

FIBRATIONS AND COFIBRED PAIRS

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If A is a subspace of X such that the inclusion $A \rightarrow X$ is a cofibration then the pair (X, A) is called a *cofibred pair*. If A is a subspace of B and $q: Y \rightarrow B$ is a map then $Y|A$ will denote the subspace $q^{-1}(A)$ of Y . It is a result of Strøm [5, theorem 12] that: if (B, A) is a cofibred pair with A closed and $q: Y \rightarrow B$ is a fibration then $(Y, Y|A)$ is a cofibred pair. We give a new “convenient category” proof of this result, using the fibred mapping projection $(qr): (YZ) \rightarrow B$ introduced in [1, p. 280]. We use the fact that the *exponential law of maps* bijection [2, theorem 1.1] is natural in the first variable (proposition 1) to show that: $(Y, Y|A)$ is a cofibred pair if and only if all extension-lifting problems of a certain type have solutions (theorem 2). The result (corollary 3) then follows.

A space X is said to be a \mathfrak{k} -space if it has the final topology with respect to all maps (= continuous functions) of compact Hausdorff spaces into X ; we work in the convenient category of \mathfrak{k} -spaces [1, p. 276] (= the category $\mathcal{H}\mathcal{G}$ of [6, p. 554]), and so all spaces and constructions should be considered in that sense (e.g. a *fibration* will mean a map between two \mathfrak{k} -spaces having the covering homotopy property with respect to all \mathfrak{k} -spaces). This category has certain technical advantages over its well known convenient subcategory of compactly generated spaces (= Hausdorff k -spaces = Kelley spaces); it is better behaved with respect to quotients, but the important thing here is that it is closed with respect to the construction of fibred mapping spaces (YZ) .

If X and Y are spaces then $M(X, Y)$ will denote the set of all maps $X \rightarrow Y$. If $p: X \rightarrow B$, $q: Y \rightarrow B$ and $f: X \rightarrow Y$ are maps such that $qf = p$ then we will write $f: p \rightarrow q$, and $M(p, q)$ will denote the set of all maps $p \rightarrow q$, $X \square Y$ will denote the pullback space of X with Y , $p_q: X \square Y \rightarrow Y$, $q_p: X \square Y \rightarrow X$ will denote the projections and $p \square q: X \square Y \rightarrow B$ the map

$$(p \square q)(x, y) = p(x) = q(y), \quad (x, y) \in X \square Y.$$

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If $r: Z \rightarrow B$ is another map then Y_b and Z_b will denote the respective fibres of q and r over $b \in B$. We define the set

$$(YZ) = \bigcup_{b \in B} M(Y_b, Z_b)$$

and the function $(qr): (YZ) \rightarrow B$ as the obvious projection on B ; we recall the definition [1, p. 280] of a topology on (YZ) , with the following important properties:

(i) a bijection

$$\theta: M(p \sqcap q, r) \rightarrow M(p, (qr)),$$

is given by

$$\theta(f)(x)(y) = f(x, y), \quad f \in M(p \sqcap q, r), \quad p(x) = q(y)$$

(the exponential law of maps = [2, theorem 1.1]);

(ii) if q and r are fibrations then so is (qr) [1, theorem 3.4].

If $f: p \rightarrow q$, and $f: X \rightarrow Y$ is a homeomorphism, then we will write $f: p \cong q$.

We list five more or less trivial facts about pullbacks and maps $q: Y \rightarrow B$:

(iii) if $s: B \times I \rightarrow B$ is the projection on the first factor then $q_s \cong q \times 1_I$;

(iv) if $f: W \rightarrow Z$ and $g: Z \rightarrow B$ are maps then $(q_g)_f \cong q_{(gf)}$;

(v) if $i: A \rightarrow B$ denotes the inclusion of subspace A in B and

$$q|A = q|(Y|A): (Y|A) \rightarrow A$$

then $q_i \cong q|A$;

(vi) if (v) is used to identify $Y \sqcap A$ with $Y|A$ then the projection $i_q: Y \sqcap A \rightarrow Y$ is essentially just the inclusion $Y|A \rightarrow Y$; and

(vii) if W is a space and $r: B \times W \rightarrow B$ is the projection on the first factor then there is a bijection $\varphi: M(Y, W) \rightarrow M(q, r)$

$$\varphi(f)(y) = (q(y), f(y)), \quad f \in M(Y, W), \quad y \in Y$$

(the inverse of φ is the function $\psi: M(q, r) \rightarrow M(Y, W)$,

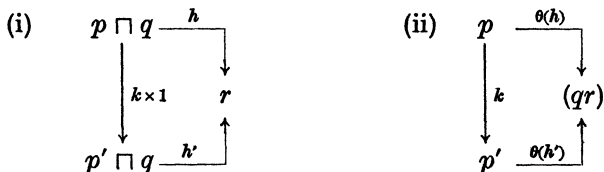
$$\psi(g) = r'g,$$

where $r': B \times W \rightarrow W$ is the projection on the second factor).

The following result describes the naturality of the exponential law

(i) with respect to the first variable p :

PROPOSITION 1. *Given maps $p: X \rightarrow B, p': X' \rightarrow B, q: Y \rightarrow B, r: Z \rightarrow B, k: p \rightarrow p', h: p \sqcap q \rightarrow r, h': p' \sqcap q \rightarrow r$. If $k \times 1: p \sqcap q \rightarrow p' \sqcap q$ is the map $(k \times 1)(x, y) = (k(x), y)$, where $p(x) = q(y)$, then diagram (i) commutes if and only if diagram (ii) commutes:*

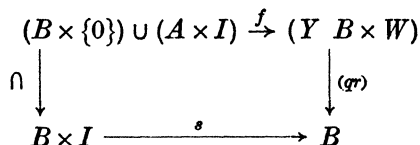


PROOF. $h(x, y) = h'(k(x), y)$ for all $(x, y) \in X \sqcap Y$ if and only if $\theta(h)(x)(y) = \theta(h')(k(x))(y)$ for all such (x, y) . This is simply the condition that $\theta(h)(x) = \theta(h')(k(x))$ for all $x \in X$, i.e. that $\theta(h) = \theta(h')k$.

Let $s: B \times I \rightarrow B$ denote the projection on the first factor.

THEOREM 2. *If $q: Y \rightarrow B$ is a map and A is a subspace of B then the following conditions are equivalent:*

- (a) $(Y, Y|A)$ is a cofibred pair;
- (b) for every space W (with associated projection $r: B \times W \rightarrow B$) and map $f: (B \times \{0\}) \cup (A \times I) \rightarrow (Y \ B \times W)$ such that the following square commutes



there exists a map $f': B \times I \rightarrow (Y \ B \times W)$ filling in the diagram, i.e. such that the triangles formed are commutative.

PROOF. This is a direct application of proposition 1, to the case where p and p' are the respective projections of $(B \times \{0\}) \cup (A \times I)$ and $B \times I$ on B , $k: (B \times \{0\}) \cup (A \times I) \rightarrow B \times I$ is the inclusion and $r: B \times W \rightarrow B$ is the usual projection; with (iii)–(vii) above being used to simplify some of the statements involved. Let us consider the following conditions:

- (c) for every space W and every map $g: X \sqcap Y \rightarrow W$ there exists a map $g': X' \sqcap Y \rightarrow W$ with $g'(k \times 1) = g$;
- (d) for every space W and every map $h: p \sqcap q \rightarrow r$ there exists a map $h': p' \sqcap q \rightarrow r$ with $h'(k \times 1) = h$; and
- (e) for every space W and every map $\theta(h): p \rightarrow (qr)$ there is a map $\theta(h'): p' \rightarrow (qr)$ such that $\theta(h')k = \theta(h)$.

It follows from remark (iii) that $X' \sqcap Y$ is essentially $Y \times I$; from (iv) that $q_p = q_{(p'k)} \cong (q_{p'})_k$, from (iii) that

$$(q_{p'})_k \cong (q \times 1_I)k,$$

and hence from (v) that $X \sqcap Y$ is essentially $(Y \times \{0\}) \cup ((Y|A) \times I)$; and from (vi) that

$$k \times 1: X \sqcap Y \rightarrow X' \sqcap Y$$

is essentially the inclusion $(Y \times \{0\}) \cup ((Y|A) \times I) \rightarrow Y \times I$. Hence (a) is equivalent to condition (c), now (c) is equivalent to (d) by (vii), and (d) to (e) by proposition 1. (e) is simply a restatement of (b) above, and so the result is proved.

To apply this theorem we notice that if q is a fibration then so is (qr) (see (ii) above). If (B, A) is a relative CW-complex then it follows by the \mathfrak{k} -space version of [3, p. 416, theorem 9] that $(Y, Y|A)$ is a cofibred pair. If we use the \mathfrak{k} -space version of the stronger theorem [4, theorem 4] then we immediately obtain the following result.

COROLLARY 3. *If (B, A) is a cofibred pair with A closed and $q: Y \rightarrow B$ is a fibration then $(Y, Y|A)$ is a cofibred pair.*

REFERENCES

1. P. I. Booth, *The section problem and the lifting problem*, Math. Z. 121 (1971), 273–287.
2. P. I. Booth, *The exponential law of maps II*, Math. Z. 121 (1971), 311–319.
3. E. H. Spanier, *Algebraic Topology*, McGraw-Hill, New York, 1966.
4. A. Strøm, *Note on cofibrations*, Math. Scand. 19 (1966), 11–14.
5. A. Strøm, *Note on cofibrations II*, Math. Scand. 22 (1968), 130–142.
6. R. Vogt, *Convenient categories of topological spaces for homotopy theory*, Arch. Math. (Basel) 22 (1971), 546–555.

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