $b=\min(\frac{1}{2},a)$. Let (5.3.3) hold true with $M_1(\xi)=\mathrm{const}(1+|\xi|)^{a-1}$. Let Φ satisfy (5.1.3) and let $\varrho\in R[e^{\theta\gamma x}]$. Let us suppose either that $\theta<1$ or that

$$0 \leqq \theta\log\theta < b/(a+1) \tag{5.3.6}$$

and that (5.3.5) holds true with $\alpha=\theta\gamma$. Then (5.2.1) is valid.

Proof. The conditions imply that (5.3.1) holds true with $M_0(\xi)=C(1+|\xi|)^a$. If $a>\frac{1}{2}$ and $\theta<1$ the result thus is a special case of Example 4. Let us consider the case $a>\frac{1}{2}$ and $\theta\geqq 1$. We may choose S_0, S_1, S, T and τ as in Example 1 and let $\bar{S}_1=S_1=CX^{a-\frac{1}{2}}$ and $h_P(x)=e^{\gamma x}$. Since (5.3.5) holds true with $\alpha=\theta\gamma$ we find that for every $\delta>0$

$$\tau(x) \leqq \exp\left(-\frac{\theta\gamma x}{a+1}+\delta x\right), \quad x\geqq x_0 .$$

By using $\lim_{x\to\infty}x^{-1}\chi_p(x)=\gamma$ and $\bar{S}_1(X)\leqq CX^{-\frac{1}{2}}S(X)$ it is easy to see that we can find a number $\beta > \gamma$ such that (4.3.11) holds true with $\mu = 1$ and $c = 0$. The result then follows from Theorem 4. If $0 < a \leqq \frac{1}{2}$ the result is derived in the same way by using the L^s-estimate in the remark to Lemma 1.

The result in Example 5 for $\theta > 1$ is of interest in connection with the question of the necessity of the conditions. Earlier a theorem was proved (see [9, Theorem 6, p. 347]), which can be restated as follows.

Let $(1+|x|^{\delta+\frac{1}{2}})F(x) \in L^1(-\infty,\infty)$ for some $\delta > 0$ and let a and c be positive constants. If F has the property that $\Phi * F(x) = O(e^{-\alpha x})$, $x \to \infty$, implies $\Phi(x) = O(e^{-\alpha x/(a+1)})$, $x \to \infty$, for every bounded function Φ satisfying (5.1.3) and every α, $0 < \alpha < c$, then there exists a function $g(\zeta)$, $\zeta = \xi + i\eta$, analytic in the strip $0 < \eta < c/(a+1)$ such that

$$1/\hat{F}(\xi) = \lim_{\eta \to 0+} g(\xi + i\eta), \qquad -\infty < \xi < \infty .$$

It follows from Example 5 that it is impossible to extend this result to the strip $0 < \eta < c$. Conversely, it follows from the theorem quoted above that the result in Example 5 cannot be extended to values of θ such that $\theta > a+1$.

The following example illustrates how other Tauberian conditions than (5.1.3) influence the result.

Example 6. Let g satisfy (5.3.1) with $M_0(\xi) = \text{const}\, e^{b\xi}, \xi \geqq 0$, for some $b > 0$ and let $\varrho(x) = O(e^{-\alpha x}), x \to \infty$, where $0 < \alpha \leqq \gamma$. Let us suppose that

$$\text{(5.3.7)} \qquad \Phi(x) - \Phi(x+y) \leqq \sigma(x), \quad 0 \leqq y \leqq x^{-1}, \quad x \geqq x_0 ,$$

where $\sigma \in R[e^{\gamma x}]$ and $\sigma(x) \geqq e^{-\beta x}, x > x_0$, for some $\beta, 0 < \beta < \alpha$. Then

$$|\Phi(x)| \leqq Ck\sigma(x), \quad x \geqq x_1 ,$$

where $C = C(\gamma)$ and $k = \max(1, b/(\alpha - \beta))$.

The choice $\sigma(x) = K/x$ thus yields a well-known result of Ganelius (see [5, Theorem 2, p. 9]).

Proof. Choose c, $b < c < 2b$. Condition (1) of Theorem 1 is satisfied with $S(X) = e^{cX}$, $\varrho(x) = \text{const}\, e^{-\alpha x}$. Let φ be defined by (2.1.16), $\lambda = (2k)^{-1}$, $t(x) = 2kx^{-1}$ and $m = \sigma$. Then t and σ belong to $\mathscr{E}$ and

$$\varrho(x) S(1/t(x)) = o(m(x)), \quad x \to \infty .$$

The result thus follows from the corollary of Theorem 1 applied with $w(x) = h_P(x) = e^{\gamma x}$, $m(x) = \sigma(x)$, since $t(x)\varphi(x) \to 0, x \to \infty$ and (4.2.10) is obviously satisfied.

It should be mentioned that the conditions for g introduced in 5.3 may be considerably weakened. Let us replace (5.1.1) by

$$1/\hat{F}(\xi) = \lim_{\eta\to 0+} g(\xi+i\eta), \quad -\infty<\xi<\infty, \tag{5.3.8}$$

and let $\hat{F}\neq 0$ on $(-\infty,\infty)$. In Examples 4, 5 and 6 it is then sufficient to impose the conditions for g in the strip $0<\eta<\gamma$ only. The conditions of Example 3 can be weakened in the same way provided that (4.4.1) is satisfied. These results will be derived from the results in Section 4 in a subsequent paper.

5.4. *The function* $1/\hat{F}$ *analytic in a domain which tapers off at infinity.*

Let us consider the case in which g is analytic in a domain Ω of the following type. Let $\gamma(\xi)\searrow$, $\xi\geqq 0$, $\gamma(\xi)\to 0$, $\xi\to\infty$, $\gamma\leqq 1$ and $\gamma(\xi)=\gamma(-\xi)$, $-\infty<\xi<\infty$. Let $\zeta=\xi+i\eta$ and let

$$\Omega = \{\zeta\,;\ |\eta|<\gamma(\xi)\}. \tag{5.4.1}$$

Let $M_0(\xi)\nearrow$, $\xi\geqq 0$. Introduce the condition

$$\begin{cases} g \text{ is analytic in } \Omega, \\ |g(\xi+i\eta)| \leqq M_0(|\xi|), \quad \xi+i\eta\in\Omega. \end{cases} \tag{5.4.2}$$

Then, by Cauchy's formula

$$|g^{(n)}(\xi)| \leqq \frac{n!\,M_0(|\xi|+1)}{(\gamma(|\xi|+1))^n}, \quad -\infty<\xi<\infty.$$

If we put $R(X)=2^{\frac{1}{2}}X^{\frac{1}{2}}M_0(X+1), G(X)=1/\gamma(X+1), X>0$, we thus have

$$M_2\{g^{(n)}\,;\ -X,X\} \leqq n!(G(X))^n R(X), \quad n=0,1,2,\ldots.$$

Now let us choose a positive sequence $(K_n)_0^\infty$ such that $K_0=1$, ${K_n}^{1/n}\nearrow$, $n\geqq 1$, ${K_n}^{1/n}\to\infty$, $n\to\infty$ and the sequence $(n!K_n)_0^\infty$ is logarithmically convex. Let

$$k(x) = \sup_n x^n/K_n, \quad x\geqq 0. \tag{5.4.3}$$

Then

$$(G(X))^n\leqq K_n k(G(X)), \quad X\geqq 0,\ n=0,1,2,\ldots,$$

and hence

$$M_2\{g^{(n)}\,;\ -X,X\} \leqq n!K_n k(G(X))R(X), \quad n=0,1,2,\ldots. \tag{5.4.4}$$

Thus (5.3.2) is satisfied with $P_n=n!K_n$, $S_0(X)=k(G(X))R(X)$, and the theorems in Section 4 can be applied.

Example 7. Let $\zeta = \xi + i\eta$ and let a, b and B denote positive constants. Let g be analytic in the domain

$$\Omega = \{\zeta\,;\ |\eta| < a/(1+|\xi|)\}$$

and

$$|g(\zeta)| \leqq \operatorname{const} e^{b|\xi|}, \qquad \zeta \in \Omega$$

Let Φ satisfy (5.1.3). Let $\varrho(x) = e^{-V(x)}$ and

(5.4.5) $$\varrho \in R[\exp(Bx)^{\frac{1}{2}}]\,.$$

Then

(5.4.6) $$\Phi(x) = O\big(V(x)^{-1}\big), \qquad x \to \infty\,.$$

Proof. The result is obtained by choosing $K_n = n^n e^{-n} B^{-2n}, n \geqq 1$. Then $k(x) \leqq \exp(B^2 x)$. If $P_n = n^n e^{-n} B^{-2n} n!$ we find that

$$h_P(x) > x^{-\frac{1}{4}} \exp(2Bx^{\frac{1}{2}}), \qquad x > x_1\,.$$

From (5.4.5) it follows that $\varrho \in R[Ch_P(\frac{1}{2}x)]$ for some constant C. The result then follows from Theorem 3 applied with $S(X) = \exp(B_1 X)$, where $B_1 > b + B^2 a^{-1}$.

The estimate (5.4.6) is best possible in the sense that it cannot be replaced by

$$\Phi(x) = O(\delta(x) V(x)^{-1}), \qquad x \to \infty$$

for any function δ such that $\delta(x) \to 0$, $x \to \infty$. This follows in the same way as the corresponding result for Example 2.

Finally, I shall consider an example connected with Lambert summability. Let

(5.4.7) $$\alpha(t) = \frac{d}{dt}\left(\frac{te^{-t}}{1-e^{-t}}\right), \qquad F(x) = e^{-x}\alpha(e^{-x})\,.$$

Then

$$\hat{F}(\xi) = i\xi\Gamma(1+i\xi)\zeta(1+i\xi)\,.$$

where ζ denotes the Riemann zetafunction.

Let $s = \sigma + it$. It is well-known that there is a region

$$1 - A(\log t)^{-\frac{2}{3}}(\log\log t)^{-\frac{1}{3}} \leqq \sigma \leqq 2, \qquad t > t_0\,,$$

where uniformly

$$1/\zeta(s) = O((\log t)^{\frac{2}{3}}(\log\log t)^{\frac{1}{3}}), \qquad t \to \infty\,,$$

(see [11, 6.15, p. 114]). It follows that $g(\zeta) = 1/\hat{F}(\zeta)$ is analytic in a domain Ω defined by (5.4.1), where $\gamma(\xi) = \text{const}$, $0 \leqq |\xi| \leqq \xi_0$,

$$\gamma(\xi) = A(\log|\xi|)^{-\frac{3}{4}}(\log\log|\xi|)^{-\frac{3}{4}}, \quad |\xi| > \xi_0$$

and

$$|g(\zeta)| \leqq e^{\frac{1}{2}\pi|\xi|}|\xi|^{-1}, \quad \zeta \in \Omega, |\xi| \geqq \xi_0 .$$

Hence

$$M_2\{g^{(n)};\ -X,X\} \leqq \text{const}\, n!\, e^{\frac{1}{2}\pi X}(G(X))^n, \quad X \geqq X_0 ,$$

where

$$G(X) = A^{-1}(\log X)^{\frac{3}{4}}(\log\log X)^{\frac{3}{4}} ,$$

Choose $\alpha, 0 < \alpha < 4/3$ and $K_n = (\log(n+e))^{n/\alpha}$, and let k be defined by (5.4.3). Then $k(x) < \exp\exp x^\alpha$ and $k(G(X)) < e^X, X > X_0$. We thus have

$$M_2\{g^{(n)};\ -X,X\} \leqq P_n S(X), \quad X \geqq X_0 ,$$

where $S(X) = \exp((1+\frac{1}{2}\pi)X)$ and $P_n = n!K_n$. Let $a = \alpha^{-1}$. Then

$$p(x) > \exp(x(\log x)^{-a}), \quad x > x_1 .$$

The following result thus follows from Theorem 3.

Example 8. Let F be defined by (5.4.7), let Φ satisfy (5.1.3) and $|\Phi * F(x)| \leqq \varrho(x)$, $x \geqq x_0$. If $\varrho \in R[v]$ where $v(x) = \exp(x(\log x)^{-a})$, $x \geqq x_1$ for some $a > \frac{3}{4}$ and $\varrho(x) = e^{-V(x)}$, then

$$\Phi(x) = O(V(x)^{-1}), \quad x \to \infty .$$

The conditions for g introduced in 5.4 can be weakened in the same way as in 5.3. For example, if $\hat{F} \neq 0$ on $(-\infty,\infty)$ and (5.3.8) and (4.4.1) are satisfied, then it is sufficient to impose the conditions for g in Example 7 in that part of the domain Ω which is situated in the upper half-plane. These results will be derived from the above theorems in a subsequent paper.

REFERENCES

1. N. I. Achieser, *Vorlesungen über Approximationstheorie,* Akademie-Verlag, Berlin 1967.
2. A. Beurling, *Sur les intégrales de Fourier absolument convergentes et leur application à une transformation fonctionnelle,* C.R. du 9e congrès des mathématiciens scandinaves à Helsingfors 1938, Helsinki 1939, 345–366.
3. L. Frennemo, *On general Tauberian remainder theorems,* Math. Scand. 17 (1965), 77–88.
4. T. Ganelius, *General and special Tauberian remainder theorems,* C.R. du 13e congrès des mathématiciens scandinaves à Helsingfors 1957, Helsinki 1958, 102–103.
5. T. Ganelius, *The remainder in Wiener's Tauberian theorem,* Acta Univ. Gothoburgensis Math. Goth. 1 (1962), 13 pp.
6. T. Ganelius, *Tauberian theorems for the Stieltjes transform,* Math. Scand. 14 (1964), 213–219.

7. T. Ganelius, *Tauberian remainder theorems*, Lecture Notes in Mathematics 232, Springer-Verlag, Berlin, Heidelberg, New York, 1971.
8. S. Lyttkens, *The remainder in Tauberian theorems*, Ark. Mat. 2 (1954), 575–588.
9. S. Lyttkens, *The remainder in Tauberian theorems* II, Ark. Mat. 3 (1956), 315–349.
10. S. P. M. Strube, *Tauberian remainder theorems*, Thesis, The University of Michigan, 1972.
11. E. C. Titchmarsh, *The theory of the Riemann zeta-function*, At the Clarendon Press, Oxford 1951.
12. G. Valiron, *General theory of integral functions*, Librairie de l'Université, Toulouse 1923.
13. N. Wiener, *Tauberian theorems*, Ann. of Math. 33 (1932), 1–100.

UNIVERSITY OF UPPSALA, SWEDEN