MATH. SCAND. 35 (1974), 159–164

ON HELICES IN THE EUCLIDEAN n-SPACE

FR. FABRICIUS-BJERRE

1. Introduction.

A *helix* in a Euclidean n-space R^n is a curve for which the tangents are inclined at a constant angle $v \neq \frac{1}{2}\pi$ to a line m. We call m and any line parallel to m a *line of reference* for the helix. The lines of reference may be determined as lines parallel to a non-zero vector $\boldsymbol{e}$ which is called a *vector of reference* for the curve.

In section 2 we show some properties of a helix γ, of its projection γ_1 on a hyperplane perpendicular to the lines of reference, and of its involutes. The properties are generalizations of well-known theorems on helices in R^3.

Section 3 contains a construction of a helix γ with given angle of inclination and whose projection on a hyperplane is a given curve γ_1. We prove that if γ_1 itself is a helix then the constructed helix γ will be situated in another hyperplane, and γ and γ_1 will be affinely connected. If γ_1 is not a helix, the curve γ does not lie in any (proper) subspace of R^n.

Finally we state in section 4 a theorem on helices in R^n, for which there exist m linearly independent vectors of reference.

2. Projections and involutes of a helix.

Let γ denote a regular curve in R^n given by its parametric equation $\overrightarrow{OP} = \boldsymbol{r}(s)$, $s \in I$, with the unit tangent vector $\boldsymbol{r}'(s) = \boldsymbol{t}$, and let $\boldsymbol{e}$ denote a non-zero vector. If the equation

$$(2.1) \qquad \boldsymbol{t}(s)\cdot \boldsymbol{e} = c ,$$

where $c \neq 0$ is a constant, is valid for any $s \in I$, then γ is a *helix* with angle of inclination v determined by $c = |\boldsymbol{e}| \cos(\boldsymbol{e},\boldsymbol{t}) = |\boldsymbol{e}| \cos v$, and with $\boldsymbol{e}$ as a vector of reference.

Let H_1 denote a hyperplane through a point P_0 of γ and perpendicular to the vector $\boldsymbol{e}$. The projection γ_1 of γ on H_1 is the intersection between

Received July 12, 1974.

the hyperplane and the projecting cylinder Γ, whose generators are lines of reference for γ. By development of Γ into a plane the generators will be developed into a pencil of parallel lines. The curves γ and γ_1 are transformed into straight lines, the first one cutting the parallel lines at the angle v, while the second one will be a normal to the pencil. Hence both curves are *geodesics* on Γ. Corresponding arclengths s and s_1 of γ and γ_1 are obviously connected by the equation

$$s_1 = s \sin v \,. \tag{2.2}$$

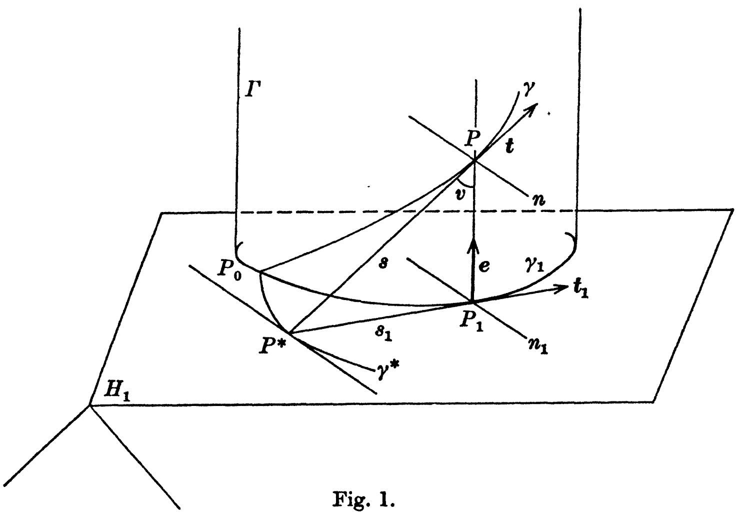

Fig. 1.

Let γ^* denote the *involute* of γ_1 starting at P_0. To a point $P \in \gamma$ corresponds a point $P_1 \in \gamma_1$ and a point $P^* \in \gamma^*$ (fig. 1). Since P^*P_1 has the length s_1 of $\widehat{P_0P_1}$ the equation (2.2) shows that P^*P has the length s of $\widehat{P_0P}$. I.e. the *involute of* γ_1 *is also involute of the helix* γ. This implies that the *involutes of a helix in* R^n *are all lying in hyperplanes orthogonal to the lines of reference.* The tangent to γ^* at P^* is parallel to the principal normal n to γ at P and to the principal normal n_1 to γ_1 at P_1, and it is perpendicular to the tangent plane to Γ along the generator through P and P_1.

In order to find a relation between the *radius of curvature* ϱ of γ at P and ϱ_1 of γ_1 at P_1 we remark, that the osculating plane of γ_1 at P_1 is the projection of the osculating plane of γ at P. Since the principal normals n and n_1 are parallel, the projection of the osculating circle at P on H_1 is an ellipse with semiaxes a and b, where $a = \varrho$ and $b = \varrho \sin v$. The radius of curvature for the ellipse at the vertex P_1 is determined by b^2/a, and the wanted relation is

$$\varrho_1 = \varrho \sin^2 v . \tag{2.3}$$

3. Construction of helices.

We shall show how a helix with given angle of inclination v and with given projection γ_1 on a hyperplane H_1 may be constructed. Let P_0 be an arbitrary point of γ_1, and let s_1 denote the arclength of γ_1 measured from P_0. From a point $P_1 = P_1(s_1)$ we draw a vector $\overrightarrow{P_1P} = s_1 \cot v \boldsymbol{e}$, where $\boldsymbol{e}$ is a *unit* normal vector to H_1 (fig. 1). When P_1 traverses γ_1, the point P traverses a curve γ lying on a cylinder Γ with the normals to H_1 as generators. By development of Γ into a plane γ is transformed into a line which intersects the developed generators at the angle v. Hence, γ is the desired helix. The vector $\boldsymbol{e}$ is a vector of reference for γ.

For a closer examination of the helix γ we must distinguish between two cases according as γ_1 is a helix in H_1 or not.

1°. Let γ_1 *be a helix* in H_1 with a vector $\boldsymbol{f}$ as vector of reference, and let U denote an $(n-2)$-dimensional subspace of H_1 passing through the point P_0 with $\boldsymbol{f}$ as normal vector. The projection of γ_1 on U is called γ_2, P_2 denotes the projection of the point P_1 and u is the angle of inclination

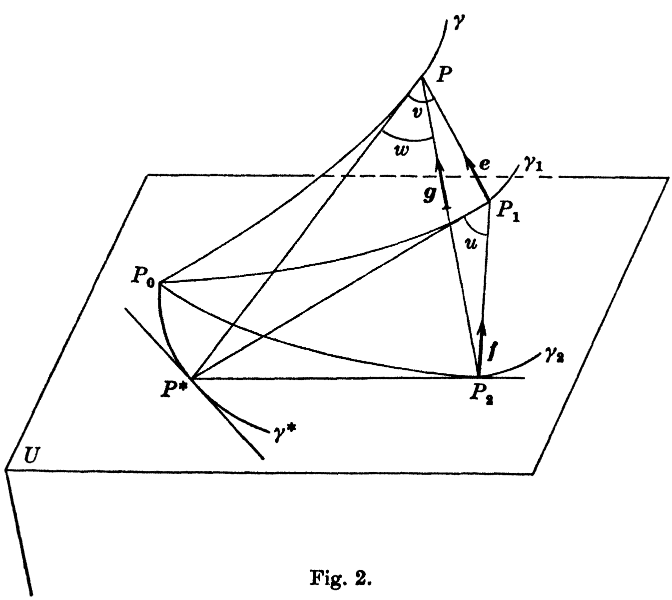

Fig. 2.

of the helix γ_1 (fig. 2). The formerly considered involute γ^* of γ_1 issuing from P_0 is now lying in the subspace U and is common involute of the three curves γ, γ_1 and γ_2. Let P^* be the point on γ^* which corresponds to the points P, P_1 and P_2. The four points P^*, P, P_1 and P_2 are vertices of a tetrahedron which has a right angle along the edge P^*P_1, since $\boldsymbol{e}$ is a normal vector to H_1, and where the angles u and v in the right triangles $P^*P_2P_1$ and P^*P_1P have constant values. Consequently, tetrahedrons corresponding to different positions of P^* are similar.

When P^* traverses γ^*, the lines P^*P, P^*P_1 and P^*P_2 will envelope the evolutes γ, γ_1 and γ_2 of γ^*. The edges P_2P_1 and P_1P keep their directions determined by the vectors $\boldsymbol{f}$ and $\boldsymbol{e}$, and this implies that also the third edge P_2P has a constant direction, fixed by a vector $\boldsymbol{g}$. Since both $\boldsymbol{e}$ and $\boldsymbol{f}$ are normal vectors to U the vector $\boldsymbol{g}$ will be a normal vector to U, and γ is then lying in the hyperplane H, which contains the subspace U and which in the pencil of hyperplanes with basis U is determined by the vector $\boldsymbol{g}$. In this hyperplane H the curve γ_2 is the projection of γ on U.

Thus we have shown that if γ_1 is a helix in H_1 then the constructed helix γ is lying in a hyperplane H. Since γ_1 is the projection of γ there is an affine connection between the two curves. The two projecting cylinders with basis γ_2 through γ or γ_1 are congruent and may be transformed into each other by rotations about the subspace U.

We can find the angle of inclination w of the helix γ with respect to the lines of reference determined by the vector $\boldsymbol{g}$. The tetrahedron $P^*PP_1P_2$ is lying in a space of three dimensions. In this space we consider a sphere with center P^* intersecting the faces of the tetrahedron with common vertex P^* in a right spherical triangle with legs $\frac{1}{2}\pi-u$ and $\frac{1}{2}\pi-v$ and with hypotenuse $\frac{1}{2}\pi-w$. Hence the angle w is determined by the equation

$$\sin w = \sin u \sin v .$$

2°. Now we assume that γ_1 is *not a helix* and in addition that γ_1 *does not lie in a subspace* of H_1. We will show that in this case the constructed helix γ does not lie in a hyperplane.

If γ did lie in a hyperplane H with normal vector $\boldsymbol{e}'$ (linearly independent of $\boldsymbol{e}$) let $\boldsymbol{t}=\boldsymbol{t}(s_1)$ as above denote the unit tangent vector to γ. Then the equation

$$\boldsymbol{t}\cdot\boldsymbol{e}' = 0 \tag{3.1}$$

is satisfied. Let k and k' denote arbitrary constants. Then (2.1) and (3.1) give

$$\boldsymbol{t}\cdot(k\boldsymbol{e}+k'\boldsymbol{e}') = kc . \tag{3.2}$$

Now the constants k and k' may be chosen such that $ke+k'e'$ is a vector f perpendicular to e, and we get

$$t \cdot f = c_1 , \tag{3.3}$$

where c_1 is a new constant.

Since e is assumed to be a unit vector we have $t \cdot e = \cos v$, and we find between the linearly dependent vectors e, t and the unit tangent vector t_1 to γ_1 (fig. 1) the relation

$$t = e \cos v + t_1 \sin v . \tag{3.4}$$

By means of (3.3) we now find

$$t_1 \cdot f = c_2 , \tag{3.5}$$

where c_2 is a constant. Since f is perpendicular to e it belongs to the vector space of H_1. The equation (3.5) shows that according to $c_2 \neq 0$ or $c_2 = 0$ the curve γ_1 is a helix in H_1 or lying in a subspace of H_1, contrary to our assumption on γ_1. Hence γ does not lie in a subspace of R^n.

4. Helices with m linearly independent vectors of reference.

In section 3.1° we have seen that the constructed helix is lying in a hyperplane and has two linearly independent vectors of reference e and g. In this section we study the general case where a helix γ, given as above by $\overrightarrow{OP} = r(s)$, has m linearly independent vectors of reference $e_1, e_2, \ldots, e_m$, corresponding to the m equations

$$t \cdot e_i = c_i \neq 0, \quad i = 1,2,\ldots,m . \tag{4.1}$$

For $m=1$, the curve γ is called an *ordinary helix* in R^n.

If $k_1, k_2, \ldots, k_m$ denote arbitrary constants we find

$$t \cdot \sum k_i e_i = \sum k_i c_i . \tag{4.2}$$

This equation shows that any vector in the vector space $V = V^m$ which is spanned by the vectors e_i, for which $\sum k_i c_i \neq 0$, is a vector of reference for γ. The equation

$$\sum k_i c_i = k_1 c_1 + k_2 c_2 + \ldots + k_m c_m = 0 \tag{4.3}$$

has $m-1$ linearly independent solutions $(k_1, k_2, \ldots, k_m)$ corresponding to $m-1$ linearly independent vectors $e'_1, e'_2, \ldots, e'_{m-1}$ spanning a vector space of $m-1$ dimensions $V' \subset V$. For any of these vectors an equation

$$t(s) \cdot e'_j = 0 \tag{4.4}$$

is valid. Integration of (4.4) shows that the curve γ is lying in a hyperplane with e'_j as normal vector. Hence γ is lying in the intersection of $m-1$ linearly independent hyperplanes, i.e. in a *subspace* R' *of* $n-m+1$ *dimensions*, with V' as normal vector space.

The vector space V contains a one-dimensional vector space $k\boldsymbol{v}$, the vectors of which are orthogonal to V' and consequently lying in the vector space that belongs to R'. Hence γ is an ordinary helix in R' with $\boldsymbol{v}$ as a vector of reference.

Thus *any helix in* R^n *may be regarded as an ordinary helix in a subspace of* R^n. The number of linearly independent vectors of reference determines the dimension of the subspace, in which the ordinary helix is lying.

The helix γ which we have considered in section 3.1° is not an ordinary helix in R^n, but it is ordinary in some subspace R' of $n-m+1$ dimensions where $m \geqq 2$. The helix γ examined in section 3.2° does not lie in any subspace of R^n and consequently it is an ordinary helix in R^n. For this helix γ we have $m=1$.

DANMARKS TEKNISKE HØJSKOLE, LYNGBY, DENMARK