MATH. SCAND. 35 (1974), 165–174

FØLNER'S CONDITION FOR EXPONENTIALLY BOUNDED GROUPS

J. W. JENKINS*

Introduction.

For a locally compact group G it is known (cf. [3]) that amenability of G is equivalent to a uniform Følner's condition, i.e.

(FC) for each compact subset K of G and each $\varepsilon > 0$ there is a compact subset of U of G with $0 < |U|_G$ such that

$$|KU \triangle U|_G < \varepsilon |U|_G .$$

($|A|_G$ denotes the left Haar measure of a subset A of G.) In [4] a method has been given for constructing sets U that satisfy FC for a given K and ε, for connected solvable groups. The question we consider here is when can the set U be chosen from the sets K^n, $n = 1, 2, \ldots$? To effectively deal with this question we must obviously restrict our attention to compact subsets K with $|K|_G > 0$. Therefore, we consider the following condition:

(F) For each compact subset K of G with $\operatorname{int}(K) \neq \emptyset$, and for each $\varepsilon > 0$ there is a positive integer n such that

$$|KK^n \triangle K^n|_G < \varepsilon |K^n|_G .$$

In order that a group G satisfy F, G must satisfy topological as well as algebraic conditions, as the following example shows.

Let Z denote the discrete group of integers and let Z_o and Z_e denote the subsets of odd and even integers respectively. If K is a finite subset of Z_o then $K^{2n+1} \subset \mathsf{Z}_o$ while $K^{2n} \subset \mathsf{Z}_e$. Thus,

$$|K^{n+1} \triangle K^n|_{\mathsf{Z}} \geqq 2 |K^n|_{\mathsf{Z}}$$

for each positive integer n.

We will show that if G is connected, $K \subset G$, and $\operatorname{int}(K) \neq \emptyset$ then for n sufficiently large $|K^{n+1} \triangle K^n|_G > 0$. Hence the pathology that arises in

* Research partially supported by National Science Foundation Grant #PO28925 and SUNY Faculty Fellowship.

Received May 21, 1973; in revised form December 14, 1973.

the above example is precluded, and the desired characterization is obtained.

A locally compact group G is said to be exponentially bounded if for each compact neighborhood U of the identity,

$$\liminf_n |U^n|_G^{1/n} = 1$$

The discrete solvable, and the connected groups that are exponentially bounded can be characterized by their subsemigroups as follows: for $a,b \in G$ let $[a,b]$ denote the subsemigroup of G generated by a and b. $[a,b]$ is free if $a[a,b] \cap b[a,b] = \emptyset$. $[a,b]$ is uniformly discrete if there is a neighborhood U of e in G such that $sU \cap tU = \emptyset$ for $s,t \in [a,b]$, $s \neq t$. The results of [6] and [9] show that if G is a discrete solvable group or if G is a connected locally compact group then G is exponentially bounded if and only if G does not contain a uniformly discrete free subsemigroup on two generators. We show that a connected, locally compact group G satisfies F if and only if G is exponentially bounded.

It is well-known (cf. [5]) that a closed subgroup of an amenable group is itself amenable. However, even the open subsemigroups of an amenable group may fail to be amenable. Using an alternate characterization of the exponentially bounded, connected, locally compact groups given in terms of the eigenvalues of operators occuring in the adjoint representation of an approximating Lie group, we show that G satisfies F if and only if each open subsemigroup of G is amenable.

Using this result we also obtain a characterization of the connected amenable groups satisfying F in terms of convolution operators. More specifically, given a locally compact group G let $L^1(G)$ and $L^\infty(G)$ be defined as usual. Let $L_c^1(G)$ denote the elements of $L^1(G)$ with compact (essential) support. We consider the following property:

(P) if $0 \neq \theta \in L^\infty(G)$, $\theta \geqq 0$, and $x \in L_c^1(G)$ then $x * \theta \geqq 0$ implies

$$\int_G x(g)\,dg \geqq 0 .$$

It follows from the results in [9] that a discrete solvable group G is exponentially bounded if and only if G satisfies P. We prove that conditions F and P are equivalent for connected, amenable, locally compact groups.

Theorems and proofs.

LEMMA 1. *Let G be an exponentially bounded, locally compact group. Let K be a compact subset of G with* $\operatorname{int}(K) \neq \emptyset$ *and assume that for some* *positive integer p there is a $g \in \operatorname{int}(K)$ such that $g^{p+1} \in K^p$. Then given $\varepsilon > 0$ there is a positive integer n such that*

$$|K^{n+1} \triangle K^n|_G < \varepsilon |K^n|_G .$$

Proof. Let g and p be as in the statement of the lemma. Let $U = g^{-1}K$ and for each integer $n \geqq 0$ set $V_n = g^{-n} U g^n$. Then

$$\begin{aligned} K^n &= gUgU \ldots gU \\ &= g^n(g^{-n+1}Ug^{n-1}) \ldots (g^{-1}Ug)U \\ &= g^n V_{n-1} V_{n-2} \ldots V_0 . \end{aligned}$$

Let W be any compact neighborhood of e in G. Then, since G is exponentially bounded,

$$\liminf_n |K^n|_G^{1/n} \leqq \liminf_n |(K \cup W)^n|^{1/n} = 1 .$$

Thus, given any positive integer l, there is a sequence of positive integers $\{n_m\}$ such that

$$\lim_m |K^{n_m+l}|_G / |K^{n_m}|_G = 1 .$$

(If not, there is a $\delta > 1$ such that for all $n > 0$

$$|K^{nl}|_G = (|K^{nl}|_G/|K^{(n-1)l}|_G) \ldots (|K^{2l}|_G/|K^l|_G)|K^l|_G \geqq \delta^n .$$

Hence

$$\liminf |(K^l \cup W)^n|_G^{1/n} \geqq \delta) .$$

Now, since $g^{p+1} \in K^p$, $g \in g^{-p}K^p = V_{p-1} \ldots V_0$. Thus, for any integer n, $g \in g^n V_{p-1} \ldots V_0 g^{-n}$. Finally,

$$\begin{aligned} g = g^{-2n}gg^{2n} \in g^{-2n}g^n V_{p-1} \ldots V_0 g^{-n} g^{2n} &= g^{-n} V_{p-1} \ldots V_0 g^n \\ &= (g^{-n} V_{p-1} g^n)(g^{-n} V_{p-2} g^n) \ldots (g^{-n} V_0 g^n) \\ &= V_{p+n-1} V_{p+n-2} \ldots V_n . \end{aligned}$$

We therefore have

$$g V_{n-1} \ldots V_0 \subset V_{n+p-1} \ldots V_0 \quad \text{for each } n \geqq 0 .$$

We can now write

$$\begin{aligned} |K^{n+1} \triangle K^n|_G &= |g^{n+1} V_n \ldots V_0 \triangle g^n V_{n-1} \ldots V_0|_G \\ &\leqq |g V_n \ldots V_0 \triangle V_n \ldots V_0|_G + |V_n \ldots V_0 \cap C(V_{n-1} \ldots V_0)|_G \\ &= 2|g V_n \ldots V_0 \cap C(V_n \ldots V_0)|_G + |V_n \ldots V_0 \cap C(V_{n-1} \ldots V_0)|_G \\ &\leqq 2|V_{n+p} \ldots V_0 \cap C(V_n \ldots V_0)|_G + |V_n \ldots V_0 \cap C(V_{n-1} \ldots V_0)|_G \end{aligned}$$

$$= 2|g^{p+n+1}K^{n+1+p} \setminus g^{n+1}K^{n+1}|_G + |g^{n+1}K^{n+1} \setminus g^n K^n|_G$$
$$= 2(|K^{n+p+1}|_G - |K^{n+1}|_G) + |K^{n+1}|_G - |K^n|_G$$
$$\leqq 3(|K^{n+p+1}|_G - |K^{n+1}|_G) .$$

Therefore, by chosing a sequence $\{n_m\}$ so that

$$\lim_m |K_{n_m+p+1}|_G / |K_{n_m+1}|_G = 1$$

we have

$$\lim_m |K^{n_m+1} \triangle K^{n_m}|_G / |K^{n_m}|_G = 0 ,$$

and the lemma is proved.

THEOREM 2. *A connected, locally compact group G satisfies* F *if and only if G is exponentially bounded.*

PROOF. It is easily seen that any locally compact group satisfying F is exponentially bounded, so we consider the converse. Assume therefore that G is exponentially bounded, and that K is a compact subset of G with $\operatorname{int}(K) \neq \emptyset$. By lemma 1, we have only to show that there is a $g \in \operatorname{int}(K)$ and a positive integer p such that $g^{p+1} \in K^p$. For this, we first assume that G is a simply connected solvable Lie group. Let $\mathfrak{g}$ be the Lie algebra of G, and let exp denote the exponential map of $\mathfrak{g}$ into G. Then $\exp(\mathfrak{g})$ is dense in G (cf. [2]). Thus

$$\exp(\mathfrak{g}) \cap \operatorname{int}(K) \neq \emptyset .$$

Hence, there is a one parameter subgroup $g(t)$ of G such that $g(t) \in \operatorname{int}(K)$ for $\alpha \leqq t \leqq \beta$ where $0 < \alpha < \beta$. By choosing a positive integer p sufficiently large, we have $\beta p > \alpha(p+1)$. Hence

$$(\alpha p, \beta p) \cap (\alpha(p+1), \beta(p+1)) \neq \emptyset .$$

Choose $\lambda \in (\alpha, \beta)$ such that $\lambda(p+1) \in (\alpha p, \beta p)$. Then, if $g = g(\lambda)$,

$$g^{p+1} = g(\lambda(p+1)) \in K^p .$$

If G is a connected, solvable Lie group and $(\tilde{G}, \pi)$ is the universal covering group of G, then by passing to $\tilde{G}$ and constructing the appropriate $\tilde{g} \in \tilde{G}$ as above, $\pi(\tilde{g})$ will have the desired properties.

Assume now that G is a connected Lie group, and that S is the solv-radical of G. It was shown in [6], that if G is exponentially bounded then G/S is compact. Let K be a compact subset of G with $\operatorname{int}(K) \neq \emptyset$. Then, if π is the canonical mapping of G into G/S, $\operatorname{int}(\pi(K)) = \emptyset$. Let W be a neighborhood of the identity in G/S and let $k \in K$ such that

$\pi(k)W \subset \mathrm{int}(\pi(K))$. Since G/S is compact there is a positive integer n such that e, the identity in G/S, belongs to $\pi(k^n)W$. Thus

$$K_0 = S \cap \mathrm{int}(K^n) \neq \emptyset .$$

Now by a simple modification of the argument used at the beginning of this proof, we can produce a $g \in K_0$ such that $g^{np+1} \in K_0{}^p \subset K^{np}$. Therefore, if G is a connected Lie group that is exponentially bounded then G satisfies F.

To complete the proof we need the following observations. Let G be any locally compact group and H a compact normal subgroup of G. Normalize the Haar measures of G, H, and G/H so that $|H|_H = 1$, and

$$\int_G f(g)\,dg = \int_{G/H}\int f(gh)\,dh\,dg$$

for each compactly supported continuous function f. Then, for compact $K \subset G$,

$$|K|_G \leqq |\pi(K)|_{G/H} ,$$

where π is the canonical homomorphism of G onto G/H, and if $\mathrm{int}(K) \neq \emptyset$, then there is a $\delta > 0$ such that

$$|K^n|_G \geqq \delta|\pi(K^{n-1})|_{G/H} ,$$

for all positive integers n. To see the latter, first note that

$$\begin{aligned} |K^n|_G &= \int_G x_{K^n}(g)\,dg \\ &= \int_{G/H}\int_H x_{K^n}(gh)\,dh\,d\bar{g} \\ &= \int_{G/H} |g^{-1}K^n \cap H|_H\,d\bar{g} . \end{aligned}$$

Let V be a neighborhood of e in G and K_0 a subset of K such that $K_0 V \subset K$. Then, $K^{n-1}K_0 V \subset K^n$ and hence, if $g \in K^{n-1}K_0$, then $g^{-1}K^n \supset V$. Thus

$$|g^{-1}K^n \cap H|_H \geqq |V \cap H|_H = \delta' > 0$$

for each $g \in K^{n-1}K_0$. Hence

$$|K^n|_G \geqq \delta'|\pi(K^{n-1}K_0)|_{G/H} .$$

Now, set

$$\delta = \delta' \inf\{\Delta_{G/H}(\pi(k)) \mid k \in K_0\} ,$$

and we are through. (Since exponentially bounded groups are unimodular, this last step was not necessary for our purpose.)

We can now complete the proof. Let G be any connected, exponentially bounded, locally compact group. Let H be a compact normal subgroup of G such that G/H is a Lie group. Then G/H is exponentially bounded (cf. [6]) and hence satisfies F. Let K be a compact subset of G with $\mathrm{int}(K) \neq \emptyset$ and choose $\delta > 0$ as in the preceeding paragraph. Then

$$|K^{n+1} \triangle K^n|_G / |K^n|_G \leqq |\pi(K^{n+1}) \triangle \pi(K^n)|_{G/H} / \delta |\pi(K^n)|_{G/H} \,.$$

It now readily follows that G satisfies F.

If G is a connected Lie group and $g \in G$, the inner-automorphism determined by g is denoted by σ_g and is a diffeomorphism of the manifold G that fixes the identity e. The differential of σ_g at e, is an automorphism of the Lie algebra $\mathfrak{g}$ of G, here identified with the tangent space of G at the identity. The mapping $g \to d\sigma_{g|e}$ is a representation of G on $\mathfrak{g}$ called the adjoint representation. We will write $\mathrm{Ad}g$ for $d\sigma_{g|e}$. G is said to be type R if for each $g \in G$, the eigenvalues of $\mathrm{Ad}g$ are of modulus one.

If G is a connected locally compact group and U is any neighborhood of e then there exist a compact, normal subgroup K of G such that $K \subset U$ and G/K is a Lie group (cf. [8]). G is said to be type R if for some compact, normal subgroup K, G/K is a type R Lie group. In [6], we proved that G is exponentially bounded if and only if G is type R. This characterization will be essential in proving the next theorem.

Let S be a subsemigroup of a locally compact group G, and let $\mathrm{LUC}(S)$ denote the space of left uniformly continuous functions on S, i.e. a bounded continuous function $f \in \mathrm{LUC}(S)$ if and only if whenever $s_\alpha \to s_0$, $s_\alpha, s_0 \in S$,

$$\sup\{|f(s_\alpha s) - f(s_0 s)| : s \in S\} \to 0 \,.$$

S is said to be amenable if there is a positive linear functional of norm one, μ, defined on $\mathrm{LUC}(S)$, such that $\mu({}_s f) = \mu(f)$ for each $f \in \mathrm{LUC}(S)$ and $s \in S$, where ${}_s f(t) = f(st)$ for each s, t, f. In [7] we proved that if G is amenable and S is an open semgroup in G then S is amenable if and only if the right ideals of S have the finite intersection property. We now have

THEOREM 3. *A connected locally compact group G satisfies* F *if and only if each open subsemigroup of G is amenable.*

PROOF. Assume that G satisfies F. Then G is exponentially bounded and hence amenable. If S is an open, nonamenable subsemigroup of G, then by the results referred to above, there exist disjoint right ideals I and J of S. Hence, if $a \in I$ and $b \in J$, $[a, b]$ is free and uniformly discrete. Thus, by [6], G is not exponentially bounded.

Conversely, assume that G does not satisfy F. Then by [6], G is not type R. Let K be a compact normal subgroup of G such that $H = G/K$

is a nontype R Lie group. If S is the solv-radical of H and S is type R then the semisimple Lie group H/S is not type R, and is thus not compact. Hence, H/S is not amenable, and so also G.

Assume then that S is not type R. If $\tilde{S}$ denotes the universal covering group of S then $\tilde{S}$ is not type R. We define the groups S_2, $S_3{}^\sigma$ and S_4 as follows: S_2 is the semidirect product of R with itself with multiplication defined by

$$(x,y)(u,v) = (x+u, e^x v + y) \quad (x,y,u,v \in \mathsf{R}) .$$

For $\sigma \in \mathsf{R}$, $\sigma \neq 0$, $S_3{}^\sigma$ is the semidirect product of R with R^2 where multiplication is defined by

$$(x,\alpha)(y,\beta) = \big(x+y, A(x)\beta + \alpha\big) \quad (\alpha,\beta \in \mathsf{R}^2;\ x,y \in \mathsf{R})$$

with

$$A(x) = e^{\sigma x}\begin{pmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{pmatrix} .$$

Finally, S_4 is the semidirect product of R^2 with itself where multiplication is defined by

$$(x,y,\alpha)(u,v,\beta) = \big(x+u, y+v, B(x,y)\beta + \alpha\big)$$
$$\big(x,y,u,v \in \mathsf{R};\ (\alpha,\beta) \in \mathsf{R}^2\big)$$

with

$$B(x,y) = e^y\begin{pmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{pmatrix} .$$

In [1], it was shown that a simply connected solvable Lie group that is not type R has a homomorphic image one of the group S_2, $S_3{}^\sigma$ $(\sigma \neq 0)$, S_4. Therefore, to complete the proof of theorem 3, we need only show that these groups contain open semigroups with disjoint right ideals, for then, $\tilde{S}$ would be homomorphic to a group containing an open semigroup with disjoint right ideals, and so, by taking preimages, there exists an open semigroup S_0 in $\tilde{S}$, with disjoint right ideals, I and J. Now if π is the covering map of $\tilde{S}$ into S then $\pi(S_0)$ is open in S and $\pi^{-1}(e) \subset C(\tilde{S})$, the center of $\tilde{S}$. Thus, if $\pi(I) \cap \pi(J) \neq \emptyset$ then $I^{-1}J \cap C(\tilde{S}) \neq \emptyset$. If $a \in I$, $b \in J$ and $a^{-1}b = z \in C(\tilde{S})$ then $ba = aza = a^2 z = ab$. But $ab \in I$, $ba \in J$ and $I \cap J = \emptyset$. This contradiction shows that $\pi(I) \cap \pi(J) = \emptyset$ and hence $\pi(S_0)$ is not amenable.

We will only consider S_4; obvious modifications in our construction will produce the desired subsemigroups in S_2 and $S_3{}^\sigma$, $\sigma \neq 0$. First note that S_4 can alternatively be described as the semidirect product of C, the complex numbers, with itself where multiplication is defined by

$$(z,w)(n,\zeta) = (z+\eta, e^z\zeta + w) \quad (z,w,\eta,\zeta \in \mathsf{C}) .$$

with this description, let

$$U^+ = \{(z,w) \in S_4 \mid \operatorname{Re}(z) < -2,\ |w-1| < e^{-2}\},$$

$$U^- = \{(z,w) \in S_4 \mid \operatorname{Re}(z) < -2,\ |w+1| < e^{-2}\},$$

and

$$S = \bigcup_{n=1}^{\infty} U_1 \ldots U_n, \quad U_i \in \{U^+, U^-\}, \quad i = 1,2,\ldots n.$$

S is an open subsemigroup of S_4. If $(\eta,\zeta) \in S$ then there exists (z_i,w_i), $1 \leqq i \leqq n$, such that $(z_i,w_i) \in U^+ \cup U^-$ and

$$(\eta,\zeta) = (z_1,w_1)\ldots(z_n,w_n).$$

Hence $\eta = \sum_{i=1}^{n} z_i$ and

$$\zeta = w_1 + e^{z_1}w_2 + \ldots + e^{z_1 + \ldots + z_{n-1}}w_n.$$

Now, if $(z_1,w_1) \in U^+$

$$|\zeta - 1| \leqq |w_1 - 1| + (1+e^{-2})(e^{-2} + \ldots + e^{-2(n-1)}) < 1,$$

while if $(z_1,w_1) \in U^-$

$$|\zeta + 1| \leqq |w_1 + 1| + (1+e^{-2})(e^{-2} + \ldots + e^{-2(n-1)}) < 1.$$

Therefore U^+S and U^-S are disjoint right ideals of S.

REMARK. Even for exponentially bounded, connected, locally compact groups, the closed subsemigroups need not be amenable.

Let $G = \mathsf{R} \times \mathrm{SO}(3,\mathsf{R})$. Then clearly G is connected and exponentially bounded. Let $\alpha,\beta \in \mathrm{SO}(3,\mathsf{R})$ such that the group generated by α and β $\langle\alpha,\beta\rangle$ is free (cf. [5] for the existence of such elements), and set $a = (1,\alpha)$ and $b = (1,\beta)$. Since for any M, the number of elements $(r,\lambda) \in [a,b]$ with $r < M$ is finite, $[a,b]$ is closed. Since $\langle\alpha,\beta\rangle$ is a free group, $[a,b]$ is a free semigroup. Finally note that $[a,b]$ is discrete in G, and hence any bounded function on $[a,b]$ is left uniformly continuous.

Now, if μ is a left invariant mean for the bounded functions on $[a,b]$ then

$$1 = \mu(\chi_{[a,b]}) \leqq \mu(\chi_{a[a,b]}) + \mu(\chi_{b[a,b]}) = 2.$$

Therefore $[a,b]$ is not amenable.

We can now prove

THEOREM 4. *For a connected, amenable locally compact group conditions* F *and* P *are equivalent.*

PROOF. Assume G is a connected, locally compact group that does not satisfy F. By theorem 3 there exists an open, nonamenable subsemigroup S of G, and by [7], S contains open disjoint right ideals I and J. Let $a \in I$, $b \in J$ and let $U = U^{-1}$ be a compact neighborhood of e such that $UaU \subset I$ and $UbU \subset J$. Let $x = x_1 - (x_2 + x_3)$ where $x_i = \chi_{c_i U}$ with $c_1 = a$, $c_2 = ab$, $c_3 = a^2$ and let $\theta = \chi_S$. First note that

$$\operatorname{supp}(x_i * \theta) \subset c_i US \quad \text{for } 1 \leqq i \leqq 3 ,$$

and that $c_2 US \cap c_3 US = \varnothing$. Furthermore, if $g \in c_i US$, $i = 2,3$ then

$$g^{-1} aU \subset S^{-1} U c_i^{-1} aU \subset S^{-1} .$$

Thus, if $g \in c_i US$, $i = 2,3$, then

$$\begin{aligned} x_1 * \theta(g) &= \int \chi_{c_1 U}(s) \chi_S(s^{-1} g) ds \\ &= |c_1 U \cap g S^{-1}|_G \\ &= |g^{-1} c_1 U \cap S^{-1}| = |U|_G . \end{aligned}$$

Now, for each $i = 1,2,3$ and any $g \in G$, $x_i * \theta(g) \leqq |U|_G$. Therefore, since

$$\operatorname{supp}(x_2 * \theta) \cap \operatorname{supp}(x_3 * \theta) = \varnothing ,$$

we have

$$(x_2 + x_3) * \theta(g) \leqq |U|_G .$$

Hence $x * \theta \geqq 0$. However,

$$\int x(g) dg = -|U|_G < 0 .$$

Conversely, suppose G satisfies F and let $0 \neq \theta \in L^\infty(G)$ such that $\theta \geqq 0$. Let $x \in L^1{}_c(G)$ for which $x * \theta \geqq 0$, and let U be a compact neighborhood of e in G such that $\operatorname{supp}(x) \subset U$ and such that, if

$$x_n = \int_{U^n} \theta(g) dg \quad (n = 1, 2, \ldots)$$

then $x_1 > 0$. Then $0 < x_n < |U^n|_G$ for all n, and hence by theorem 2,

$$\liminf_n (x_n)^{1/n} = 1 .$$

Thus, there exist a subsequence $\{n_k\}$ with

$$\lim_k x_{n_k+1} / x_{n_k} = 1 .$$

Now, since $x * \theta \geqq 0$, for each k

$$\begin{aligned} 0 &\leqq \int_{U^{n_k}} x * \theta(g) dg \\ &= \int_{U^{n_k}} \left(\int_U x(s) \theta(s^{-1} g) ds \right) dg \\ &= \int_U x(s) \left(\int_{U^{n_k}} \theta(s^{-1} g) dg \right) ds . \end{aligned}$$

Thus, for each k

$$0 \leqq \int_U x(s)\{(x_{n_k})^{-1} \int_{U^{n_k}} \theta(s^{-1}g)dg\}ds \,.$$

But, for $s \in U$, $U^{n_k-1} \subset s^{-1}U^{n_k} \subset U^{n_k+1}$, and hence

$$x_{n_k-1}/x_{n_k} \leqq A_k(s) = (x_{n_k})^{-1} \int_{U^{n_k}} \theta(s^{-1}g)dg \leqq x_{n_k+1}/x_{n_k} \,.$$

Consequently, $\lim_k A_k(s) = 1$ uniformly for $s \in U$ and thus

$$0 \leqq \lim_k \int_U x(s)A_k(s)ds = \int_U x(s)ds \,.$$

REFERENCES

1. L. Auslander and C. Moore, *Unitary representations of solvable Lie groups*, Memoirs Amer. Math. Soc. 62 (1966).
2. J. Diximier, *L'application exponentielle dans les groupes de Lie resolubles*, Bull. Soc. Math. France, 85 (1957), 325–388.
3. W. Emerson and F. Greenleaf, *Covering properties and Følner conditions for locally compact groups*, Math. Z. 102 (1967), 370–384.
4. F. Greenleaf, *Ergodic theorems and the construction of summing sequences in amenable locally compact groups*, (to appear).
5. F. Greenleaf, *Invariant means on topological groups*, Van Nostrand, Princeton, N.J., 1969.
6. J. Jenkins, *Growth of connected locally compact groups*, (to appear, J. Functional Analysis).
7. J. Jenkins, *Amenable subsemigroups of a locally compact group*, Proc. Amer. Math. Soc. 25 (1970), 766–770.
8. D. Montgomery and L. Zippin, *Topological transformation groups*, Interscience Publishers, New York, 1955.
9. J. Rosenblatt, *Invariant linear functionals and counting conditions*, Thesis, University of Washington, Seattle, Washington, 1972.

STATE UNIVERSITY OF NEW YORK AT ALBANY, U.S.A.