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ON THE SPACE OF MAPS OF A CLOSED SURFACE INTO THE 2-SPHERE

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1. Introduction and statements of results.

In this paper we compute (up to a central extension) the fundamental group of an arbitrary (path-) component in the space of (continuous) maps of a given closed surface into the 2-sphere S^2. As an application we solve the homotopy problem for the countable number of components in the space of maps of a closed, orientable surface into S^2.

Let C be an arbitrary closed surface. In case C is orientable, we fix an orientation of C. Denote by $M(C,S^2)$ the space of maps of C into S^2 equipped with the compact-open topology. By the Hopf classification theorem, $M(C,S^2)$ has a countable number of components if C is orientable, and exactly 2 components if C is non-orientable. In the two cases, C orientable, respectively non-orientable, the components in $M(C,S^2)$ are enumerated by the degree, respectively the degree mod 2 of maps of C into S^2. We denote by $M_k(C,S^2)$ that component which contains the maps of degree k, respectively degree k mod 2.

For any non-negative integer m we denote by Z_m the cyclic group of infinite order if $m=0$ and of order m if $m>0$. Similarly, Z^m denotes the trivial group if $m=0$ and the free abelian group of rank m if $m>0$. In the orientable case we shall prove

THEOREM 1. *Let T_g be a closed, orientable surface of genus $g \geqq 0$. For each degree k, there exists a short exact sequence*

$$0 \to Z_{2|k|} \to \pi_1(M_k(T_g,S^2)) \to Z^{2g} \to 0 .$$

For $g=0$, $T_g=S^2$, and Theorem 1 states that $\pi_1(M_k(S^2,S^2)) \cong Z_{2|k|}$. This is a theorem of Hu [6, Theorem 5.3], see also Koh [7, Lemma 3.9]. We use the theorem of Hu in the proof of theorem 1.

In the non-orientable case we shall prove

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THEOREM 2. *Let P_h be a closed, non-orientable surface with h crosscaps, $h>0$. For each degree k* mod 2 *there exists a short exact sequence*

$$0 \to \mathsf{Z}_2 \to \pi_1(M_k(P_h,S^2)) \to \mathsf{Z}^{h-1} \to 0 \,.$$

For $k=0$, Theorem 1 and Theorem 2 are due to Dyer [2, p.1288]. Hans J. Munkholm has observed that the extensions described by the short exact sequences in Theorem 1 and Theorem 2 are always central extensions. Normally, they are, however, non-trivial. We shall discuss this in Section 5.

For any closed, orientable surface T_g of genus $g \geqq 0$ and any degree $k \neq 0$, the two components $M_k(T_g,S^2)$ and $M_{-k}(T_g,S^2)$ are homeomorphic. A homeomorphism can be constructed by composition with a fixed orientation reversing homeomorphism on T_g. Our original interest in Theorem 1 can then be expressed in the following

COROLLARY. *Two components in $M(T_g,S^2)$ corresponding to degrees m and n have the same homotopy type if and only if $m = \pm n$.*

The problem behind this corollary, namely to divide the set of components in a given space of maps into homotopy types, was solved in [3] for the space of self-mappings on the n-sphere S^n for $n \geqq 1$ and for various other spaces of maps between spheres. The methods in [3], and in the related paper [4], use extensively constructions involving a suspension parameter in the domain. The corollary is therefore interesting, since it deals with a situation, where the domain is not a suspension.

2. Preliminaries.

All topological spaces will be equipped with a base point. For any pair of based spaces A and B, we denote by $\pi(A,B)$ the set of based homotopy classes of based maps of A into B. If $A = S^n$, the n-sphere, we use mostly the standard notation $\pi_n(B) = \pi(S^n,B)$. For any based space X, ΩX and ΣX denotes respectively the space of loops on X and the (reduced) suspension of X. $\vee$ and $\wedge$ between based spaces shall denote respectively wedge product and smash product. All mapping spaces will be equipped with the compact-open topology.

For any closed surface C we denote by $F(C,S^2)$ the space of based maps of C into S^2. The components in $F(C,S^2)$ are enumerated by the degree, respectively the degree mod 2 of maps of C into S^2 according to C orientable, respectively non-orientable, and a component in $F(C,S^2)$ is denoted similarly to the corresponding component in the space of maps $M(C,S^2)$ with no restrictions on base points. Since S^2 is simply connected, and therefore a simple space, the Hurewich fibration p: $M_k(C,S^2) \to S^2$ defined by evaluation at the base point of C has $F_k(C,S^2)$ as fibre.

For each closed surface C we choose now an embedded 2-disc D^2, such that the base point in C belongs to the boundary ∂D^2 of D^2. Collapsing ∂D^2 to the base point defines a map $\nu: C \to C \vee S^2$. Let also $\nabla: S^2 \vee S^2 \to S^2$ denote the folding map. For any pair of based maps $f: C \to S^2$ and $g: S^2 \to S^2$ we can then define the map $f+g: C \to S^2$ as the composite map $f+g = \nabla \circ (f \vee g) \circ \nu$.

For each degree k we choose a fixed based map $g_k: S^2 \to S^2$ of degree k. It is then easy to prove that the map $\theta: F_0(C,S^2) \to F_k(C,S^2)$ defined by $\theta(f) = f + g_k$ is a homotopy equivalence with an inverse $\psi: F_k(C,S^2) \to F_0(C,S^2)$ defined by $\psi(h) = h + g_{-k}$. For C non-orientable the degree is understood to be counted mod 2.

In particular we get then

PROPOSITION 1. *For any closed surface C, all the components in $F(C,S^2)$ have the same homotopy type.*

3. Proofs of Theorem 1 and its corollary.

Let T_g be a closed, orientable surface of genus $g \geqq 0$. Denote by A_g the wedge of $2g$ circles (1-spheres) for $g > 0$ and a point for $g = 0$. Then $\pi_1(A_g)$ is isomorphic to the free group on $2g$ generators. To be specific, let $\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g$ be the system of generators for $\pi_1(A_g)$ represented by the inclusion maps into A_g of the $2g$ circles in A_g. Denote by $\prod_{i=1}^{g} [\alpha_i, \beta_i]$ the product of the commutators $[\alpha_i, \beta_i]$. Let $\varphi: S^1 \to A_g$ be a based map with the homotopy class $\prod_{i=1}^{g} [\alpha_i, \beta_i]$. Then it is well-known that T_g is homotopy equivalent to the mapping cone of φ. Hence we get a mapping sequence

$$S^1 \xrightarrow{\varphi} A_g \longrightarrow T_g \xrightarrow{q} S^2 \xrightarrow{\Sigma\varphi} \Sigma A_g \to \ldots ,$$

where $q: T_g \to S^2$ is the map defined by collapsing A_g to the base point. Clearly q is a based map of degree 1.

For any based space X this mapping sequence induces an exact homotopy sequence

$$\pi(\Sigma A_g, X) \xrightarrow{(\Sigma\varphi)^*} \pi(S^2, X) \longrightarrow \pi(T_g, X) \longrightarrow \pi(A_g, X) \xrightarrow{\varphi^*} \pi(S^1, X) .$$

Proposition 2. *$(\Sigma\varphi)^*$ is always the zero map. If $\pi_1(X)$ is abelian, then φ^* is also the zero map, and we get a short exact sequence*

$$0 \to \pi_2(X) \to \pi(T_g,X) \to \pi(A_g,X) \to 0 .$$

Here $\pi(A_g,X) \cong \oplus_{i=1}^{2g}(\pi_1(X))_i$, the direct sum of $2g$ copies of $\pi_1(X)$.

Proof. The homomorphism $(\Sigma\varphi)^*: \pi(\Sigma A_g,X) \to \pi(S^2,X)$ is equivalent to a homomorphism $\pi(A_g,\Omega X) \to \pi(S^1,\Omega X)$. Using this equivalence it is clear that for any $\gamma \in \pi(A_g,\Omega X)$, we get

$$(\Sigma\varphi)^*(\gamma) = \prod_{i=1}^{g}[\gamma\circ\alpha_i,\gamma\circ\beta_i] ,$$

where $\gamma\circ\alpha_i$ and $\gamma\circ\beta_i$ are the homotopy classes defined by composition. Since $\pi(S^1,\Omega X) \cong \pi_2(X)$ is abelian, $(\Sigma\varphi)^*(\gamma) = 0$. This proves that $(\Sigma\varphi)^*$ is the zero map.

When $\pi_1(X)$ is abelian, an analogous argument shows that $\varphi^*: \pi(A_g,X) \to \pi(S^1,X)$ is the zero map.

Since the remaining assertions are now obvious, Proposition 2 is proved.

For any degree k, the above map $q: T_g \to S^2$ of degree 1 induces a map between fibrations by composition of maps,

$$\begin{array}{ccc} F_k(S^2,S^2) & \longrightarrow & F_k(T_g,S^2) \\ \downarrow & & \downarrow \\ M_k(S^2,S^2) & \xrightarrow{q^*} & M_k(T_g,S^2) \\ {\scriptstyle p_S}\downarrow & & \downarrow{\scriptstyle p_T} \\ S^2 & \xrightarrow{1_{S^2}} & S^2 . \end{array}$$

The Hurewicz fibrations p_S and p_T are defined by evaluation at the base points in respectively S^2 and T_g. 1_{S^2} denotes the identity map on S^2.

This map between fibrations induces a map between homotopy sequences

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & \pi_2(S^2) & \xrightarrow{\partial_S} & \pi_1(F_k(S^2,S^2)) & \longrightarrow & \pi_1(M_k(S^2,S^2)) & \longrightarrow & 0 \\ & & {\scriptstyle 1}\downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & \pi_2(S^2) & \xrightarrow{\partial_T} & \pi_1(F_k(T_g,S^2)) & \longrightarrow & \pi_1(M_k(T_g,S^2)) & \longrightarrow & 0 . \end{array}$$

Consider now the homotopy equivalences

$$\theta_S: F_0(S^2,S^2) \to F_k(S^2,S^2) \quad \text{and} \quad \theta_T: F_0(T_g,S^2) \to F_k(T_g,S^2)$$

defined in Section 2. To define θ_T we choose the 2-cell, which goes into the definition, such that it is contained in the 2-cell we attach to A_g to obtain T_g, and such that the boundaries of these 2-cells have just the base point in T_g in common. Choosing the constant based maps as base points in $F_0(S^2, S^2)$ and $F_0(T_g, S^2)$ and base points in $F_k(S^2, S^2)$ and $F_k(T_g, S^2)$ accordingly we get then a commutative diagram

$$\begin{array}{ccccccc}
\pi(S^2, \Omega S^2) & \xrightarrow[\cong]{} & \pi(S^1 \wedge S^2, S^2) & \xrightarrow[\cong]{} & \pi_1(F_0(S^2, S^2)) & \xrightarrow[\cong]{(\theta_S)_*} & \pi_1(F_k(S^2, S^2)) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\pi(T_g, \Omega S^2) & \xrightarrow{\cong} & \pi(S^1 \wedge T_g, S^2) & \xrightarrow{\cong} & \pi_1(F_0(T_g, S^2)) & \xrightarrow[\cong]{(\theta_T)_*} & \pi_1(F_k(T_g, S^2)) .
\end{array}$$

All the vertical maps in this diagram are induced by the map $q\colon T_g \to S^2$. The unnamed horizontal isomorphisms are the obvious adjoint isomorphisms.

Combining the short exact sequence from Proposition 2 and the above commutative diagram with the map between the homotopy sequences for the fibrations p_S and p_T, we get an induced commutative diagram of exact sequences

$$\begin{array}{ccccccc}
 & & 0 & & & & \\
 & & \downarrow & & & & \\
\pi_2(S^2) & \xrightarrow{\partial'_S} & \pi(S^2, \Omega S^2) & \longrightarrow & \pi_1(M_k(S^2, S^2)) & \longrightarrow & 0 \\
{\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle q^*} & & \downarrow & & \\
\pi_2(S^2) & \xrightarrow{\partial'_T} & \pi(T_g, \Omega S^2) & \longrightarrow & \pi_1(M_k(T_g, S^2)) & \longrightarrow & 0 \\
 & & \downarrow & & & & \\
 & & \pi(A_g, \Omega S^2) & & & & \\
 & & \downarrow & & & & \\
 & & 0 & & & & .
\end{array}$$

A simple diagram chasing in this diagram provides us now with a short exact sequence

$$0 \to \pi_1(M_k(S^2, S^2)) \to \pi_1(M_k(T_g, S^2)) \to \pi(A_g, \Omega S^2) \to 0 .$$

In this exact sequence $\pi_1(M_k(S^2, S^2)) \cong \mathbf{Z}_{2|k|}$ by the theorem of Hu [6, Theorem 5.3], see also Koh [7, Lemma 3.9]. Since

$$\pi(A_g, \Omega S^2) \cong \oplus_{i=1}^{2g}(\pi_1(\Omega S^2))_i \cong \mathbf{Z}^{2g} ,$$

there exists therefore a short exact sequence

$$0 \to \mathbf{Z}_{2|k|} \to \pi_1(M_k(T_g, S^2)) \to \mathbf{Z}^{2g} \to 0$$

as asserted in Theorem 1.

From this short exact sequence it follows that $\pi_1(M_k(T_g, S^2))$ for $k \neq 0$ contains an element of order $2|k|$. A simple argument involving orders of elements proves then that

$$\pi_1(M_m(T_g, S^2)) \not\cong \pi_1(M_n(T_g, S^2))$$

if $m \neq \pm n$, and hence the corollary follows.

4. Proof of Theorem 2.

Let P_h be a closed, non-orientable surface with h crosscaps, $h > 0$. Let B_h denote the wedge of h circles, and let B_0 be a point. Then $\pi_1(B_h)$ is isomorphic to the free group on h generators. Let $\gamma_1, \ldots, \gamma_h$ be the system of generators of $\pi_1(B_h)$ represented by the inclusion maps into B_h of the h circles in B_h. If $\psi\colon S^1 \to B_h$ denotes a based map with the homotopy class

$$\prod_{i=1}^h \gamma_i^2 = \gamma_1^2 \cdots \gamma_h^2 ,$$

then it is well-known that P_h is homotopy equivalent to the mapping cone of ψ. Hence we get a mapping sequence

$$S^1 \xrightarrow{\psi} B_h \longrightarrow P_h \xrightarrow{q} S^2 \xrightarrow{\Sigma\psi} \Sigma B_h \longrightarrow \cdots ,$$

where $q\colon P_h \to S^2$ is the map defined by collapsing B_h to the base point. Clearly q is a based map of degree 1 mod 2.

For any based space X this mapping sequence induces an exact homotopy sequence

$$\pi(\Sigma B_h, X) \xrightarrow{(\Sigma\psi)^*} \pi(S^2, X) \longrightarrow \pi(P_h, X) \longrightarrow \pi(B_h, X) \xrightarrow{\psi^*} \pi(S^1, X) .$$

PROPOSITION 3. *In the above exact sequence,* coker $(\Sigma\psi)^* \cong \pi_2(X) \otimes \mathbf{Z}_2$. *If we assume that $\pi_1(X)$ is abelian and uniquely divisible by 2, then* $\ker \psi^* \cong \pi(B_{h-1}, X)$, *and we get a short exact sequence*

$$0 \to \pi_2(X) \otimes \mathbf{Z}_2 \to \pi(P_h, X) \to \pi(B_{h-1}, X) \to 0 .$$

Here $\pi(B_{h-1}, X) \cong \oplus_{i=1}^{h-1} (\pi_1(X))_i$, the direct sum of $h-1$ copies of $\pi_1(X)$.

PROOF. The homomorphism $(\Sigma\psi)^*\colon \pi(\Sigma B_h, X) \to \pi(S^2, X)$ is equivalent to a homomorphism $\pi(B_h, \Omega X) \to \pi(S^1, \Omega X)$. Using this equivalence it is clear that for any $\alpha \in \pi(B_h, \Omega X)$ we get

$$(\Sigma\psi)^*(\alpha) = \prod_{i=1}^h (\alpha \circ \gamma_i)^2 = 2\cdot(\alpha \circ \gamma_1) + \ldots + 2\cdot(\alpha \circ \gamma_h) ,$$

where we can use additive notation, since $\pi(S^1,\Omega X)\cong\pi_2(X)$ is abelian. Therefore it is clear that the image of $(\Sigma\psi)^*$ in $\pi(S^2,X)=\pi_2(X)$ must be the subgroup $2\cdot\pi(S^2,X)=2\cdot\pi_2(X)$. Hence

$$\operatorname{coker}(\Sigma\psi)^* \cong \pi_2(X)/2\cdot\pi_2(X) \cong \pi_2(X)\otimes \mathrm{Z}_2 .$$

Assume now that $\pi_1(X)$ is abelian and uniquely divisible by 2. For any $\alpha\in\pi(B_h,X)$ we have

$$\psi^*(\alpha) = \textstyle\prod_{i=1}^h(\alpha\circ\gamma_i)^2 = 2\cdot(\alpha\circ\gamma_1)+\ldots+2\cdot(\alpha\circ\gamma_h) ,$$

where we can use additive notation since $\pi_1(X)$ is abelian. Since $\pi_1(X)$ is uniquely divisible by 2, the equation

$$2\cdot(\alpha\circ\gamma_1)+\ldots+2\cdot(\alpha\circ\gamma_h) = 0$$

determines $\alpha\circ\gamma_h$ uniquely from $\alpha\circ\gamma_1,\ldots,\alpha\circ\gamma_{h-1}$. Hence $\ker\psi^*\cong \pi(B_{h-1},X)$.

The remaining assertions are obvious, and hence Proposition 3 is proved.

For $k=0,1$, the above map $q\colon P_h\to S^2$ of degree 1 mod 2 induces a map between fibrations by composition of maps,

$$\begin{array}{ccc}
F_k(S^2,S^2) & \longrightarrow & F_k(P_h,S^2)\\
\downarrow & & \downarrow\\
M_k(S^2,S^2) & \xrightarrow{q^*} & M_k(P_h,S^2)\\
{\scriptstyle p_S}\downarrow & & \downarrow{\scriptstyle p_P}\\
S^2 & \xrightarrow{1_{S^2}} & S^2
\end{array}$$

Proceeding exactly as in Section 4, we end up with the following commutative diagram of exact sequences

$$\begin{array}{ccccccc}
 & & \pi(\Sigma B_h,\Omega S^2) & & & & \\
 & & {\scriptstyle (\Sigma\psi)^*}\downarrow & & & & \\
\pi_2(S^2) & \xrightarrow{\partial'_S} & \pi(S^2,\Omega S^2) & \longrightarrow & \pi_1(M_k(S^2,S^2)) & \longrightarrow & 0\\
{\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle q^*} & & \downarrow & & \\
\pi_2(S^2) & \xrightarrow[\partial'_P]{} & \pi(P_h,\Omega S^2) & \longrightarrow & \pi_1(M_k(P_h,S^2)) & \longrightarrow & 0\\
 & & \downarrow & & & & \\
 & & \pi(B_{h-1},\Omega S^2) & & & & \\
 & & \downarrow & & & & \\
 & & 0 & & & &
\end{array}$$

If we choose generators $\iota_2 \in \pi_2(S^2) \cong \mathsf{Z}$ and $\eta \in \pi(S^2, \Omega S^2) \cong \pi_3(S^2) \cong \mathsf{Z}$ appropriately, then we can assume that $\partial_S'(\iota_2) = 2k \cdot \eta$. See Hu [6, Theorem 5.3]. With such choices of generators, $\operatorname{im} \partial_S' = 0$ for $k = 0$ and $\operatorname{im} \partial_S' = 2\mathsf{Z} \cdot \eta$ for $k = 1$.

From the description of $(\Sigma\psi)^*$ given in the proof of Proposition 3, it follows that $\operatorname{im}(\Sigma\psi)^* = 2\mathsf{Z} \cdot \eta$.

Thus $\operatorname{im} \partial_S' \subseteq \operatorname{im}(\Sigma\psi)^*$ for both $k = 0$ and 1. This implies that $\partial_P' = 0$ and hence that

$$\pi_1(M_k(P_h, S^2)) \cong \pi(P_h, \Omega S^2)$$

for both $k = 0$ and 1. Therefore we get for all degrees k mod 2 a short exact sequence.

$$0 \to \operatorname{coker}(\Sigma\psi)^* \to \pi_1(M_k(P_h, S^2)) \to \pi(B_{h-1}, \Omega S^2) \to 0 .$$

Since $\operatorname{coker}(\Sigma\psi)^* \cong \mathsf{Z}_2$ and $\pi(B_{h-1}, \Omega S^2) \cong \mathsf{Z}^{h-1}$ this sequence is equivalent to a short exact sequence

$$0 \to \mathsf{Z}_2 \to \pi_1(M_k(P_h, S^2)) \to \mathsf{Z}^{h-1} \to 0 .$$

This proves Theorem 2.

5. Centrality of the extensions in Theorem 1 and Theorem 2. Two problems.

Hans J. Munkholm has observed that the extensions described by the short exact sequences in Theorem 1 and Theorem 2 are always central extensions. I am indepted to him for a conversation, which led to the following proof of this fact for the exact sequence in Theorem 1. Using the terminology from Section 3 it is clearly sufficient to prove the

ASSERTION. *The extension described by the short exact sequence*

$$0 \longrightarrow \pi(S^2, \Omega S^2) \xrightarrow{q^*} \pi(T_g, \Omega S^2) \longrightarrow \pi(A_g, \Omega S^2) \longrightarrow 0$$

is central.

PROOF. Consider an arbitrary pair of homotopy classes $\alpha \in \pi(T_g, \Omega S^2)$ and $\beta \in \pi(S^2, \Omega S^2)$. Choose an embedded 2-cell $\bar{D}^2$ in T_g contained in the interior of the 2-cell we attach to A_g to obtain T_g. Since the pair $(T_g, \bar{D}^2)$ has the homotopy extension property, we can represent α by a based map $f\colon T_g \to \Omega S^2$, which restricts to the constant loop on $\bar{D}^2$. Since $q(T_g \setminus \operatorname{int} \bar{D}^2)$ is an embedded 2-cell in S^2, it is clear, that we can represent β by a based map $h\colon S^2 \to \Omega S^2$, such that $h \circ q\colon T_g \to \Omega S^2$ restricts to the constant loop on $T_g \setminus \operatorname{int}\bar{D}^2$. After a change by a homotopy we can assume, that the loop multiplication

$$\mu\colon \Omega S^2 \times \Omega S^2 \to \Omega S^2$$

has the constant loop as a strict unit element. This is possible, since the pair $(\Omega S^2 \times \Omega S^2, \Omega S^2 \vee \Omega S^2)$ has the homotopy extension property. With such a choice of μ it is clear, that the diagram

$$\begin{array}{ccc} T_g & \xrightarrow{f\times(h\circ q)} & \Omega S^2 \times \Omega S^2 \\ {\scriptstyle (h\circ q)\times f}\downarrow & & \downarrow{\scriptstyle \mu} \\ \Omega S^2 \times \Omega S^2 & \xrightarrow{\mu} & \Omega S^2 \end{array}$$

is commutative. Since the group structures in the various groups in the exact sequence are all induced by μ, this shows, that the homotopy classes α and $q^*(\beta)$ commute in $\pi(T_g, \Omega S^2)$. This proves the assertion.

A similar argument will show, that the extension described by the short exact sequence in Theorem 2 is central.

Consider now an arbitrary closed surface C and let k be an arbitrary degree if C is orientable, and degree mod 2 if C is non-orientable. Since it is central, it is clear, that the extension described by the short exact sequence in Theorem 1, respectively Theorem 2, is trivial if and only if $\pi_1(M_k(C,S^2))$ is abelian. Normally, $\pi_1(M_k(C,S^2))$ is, however, non-abelian, in which case the corresponding extension is non-trivial. To mention a concrete example it is shown in Barratt [2, p. 95] and Federer [8, p. 358], that $\pi_1(M_0(C,S^2))$ is non-abelian for $C = T_1$, the 2-dimensional torus.

The above discussion raises the following

PROBLEM 1. Determine the extensions in Theorem 1 and Theorem 2. In particular: a) What is the characteristic class of such an extension? b) Is the fundamental group $\pi_1(M_k(C,S^2))$ abelian, or equivalently, is the corresponding extension trivial, in other cases than $C = S^2 (g=0)$ or $C = \mathrm{RP}^2$, the projective plane $(h=1)$?

In [5, Theorem 2] we computed the fundamental group of an arbitrary space of maps of a finite CW-complex into an Eilenberg-MacLane space of type $(\pi, 1)$. Taking this together with Theorem 1 and Theorem 2 we find, that we have computed (at least up to an extension) the fundamental group of any space of maps between closed surfaces, except when the target is the projective plane RP^2. Hence

PROBLEM 2. Compute the fundamental group of the various components in $M(C, \mathsf{RP}^2)$ for an arbitrary closed surface C.

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