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INVARIANT FOURIER INTEGRAL OPERATORS ON LIE GROUPS

BØRGE P. D. NIELSEN and HENRIK STETKÆR

1. Introduction.

This paper follows the notations of Hörmander [3] to which we refer for the definition and proofs of properties of Fourier integral operators.

In Section 3 we show that a necessary and sufficient condition for a class of Fourier integral operators on a Lie group G (i.e. a class $I_\varrho^m(G\times G,\Lambda)$ of Fourier integral distributions on $G\times G$) to be left-invariant is that the Lagrangean submanifold Λ of $T^*(G\times G)\setminus 0$ is left-invariant. The analysis of the set of closed conic Lagrangean submanifolds of $T^*(G\times G)\setminus 0$ which are left-invariant, is carried out in Section 4.

In Section 5 we prove that up to a constant factor there is a canonical isomorphism from the set of left-invariant operators in a left-invariant class $I_\varrho^m(G\times G,\Lambda)$ onto the class of Fourier integral distributions $I_\varrho^{m+\dim G/4}(G,\Lambda_e)$ on G. The isomorphism is given by a kind of point evaluation at the identity element $e\in G$. Here $\dim G$ enters due to conventions, and Λ_e denotes the Lagrangean submanifold of $T^*G\setminus 0$ which arises by the transversal intersection of Λ and the part of $T^*(G\times G)$ lying above $\{e\}\times G$. Also the connection between the principal symbols of operators related by this isomorphism is explicitly described.

In Section 6 we briefly discuss the set of bi-invariant operators in a left-invariant class. Examples show there do exist non-trivial, bi-invariant Fourier integral operatos, in contrast to the case of pseudodifferential operators, cfr. A. Melin [4], L. P. Rothschild [5] and H. Stetkær [6].

Finally we wish to thank A. Melin for advice that led to considerable improvements of the exposition.

2. Notations.

By a manifold we shall understand a C^∞ paracompact manifold, and by a submanifold an imbedded submanifold. A smooth map means a C^∞ map.

The cotangent bundle of a manifold M will be denoted T^*M, its zero-

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section 0, its projection map $\pi_M\colon T^*M \to M$, the canonical 1-form Θ_M and the canonical (symplectic) 2-form ω_M, cfr. Abraham [1, p. 96]. When

$$f\colon M_1 \to M_2$$

is a diffeomorphism between the manifolds M_1 and M_2 then

$$f^*\colon T^*M_2 \to T^*M_1$$

denotes the induced diffeomorphism of the cotangent bundles. Note that $f^*(T^*M_2 \setminus 0) = T^*M_1 \setminus 0$.

We shall identify $T^*(M_1 \times M_2)$ and $T^*M_1 \times T^*M_2$ in the canonical way; thus a point in $T^*_{(m_1,m_2)}(M_1 \times M_2)$ will be written (m_1,m_2,ξ_1,ξ_2), where $\xi_j \in T^*_{m_j}(M_j)$, $j=1,2$. We also choose to identify a manifold with the zero section of its cotangent bundle, so that in particular

$$M_1 \times T^*M_2 \subseteq T^*(M_1 \times M_2)\ .$$

The line bundle of densities of order α, $\alpha \in \mathsf{R}$, (cfr. Hörmander [3, pp. 117–118]) on a manifold M is denoted $\Omega_\alpha(M)$, the vector space of its smooth sections by $C^\infty(M,\Omega_\alpha)$, the vector space of smooth sections with compact support by $C_0^\infty(M,\Omega_\alpha)$ and its dual space by $\mathscr{D}'(M,\Omega_{1-\alpha})$. As customary we view $C^\infty(M,\Omega_\alpha)$ as a subspace of $\mathscr{D}'(M,\Omega_\alpha)$.

It is well-known how a diffeomorphism $f\colon M_1 \to M_2$ between two manifolds M_1 and M_2 induces a line bundle equivalence

$$\Omega^\alpha(f)\colon \Omega_\alpha(M_2) \to \Omega_\alpha(M_1) \quad \text{for any } \alpha \in \mathsf{R}\ .$$

The corresponding map of sections

$$f^\alpha\colon C_0^\infty(M_2,\Omega_\alpha) \to C_0^\infty(M_1,\Omega_\alpha)$$

induces by transposition an isomorphism, viz.

$$f_\alpha := {}^t f^{1-\alpha}\colon \mathscr{D}'(M_1,\Omega_\alpha) \to \mathscr{D}'(M_2,\Omega_\alpha)\ .$$

Let L_1 and L_2 be complex line bundles over manifolds M_1 and M_2 with structure groups H_1 and H_2. Let $\pi_1\colon M_1 \times M_2 \to M_1$ and $\pi_2\colon M_1 \times M_2 \to M_2$ be the projections. The exterior tensor product of L_1 and L_2 is defined by

$$L_1 \boxtimes L_2 := \pi_1^*L_1 \otimes \pi_2^*L_2\ ,$$

and is a line bundle over $M_1 \times M_2$ with structure group $H_1 \otimes H_2$.

If $d_1\colon M_1 \to L_1$ and $d_2\colon M_2 \to L_2$ are sections we let $d_1 \boxtimes d_2$ denote the obvious section in $L_1 \boxtimes L_2$.

Let us note that we may identify the bundles $\Omega_{\frac12}(M_1) \boxtimes \Omega_{\frac12}(M_2)$ and $\Omega_{\frac12}(M_1 \times M_2)$ by a map I as follows:

An element $d_j \in \Omega_{\frac{1}{2}}(M_j)$, $j=1,2$, in the fiber over the point $p_j \in M_j$ is a map

$$d_j\colon \Lambda^{n_j}(T_{p_j}M_j) \setminus \{0\} \to \mathbf{C} \quad (n_j = \dim M_j)$$

with the property that

$$d_j(s\sigma) = |s|^{\frac{1}{2}} d_j(\sigma) \quad \text{for } s \in \mathbf{R} \setminus \{0\}$$

$$\text{and } \sigma \in \Lambda^{n_j}(T_{p_j}M_j) \setminus \{0\} .$$

The map

$$I\colon \Omega_{\frac{1}{2}}(M_1) \boxtimes \Omega_{\frac{1}{2}}(M_2) \to \Omega_{\frac{1}{2}}(M_1 \times M_2)$$

defined by

$$I(d_1 \boxtimes d_2)(\sigma_1 \wedge \sigma_2) = d_1(\sigma_1) d_2(\sigma_2) \quad \text{for } \sigma_j \in \Lambda^{n_j}(T_{p_j}M_j) \setminus \{0\}$$

is clearly fiber preserving and an isomorphism in each fiber, hence it is a bundle isomorphism.

If d_j, $j=1,2$, denote sections in $\Omega_{\frac{1}{2}}(M_j)$ having function representatives a_j in the charts $\varkappa_j$, then $I \circ (d_1 \boxtimes d_2)$ is the section in $\Omega_{\frac{1}{2}}(M_1 \times M_2)$ that is represented by the function $(a_1 \circ \pi_1)(a_2 \circ \pi_2)$ in the product chart $\varkappa_1 \times \varkappa_2$.

To avoid excessive notation we write

$$d_1 \otimes d_2 := I(d_1 \boxtimes d_2)$$

analogous to the case of functions.

Let us note that a (paracompact) manifold always has a nowhere vanishing density, so that it follows that any $u \in C_0^\infty(M_1 \times M_2, \Omega_{\frac{1}{2}})$ can be written in the form

$$u = \tilde{u} d_1 \otimes d_2 ,$$

where $\tilde{u} \in C_0^\infty(M_1 \times M_2)$ and where $d_j \in C^\infty(M_j, \Omega_{\frac{1}{2}})$, $j=1,2$, never vanish.

The tensor product $A \otimes B$ of $A \in \mathscr{D}'(M_1, \Omega_{\frac{1}{2}})$ and $B \in \mathscr{D}'(M_2, \Omega_{\frac{1}{2}})$ can now be defined as an element of $\mathscr{D}'(M_1 \times M_2, \Omega_{\frac{1}{2}})$ as follows:

If $u = \tilde{u} d_1 \otimes d_2 \in C_0^\infty(M_1 \times M_2, \Omega_{\frac{1}{2}})$ where $\tilde{u} \in C_0^\infty(M_1 \times M_2)$ and $d_j \in C^\infty(M_j, \Omega_{\frac{1}{2}})$, $j=1,2$, then

$$\langle A \otimes B, u \rangle := \langle B_y, \langle A_x, \tilde{u}(x,y) d_1 \rangle d_2 \rangle .$$

It is easy to see that this is independent of the way u is written.

If in particular $d_j \in C_0^\infty(M_j, \Omega_{\frac{1}{2}})$ then we find

$$\langle A \otimes B, d_1 \otimes d_2 \rangle = \langle A, d_1 \rangle \langle B, d_2 \rangle .$$

That could of course also have been taken as a basis for the definition of $A \otimes B$.

A Fourier integral distribution

$$A \in I_\varrho^m(M_1 \times M_2, \Lambda) \subset \mathscr{D}'(M_1 \times M_2, \Omega_{\frac{1}{2}}) ,$$

where Λ is a conic, closed Lagrangean submanifold of $T^*(M_1 \times M_2) \setminus 0$ defines a continuous bilinear form on $C_0^\infty(M_1, \Omega_{\frac{1}{2}}) \times C_0^\infty(M_2, \Omega_{\frac{1}{2}})$ and thus defines a continuous linear map from $C_0^\infty(M_2, \Omega_{\frac{1}{2}})$ to $\mathscr{D}'(M_1, \Omega_{\frac{1}{2}})$, also denoted by A and referred to as a Fourier integral operator.

In this paper G will always mean an n-dimensional connected Lie group with identity element e. For $g \in G$ we let $L(g): G \to G$ ($R(g): G \to G$) denote left- (right-) translation by g, i.e.

$$L(g)h = gh \quad \text{for every } h \in G$$

$$R(g)h = hg \quad \text{for every } h \in G .$$

We also use these notations when working with $G \times G$, i.e. $L(g)(h_1, h_2) = (gh_1, gh_2)$ etc.

Note that $L(g)^*$ maps $T^*_h G$ onto $T^*_{g^{-1}h} G$ for every h in G.

DEFINITION 2.1. A subset $\Lambda \subseteq T^*(G \times G)$ is said to be *left-invariant* if

$$L(g)^* \Lambda \subseteq \Lambda \quad \text{for every } g \in G ,$$

right-invariant if

$$R(g)^* \Lambda \subseteq \Lambda \quad \text{for every } g \in G ,$$

and *bi-invariant* if it is left- and right-invariant.

Finally $\mathrm{id}_S: S \to S$ denotes the identity map on the set S. When S is obvious from the context we shall drop the suffix.

3. Fourier integral operators on manifolds.

In this section we collect what we will need of general facts about Fourier integral operators on manifolds.

Let $f: M_1 \to M_2$ be a diffeomorphism between two manifolds M_1 and M_2. Then $f_{\frac{1}{2}}$ transforms the Fourier integral distributions in $I_\varrho^m(M_1, \Lambda)$ to elements in $\mathscr{D}'(M_2, \Omega_{\frac{1}{2}})$. To determine the image we note that Λ is a conic, closed Lagrangean submanifold of $T^*M_1 \setminus 0$ if and only if $(f^*)^{-1}(\Lambda)$ is a conic closed Lagrangean submanifold of $T^*M_2 \setminus 0$. The very definition of Fourier integral distributions (Hörmander [3, p. 147]) then yields the following result:

PROPOSITION 3.1. *Let $f\colon M_1 \to M_2$ be as above. The restriction of the map*

$$f_{\frac{1}{2}} : \mathcal{D}'(M_1,\Omega_{\frac{1}{2}}) \to \mathcal{D}'(M_2,\Omega_{\frac{1}{2}})$$

to $I_\varrho{}^m(M_1,\Lambda)$ is an isomorphism of $I_\varrho{}^m(M_1,\Lambda)$ onto $I_\varrho{}^m(M_2,(f^*)^{-1}(\Lambda))$.

Let us turn to the case of a diffeomorphism $f\colon M \to M$ of a single manifold M. The class $I_\varrho{}^m(M,\Lambda)$ completely determines Λ, since

$$\Lambda = \bigcup \{\mathrm{WF}(A) \mid A \in I_\varrho{}^m(M,\Lambda)\}$$

so by Proposition 3.1 the class $I_\varrho{}^m(M,\Lambda)$ is invariant under f if and only if Λ is invariant under f^*. In the special case $M = G\times G$ we have as a corollary:

PROPOSITION 3.2. *The class $I_\varrho{}^m(G\times G,\Lambda)$ is invariant under all left translations on G if and only if Λ is.*

Similarly for right translations.

The above result partly motivates that the next section is devoted the study of some properties of invariant Lagrangean submanifolds of $T^*(G\times G)\setminus 0$.

We recall the connection between phase functions and Lagrangean manifolds.

Let Σ be a fiber space over a manifold M with projection $\pi\colon \Sigma \to M$. The projection is thus surjective and has surjective differential so that the fibers $\pi^{-1}(p)$, $p \in M$, are submanifolds of Σ. We let d_2 denote the differential along the fibers. Let finally $\varphi \in C^2(\Sigma)$. For σ in

$$C_\varphi := \{\sigma \in \Sigma \mid d_2\varphi(\sigma) = 0\}$$

we may without ambiguity define the horizontal component of $d\varphi(\sigma)$, denoted

$$l_\varphi(\sigma) \in T^*{}_{\pi(\sigma)}M$$

by

$$l_\varphi(\sigma)(\pi_* X) := d\varphi(\sigma)(X) \quad \text{for all } X \in T_\sigma\Sigma\,.$$

In the case we will consider, Σ will be an open conic subset of $M \times (\mathsf{R}^N\setminus\{0\})$ and φ will be a phase function on Σ. In that case l_φ takes the form

$$l_\varphi(x,\theta) = (x, \varphi_x{}'(x,\theta))$$

which is familiar from [3].

If φ is a non-degenerate phase function then C_φ is a submanifold of Σ, and the map $l_\varphi: C_\varphi \to T^*M$ is an immersion (cfr. Hörmander [3, p. 134]). So l_φ defines locally a submanifold Λ_φ of T^*M. It turns out that Λ_φ is a Lagrangean submanifold of $T^*M \setminus 0$. We shall say that φ *describes* a Lagrangean submanifold Λ of $T^*M \setminus 0$ if l_φ is a diffeomorphism of C_φ onto Λ.

The following two easy lemmas will be needed later.

LEMMA 3.3. *Let $\pi_1: \Sigma_1 \to M_1$ and $\pi_2: \Sigma_2 \to M_2$ be cone bundles over manifolds M_1 and M_2. Let (F,f) be a cone bundle equivalence so that the following diagram commutes*

$$\begin{array}{ccc} \Sigma_1 & \xrightarrow{F} & \Sigma_2 \\ {\scriptstyle\pi_1}\downarrow & & \downarrow{\scriptstyle\pi_2} \\ M_1 & \xrightarrow[f]{} & M_2 \end{array}$$

Let $\psi_2 \in C^2(\Sigma_2)$ be a non-degenerate phase function.

Then $\psi_1 := \psi_2 \circ F$ is also a non-degenerate phase function,

$$FC_{\psi_1} = C_{\psi_2} \quad \text{and} \quad l_{\psi_1} = f^* \circ l_{\psi_2} \circ F \ \text{on} \ C_{\psi_1}\,.$$

If ψ_2 describes $\Lambda_2 \subseteq T^*M_2 \setminus 0$ then ψ_1 describes $f^*(\Lambda_2) \subseteq T^*(M_1) \setminus 0$.

LEMMA 3.4. *Let M_1 and M be manifolds, and let $\pi: \Sigma \to M$ be a fiber space over M. Let $\varphi \in C^2(\Sigma)$ and define $\psi \in C^2(M_1 \times \Sigma)$ by*

$$\psi(m_1, \sigma) := \varphi(\sigma) \qquad \text{for all } (m_1, \sigma) \in M_1 \times \Sigma\,.$$

Let us finally view $M_1 \times \Sigma$ as a fiber space over $M_1 \times M$.

Then

$$C_\psi = M_1 \times C_\varphi \quad \text{and} \quad l_\psi = \mathrm{id} \times l_\varphi\,.$$

If Σ is a cone bundle and φ a non-degenerate phase function describing a Lagrangean submanifold $\Lambda_0 \subseteq T^*M \setminus 0$ then ψ is a non-degenerate phase function describing

$$M_1 \times \Lambda_0 \subseteq T^*(M_1 \times M) \setminus 0\,.$$

We next give an explicit description of the Keller-Maslov line bundle and how it transforms under diffeomorphisms.

Let M be a manifold and let Λ be a conic Lagrangean submanifold of $T^*M \setminus 0$. Let Φ be the set of all non-degenerate phase functions describing open conic subsets of Λ, and let us use the following notation: Each $\varphi \in \Phi$ is defined on the open conic subset $\Gamma(\varphi)$ of $M \times (\mathbb{R}^{N(\varphi)} \setminus \{0\})$

and describes the subset $U(\varphi)$ of Λ. For any two elements $\varphi, \psi \in \Phi$ we let $\sigma(\varphi,\psi)$ denote the function

$$\sigma(\varphi,\psi) := \tfrac{1}{2}[(\operatorname{sgn}\psi''_{\theta\theta} - N(\psi)) \circ l_\psi^{-1} - (\operatorname{sgn}\varphi''_{\theta\theta} - N(\varphi)) \circ l_\varphi^{-1}]$$

which is integer-valued and defined on $U(\varphi) \cap U(\psi)$.

The Keller–Maslov line bundle L on Λ can now be defined as follows (cfr. Hörmander [3, p. 148]):

In the disjoint union

$$\mathscr{L} := \bigcup_{\varphi\in\Phi} U(\varphi) \times \mathbf{C}$$

we identify the points

$$(\varphi,\lambda,z) \quad \text{and} \quad \big(\psi,\lambda, \exp\big(\tfrac{1}{2}\pi i\sigma(\psi,\varphi)\big)z\big)$$

for $\lambda \in U(\varphi) \cap U(\psi)$. Then L is the identification space. If we denote the equivalence class of the element $(\varphi,\lambda,z) \in \mathscr{L}$ by $[\varphi,\lambda,z] \in L$ then the local trivializations of L are given by the maps

$$(\lambda,z) \mapsto [\varphi,\lambda,z]$$

of $U(\varphi) \times \mathbf{C}$ into L. The structure group of L is Z_4.

Let next $f\colon M_1 \to M_2$ be a diffeomorphism between two manifolds M_1 and M_2. Assume Λ_2 is a conic Lagrangean submanifold of $T^*M_2 \setminus 0$ and let $\Lambda_1 := f^*(\Lambda_2)$ be the corresponding conic Lagrangean submanifold of $T^*M_1 \setminus 0$.

For any $\varphi_2 \in \Phi_2$, defined on an open conic subset of $M_2 \times (\mathbf{R}^{N(\varphi_2)} \setminus \{0\})$ we introduce

$$\varphi_1 := \varphi_2 \circ (f \times \mathrm{id}) \in \Phi_2 .$$

This establishes a bijection between Φ_1 and Φ_2. It is easy to prove that

$$\sigma(\varphi_2,\psi_2) = \sigma(\varphi_1,\psi_1) \circ f^* \quad \text{for all } \varphi_2,\psi_2 \in \Phi_2 ,$$

and from there that the Keller–Maslov line bundles L_1 on Λ_1 and L_2 on Λ_2 are equivalent as fiber bundles under the induced map $f^L\colon L_2 \to L_1$, given by

$$f^L([\varphi_2,\lambda_2,z]) := [\varphi_1, f^*(\lambda_2), z] \quad \text{for } [\varphi_2,\lambda_2,z] \in L_2 .$$

LEMMA 3.5. *Let M_1 and M_2 be manifolds and let Λ be a conic Lagrangean submanifold of $T^*(M_2) \setminus 0$. Let L be the Keller–Maslov line bundle over Λ.*

Then $M_1 \times \Lambda$ is a conic Lagrangean submanifold of $T^*(M_1 \times M_2) \setminus 0$, and its Keller–Maslov line bundle $L_{M_1\times\Lambda}$ may be identified with $M_1 \times L$.

PROOF. That $M_1 \times \Lambda$ is a conic Lagrangean submanifold of $T^*(M_1 \times M_2) \setminus 0$ is immediate by Lemma 3.4. Let φ and ψ correspond as there. Let I_φ be that map from $M_1 \times L$ to $L_{M_1 \times \Lambda}$ which in the local trivializations with respect to φ and ψ is given by

$$(m_1, (\lambda, z)) \mapsto ((m_1, \lambda), z) .$$

It is easy to see that the I_φ patch together to a bundle isomorphism. This is the desired identification.

Corollary 3.6. *There is a fiber bundle isomorphism*

$$I\colon \Omega_{\frac{1}{2}}(M_1 \times \Lambda) \otimes L_{M_1 \times \Lambda} \to \Omega_{\frac{1}{2}}(M_1) \boxtimes (\Omega_{\frac{1}{2}}(\Lambda) \otimes L) ,$$

viz. I given by

$$I\big(\{d_1(m_1) \otimes d_2(\lambda)\} \otimes (m_1, l(\lambda))\big) = d_1(m_1) \otimes \{d_2(\lambda) \otimes l(\lambda)\}$$

for $d_1(m_1) \otimes d_2(\lambda) \in \Omega_{\frac{1}{2}}(M_1 \times \Lambda)_{(m_1, \lambda)}$ and $(m_1, l(\lambda)) \in (L_{M_1 \times \Lambda})_{(m_1, \lambda)}$.

Remark 3.7. We have already in Proposition 3.1 noted that

$$f_{\frac{1}{2}}(A) \in I_\varrho{}^m(M_2, \Lambda_2) \quad \text{if } A \in I_\varrho{}^m(M_1, \Lambda_1) .$$

That the principal symbol of a Fourier integral operator is an invariantly defined object means the following:

If σ_A is a principal symbol of A then the map $\sigma_{f_{\frac{1}{2}}(A)}$ that makes the diagram

$$\begin{array}{ccc} \Omega_{\frac{1}{2}}(\Lambda_1) \otimes L_1 & \xleftarrow{\Omega^{\frac{1}{2}}(f^{*-1}) \otimes f^L} & \Omega_{\frac{1}{2}}(\Lambda_2) \otimes L_2 \\ \uparrow{\scriptstyle \sigma_A} & & \uparrow{\scriptstyle \sigma_{f_{\frac{1}{2}}(A)}} \\ \Lambda_1 & \xleftarrow[f^*]{} & \Lambda_2 \end{array}$$

commute, is a principal symbol of $f_{\frac{1}{2}}(A)$.

The tensor product of two Fourier integral distributions is in general not a Fourier integral distribution. However, it is true in the following special case:

Theorem 3.8. *Let a half-density on a manifold M_1 of dimension n_1, and let $B \in I_\varrho^{m+n_1/4}(M_2, \Lambda)$ be a Fourier integral distribution on a manifold M_2. Let σ_B be a principal symbol of B.*

Then $a \otimes B \in \mathscr{D}'(M_1 \times M_2, \Omega_{\frac{1}{2}})$ is a Fourier integral distribution,

$$a \otimes B \in I_\varrho{}^m(M_1 \times M_2, M_1 \times \Lambda) ,$$

and

$$\sigma_{a \otimes B} := (2\pi)^{n_1/4} a \otimes \sigma_B$$

is a principal symbol of it.

Here we have used the identification of Corollary 3.6.

PROOF. Let n_2 denote the dimension of M_2. Let $\varkappa_j\colon V_j \to U_j \subset \mathbf{R}^{n_j}$, $j=1,2$, be charts on open subsets V_j of M_j and denote the coordinates of $\varkappa_1$ by x and those of $\varkappa_2$ by y. In any of the charts $\varkappa=\varkappa_1,\varkappa_2$, $\varkappa_1\times\varkappa_2$ we shall denote the coordinate expression of a half-density u with respect to the squareroot of Lebesgue measure by $\tilde{u}\circ\varkappa^{-1}$.

Let φ be a non-degenerate phase function in an open conic subset Γ of $U_2\times(\mathbf{R}^N\setminus\{0\})$ describing an open subset of Λ. Let us first assume that B is of the simple form

$$\langle B,v\rangle = (2\pi)^{-(n_2+2N)/4}\iint e^{i(\varphi(y,\theta)-\pi N/4)}b(y,\theta)\tilde{v}(\varkappa_2^{-1}(y))\,dyd\theta$$

$$\text{for } v\in C_0^\infty(V_2,\Omega_{\frac{1}{2}})$$

with a symbol $b\in S_\varrho^{m+n_1/4+(n_2-2N)/4}(\Gamma)$ (cfr. Hörmander [3, p. 147]).

If $u\in C_0^\infty(V_1\times V_2,\Omega_{\frac{1}{2}})$ then

$$\begin{aligned}&\langle a\otimes B,u\rangle\\ &= (2\pi)^{-(n_1+n_2+2N)/4}\iiint e^{i(\varphi(y,\theta)-\pi N/4)}(2\pi)^{n_1/4}\tilde{a}(\varkappa_1^{-1}(x))\,b(y,\theta)\\ &\qquad\qquad \tilde{u}(\varkappa_1^{-1}\times\varkappa_2^{-1}(x,y))\,dx\,dy\,d\theta\,.\end{aligned}$$

According to Lemma 3.4 the function $\psi(x,y,\theta)=\varphi(y,\theta)$ is a non-degenerate phasefunction in $U_1\times\Gamma$ describing an open conic subset of the Lagrangean submanifold $M_1\times\Lambda\subset T^*(M_1\times M_2)\setminus 0$. Since the function $(x,y,\theta)\mapsto \tilde{a}(\varkappa_1^{-1}(x))b(y,\theta)$ is an element of $S_\varrho^{m+(n_1+n_2-2N)/4}(U\times\Gamma)$ we conclude that

$$a\otimes B\in I_\varrho^{m}(M_1\times M_2, M_1\times\Lambda)\,.$$

In general a Fourier integral distribution B is a locally finite sum $B=\sum_j B_j$ of Fourier integral distributions B_j of the simple form above. Since $a\otimes B=\sum_j a\otimes B_j$ is a locally finite sum the result follows from the above.

This proves the theorem except for the claim about the principal symbols. The verification of that is a straightforward application of the definition of principal symbol, cfr. Hörmander [3, p. 143]. Just note that in his notation $C_\psi=U_1\times C_\varphi$, $d_{C_\psi}=dx\otimes d_{C_\varphi}$ and $l_\psi\colon C_\psi\to M_1\times\Lambda$ may be written $l_\psi=\varkappa_1^{-1}\times l_\varphi$.

4. Properties of invariant Lagrangean submanifolds.

The existence of invariant Lagrangean submanifolds on $G\times G$ is ensured by the following example.

EXAMPLE 4.1. The normal bundle

$$N(\Delta) = \{(g,g,\xi,-\xi) \mid \xi \in T^*{}_g G, g \in G\}$$

of the diagonal $\Delta \subseteq G \times G$ is a bi-invariant conic Lagrangean submanifold of $T^*(G \times G)$. More generally, the normal bundle of a right (respectively left-) translate $(R(a) \times \mathrm{id})\Delta$ $\big($respectively $(L(a) \times \mathrm{id})\Delta\big)$, $a \in G$, of the diagonal is a conic left- (respectively right-) invariant Lagrangean submanifold of $T^*(G \times G)$.

We shall in the sequel concentrate on left-invariant Lagrangean submanifolds, but it should be mentioned that quite analogous results are valid in the right-invariant case.

It is natural, when studying invariant objects in $G \times G$, to consider the map

$$s \colon G \times G \to G \times G$$

defined by

$$s(g,h) := (g,gh) \quad \text{for } g,h \in G\,.$$

The map s lifts to a symplectomorphism $S' := (s^{-1})^*$ making the following diagram commute:

$$\begin{array}{ccc} T^*(G \times G) & \xrightarrow{S'} & T^*(G \times G) \\ \Big\downarrow \text{proj.} & & \Big\downarrow \text{proj.} \\ G \times G & \xrightarrow[s]{} & G \times G \end{array}$$

We shall need the embedding

$$S := S'|_{G \times T^*G} \colon G \times T^*G \to T^*(G \times G)$$

which equivalently can be defined by

$$S(g,(h,\eta)) = (g, gh, -R(h)^* L(g^{-1})^*\eta, L(g^{-1})^*\eta)$$
$$\text{for } (g,(h,\eta)) \in G \times T^*G\,.$$

Note that S' and hence S commutes with left-translation in the sense that

$$L(g)^* \circ S' = S' \circ (L(g) \times \mathrm{id})^* \quad \text{for all } g \in G\,.$$

LEMMA 4.2. *Let $\pi_2 \colon G \times T^*G \to T^*G$ be the projection on the second factor. Then*

$$S^*(\Theta_{G \times G}) = \pi_2^*(\Theta_G) \quad \text{and} \quad S^*(\omega_{G \times G}) = \pi_2^*(\omega_G)\,.$$

PROOF. The lemma follows easily from the fact that S' is a symplectomorphism.

LEMMA 4.3. *Let M be a subset of $T^*G \setminus 0$. Then $S(G \times M)$ is a left-invariant subset of $T^*(G \times G) \setminus 0$. Furthermore, M is a (closed, conic) Lagrangean submanifold of $T^*G \setminus 0$ if and only if $S(G \times M)$ is a (closed, conic) Lagrangean submanifold of $T^*(G \times G) \setminus 0$.*

PROOF. The left-invariance is trivial since S commutes with left-translation. Since S is an embedding and S' an isomorphism on the fibers the only problem is whether M and $S(G \times M)$ are Lagrangean simultaneously. But that follows from Lemma 4.2.

THEOREM 4.4. *The map*

$$\Lambda_e \to S(G \times \Lambda_e)$$

is a bijection of the set of closed, conic Lagrangean submanifolds of $T^*G \setminus 0$ onto the set of closed, conic left-invariant Lagrangean submanifolds of $T^*(G \times G) \setminus 0$.

PROOF. Let Λ be any closed, conic left-invariant Lagrangean submanifold of $T^*(G \times G) \setminus 0$. We start by proving that

$$S'^{-1}(\Lambda) \subseteq G \times T^*G .$$

Since S' commutes with left-translation and Λ is left-invariant

$$\Lambda' = S'^{-1}(\Lambda)$$

satisfies

$$\Lambda' = (L(g) \times \mathrm{id})^*(\Lambda') \quad \text{for all } g \in G .$$

Now, Λ' is a Lagrangean submanifold of $T^*(G \times G)$. Since it is also conic the canonical 1-form $\Theta_{G \times G}$ vanishes on its tangent vectors (cfr. Hörmander [3, p. 135]).

If $t \to g(t)$ is any C^∞-curve in G with $g(0) = e$ and $\lambda = (g_1, g_2, \xi_1, \xi_2) \in \Lambda'$ then

$$t \to \lambda(t) := \big(g(t)g_1, g_2, L(g(t)^{-1})^*\xi_1, \xi_2\big)$$

is a curve in Λ' through λ. Hence

$$\begin{aligned} 0 &= \langle (\Theta_{G\times G})_\lambda, \lambda'(0) \rangle \\ &= \langle \lambda, \big(t \to (g(t)g_1, g_2)\big)'(0) \rangle \\ &= \langle \xi_1, (t \to g(t)g_1)'(0) \rangle \\ &= \langle \xi_1, R(g_1)_*(g'(0)) \rangle , \end{aligned}$$

but since $t \to g(t)$ is arbitrary, $\xi_1 = 0$. So each element of Λ' has the form $(g_1, g_2, 0, \xi_2)$ as desired.

Using once more the left-invariance of Λ we conclude that $S'^{-1}(\Lambda)$ is of the form

$$S'^{-1}(\Lambda) = G \times \Lambda_e .$$

In fact,

$$\Lambda_e = \{\lambda \in T^*G \mid (e,\lambda) \in S'^{-1}(\Lambda)\} .$$

Hence $\Lambda = S(G \times \Lambda_e)$. The theorem is then immediate by Lemma 4.3.

The following corollary shows that the standard assumption of Hörmander [3, Chapter 4] is satisfied.

COROLLARY 4.5. *Any left-invariant conic Lagrangean submanifold of* $T^*(G \times G) \setminus 0$ *is contained in* $(T^*G \setminus 0) \times (T^*G \setminus 0)$.

PROOF. By the alternative definition of S it follows that

$$S(G \times (T^*G \setminus 0)) \subseteq (T^*G \setminus 0) \times (T^*G \setminus 0) .$$

EXAMPLE 4.6. If $a \in G$ and $\pi : T^*G \to G$ denotes the projection we set

$$\Lambda_e^a = \pi^{-1}(a) \setminus \{0\} .$$

Then Λ_e^a is a closed conic Lagrangean submanifold of $T^*G \setminus 0$. The corresponding left-invariant Lagrangean submanifold of $T^*(G \times G) \setminus 0$ is the normal bundle of the translated diagonal

$$(R(a) \times \mathrm{id})\Delta \subset G \times G$$

(with the zero-section deleted). If in particular $a = e$ then Λ^a is the normal bundle of the diagonal (cfr. Example 4.1).

THEOREM 4.7. *Let* Λ_e *be a conic Lagrangean submanifold of* $T^*G \setminus 0$ *described by a non-degenerate phase function* $\varphi : V \to \mathsf{R}$, *where* V *is an open conic subset of* $G \times (\mathsf{R}^N \setminus \{0\})$. *Let*

$$W = \{(x,y,\theta) \in G \times G \times (\mathsf{R}^N \setminus \{0\}) \mid (x^{-1}y, \theta) \in V\}$$

and define $\psi : W \to \mathsf{R}$ *by*

$$\psi(x,y,\theta) = \varphi(x^{-1}y, \theta) \quad \text{for } (x,y,\theta) \in W .$$

Then ψ *is a non-degenerate phase function describing the conic Lagrangean submanifold*

$$\Lambda := S(G \times \Lambda_e) \subset T^*(G \times G) \setminus 0 .$$

PROOF. Define $\psi_2 : G \times V \to \mathsf{R}$ by

$$\psi_2(x,y,\theta) := \varphi(y,\theta) \quad \text{for } x \in G,\ (y,\theta) \in V .$$

Then ψ_2 is by Lemma 3.4 a non-degenerate phase function which describes $G \times \Lambda_e$.

Next let us note that $\psi = \psi_2 \circ F$, where $F : W \to G \times V$ is the diffeomorphism given by

$$F(x,y,\theta) = (x,x^{-1}y,\theta) \quad \text{for } (x,y,\theta) \in W .$$

Since the diagram

$$\begin{array}{ccc} W & \xrightarrow{F} & G \times V \\ \big\downarrow & & \big\downarrow \\ G \times G & \xrightarrow[s^{-1}]{} & G \times G \end{array}$$

with the obvious vertical projections commute we conclude by Lemma 3.3 that $\psi = \psi_2 \circ F$ describes $(s^{-1})^*(G \times \Lambda_e) = S(G \times \Lambda_e)$.

5. Left invariant Fourier integral operators.

We shall in this section find all left invariant Fourier integral operators on a Lie group G, corresponding to a given left-invariant, closed, conic Lagrangean submanifold Λ of $T^*(G \times G) \setminus 0$.

In all of this section we fix G and Λ as above, $\varrho \in]\frac{1}{2},1]$ and $m \in \mathsf{R}$. Furthermore we fix a smooth, nowhere vanishing density of order $\frac{1}{2}$ on G, namely $d = \sqrt{d\mu}$ where $d\mu$ is a left Haar measure on G.

The map $\tilde{u} \to u := \tilde{u}d$ is an isomorphism of $C^\infty(G)$ onto $C^\infty(G,\Omega_{\frac{1}{2}})$. We define the "point evaluation" A_e of any continuous linear map

$$A : C_0^\infty(G,\Omega_{\frac{1}{2}}) \to C^\infty(G,\Omega_{\frac{1}{2}})$$

by

$$\langle A_e, v\rangle := (Av)^\sim(e) \quad \text{for } v \in C_0^\infty(G,\Omega_{\frac{1}{2}}) .$$

Obviously $A_e \in \mathscr{D}'(G,\Omega_{\frac{1}{2}})$.

Let us note that the above can be applied to elements $A \in I_\varrho^{\,m}(G \times G, \Lambda)$: Indeed, the assumptions of Theorem 4.1.1 of Hörmander [3] are satisfied according to Corollary 4.5 so that A induces a continuous linear map

$$A : C_0^\infty(G,\Omega_{\frac{1}{2}}) \to C^\infty(G,\Omega_{\frac{1}{2}}) .$$

The function $(Av)^\sim$ where $v \in C_0^\infty(G,\Omega_{\frac{1}{2}})$, is determined by

$$\int (Av)^\sim ud = \langle A, u \otimes v\rangle \quad \text{for all } u \in C_0^\infty(G,\Omega_{\frac{1}{2}}) .$$

DEFINITION 5.1. An element $A \in \mathscr{D}'(G\times G,\Omega_{\frac{1}{2}})$ is said to be *left-invariant* if

$$L(g)_{\frac{1}{2}} A = A \quad \text{for all } g \in G ,$$

and *right-invariant* if

$$R(g)_{\frac{1}{2}} A = A \quad \text{for all } g \in G .$$

An operator $A: C_0^{\infty}(G,\Omega_{\frac{1}{2}}) \to \mathscr{D}'(G,\Omega_{\frac{1}{2}})$ is said to be *left-invariant* if

$$A\circ L(g^{-1})^{\frac{1}{2}} = L(g)_{\frac{1}{2}}\circ A \quad \text{for all } g \in G .$$

It is easy to see that an element $A \in \mathscr{D}'(G\times G,\Omega_{\frac{1}{2}})$ is left-invariant if and only if the corresponding operator (which will also be denoted A) is left-invariant. Note that the defining relation in case A happens to map $C_0^{\infty}(G,\Omega_{\frac{1}{2}})$ into $C^{\infty}(G,\Omega_{\frac{1}{2}})$ takes the familiar form

$$A\circ L(g)^{\frac{1}{2}} = L(g)^{\frac{1}{2}}\circ A \quad \text{for all } g \in G .$$

The next theorem is one of the main results of this paper:

THEOREM 5.2. *Let Λ be a closed conic, left-invariant Lagrangean submanifold of $T^*(G\times G)\setminus 0$ and let Λ_e be the corresponding Lagrangean submanifold of $T^*G\setminus 0$. The map $A\mapsto A_e$ is an isomorphism of the vector space of left-invariant elements of $I_\varrho^m(G\times G,\Lambda)$ onto $I_\varrho^{m+n/4}(G,\Lambda_e)$, where $n=\dim G$.*

The inverse map is $A_e\mapsto s_{\frac{1}{2}}(d\otimes A_e)$.

PROOF. Let us denote the map $A\mapsto A_e$ by E. We will first prove that $A_e\in I_\varrho^{m+n/4}(G,\Lambda_e)$ if $A\in I_\varrho^m(G\times G,\Lambda)$.

Since the local finiteness of a sum $A=\sum A_j$ carries over by E, we may as well take A of the simpler form

$$\langle A,u\rangle = (2\pi)^{-(2n+2N)/4}\iiint e^{i\psi(x,y,\theta)-i\pi N/4}a(x,y,\theta)\,u_\varkappa(x,y)\,dx\,dy\,d\theta \quad \text{for } u\in C_0^{\infty}(G\times G,\Omega_{\frac{1}{2}}) .$$

Here $u_\varkappa$ is the coordinate expression of u with respect to a product chart $\varkappa=\varkappa_1\times\varkappa_2$, and $a\in S_\varrho^{m+(2n-2N)/4}(\mathrm{R}^{2n}\times\mathrm{R}^N)$.

According to Theorem 4.7 one can describe Λ by non-degenerate phase functions of the form $(g,h,\theta)\mapsto\varphi(g^{-1}h,\theta)$ so we may assume that the ψ above is of the form

$$\psi(x,y,\theta) = \varphi\big((\varkappa_1^{-1}(x))^{-1}\varkappa_2^{-1}(y),\theta\big) .$$

In particular ψ is a non-degenerate phase function for each fixed $x\in\operatorname{image}(\varkappa_1)$, and $\psi(\varkappa_1(e),.,.)$ describes an open conic subset of Λ_e.

It is now a simple matter to check that

$$\langle A_e, v\rangle = \text{const.} \iint e^{i\varphi(\varkappa_1(e),y,\theta)} a(\varkappa_1(e),y,\theta)\, v_{\varkappa_2}(y)\, dy\, d\theta$$

if $v \in C_0^\infty(G,\Omega_{\frac12})$ in the chart $\varkappa_2$ is given by the function $v_{\varkappa_2}$. But that shows $A_e \in I^{m+n/4}(G,\Lambda_e)$.

The injectivity of E is clear: If A_e is given, then A is known everywhere by left-invariance.

To establish the surjectivity we will produce a right-inverse and hence an inverse of E. Defining

$$A = s_{\frac12}(d \otimes A_0)$$

for $A_0 \in I_\varrho^{m+n/4}(G,\Lambda_e)$ we conclude from Theorem 3.8 and Proposition 3.1 that $A \in I_\varrho^m(G \times G, \Lambda)$. Now

$$L(g) \circ s = s \circ (L(g) \times \mathrm{id}) \quad \text{for } g \in G .$$

Hence,

$$L(g)_{\frac12} A = L(g)_{\frac12} s_{\frac12}(d \otimes A_0) = s_{\frac12}(L(g)_{\frac12} d \otimes A_0) .$$

Since $d = \sqrt{d\mu}$ is left-invariant we get

$$L(g)_{\frac12} A = s_{\frac12}(d \otimes A_0) = A ,$$

which shows A is left-invariant.

To prove $EA = A_0$ we note from above that EA, the operator sending v to $(Av)^\sim(e)$, is determined by

$$\int (Av)^\sim u d = \langle A, u \otimes v\rangle \quad \text{for all } u,v \in C_0^\infty(G,\Omega_{\frac12}) .$$

Now,

$$\begin{aligned}\langle A, u\otimes v\rangle &= \langle s_{\frac12}(d\otimes A_0), u\otimes v\rangle \\ &= \langle d\otimes A_0, s^{\frac12}(u\otimes v)\rangle .\end{aligned}$$

Since $(s^{\frac12}(u\otimes v))(x,y) = u(x)\otimes (L(x)^{\frac12} v)(y)$ we find further

$$\langle A, u\otimes v\rangle = \int d(x) u(x) \langle A_0, L(x)^{\frac12} v\rangle ,$$

so that

$$(Av)^\sim(x) = \langle A_0, L(x)^{\frac12} v\rangle .$$

In particular $A_e = A_0$.

Example 5.3. (Cfr. Example 4.6). Let A_e be the δ-distribution at $a \in G$, that is,

$$\langle A_e, u\rangle = \tilde u(a) \quad \text{for } u = \tilde u \sqrt{d\mu} \in C_0^\infty(G,\Omega_{\frac12}) .$$

Then $A_e \in I^{n/4}(G,\Lambda_e{}^a)$. The corresponding left-invariant operator $A \in I^0(G \times G, \Lambda^a)$ is the map

$$(\Delta(a))^{-\frac12} R(a)^{\frac12} : C_0^\infty(G,\Omega_{\frac12}) \to C_0^\infty(G,\Omega_{\frac12}) ,$$

where Δ here denotes the modular function of G, and $R(a)^{\frac{1}{2}}$ is defined in Section 2. In particular we may conclude that $R(a)^{\frac{1}{2}}$ is a left-invariant Fourier integral operator on G for any $a \in G$.

The relation between the principal symbols of the two distributions A and A_e above may now be expressed by the following theorem

THEOREM 5.4. *Let $A \in I_\varrho^m(G \times G, \Lambda)$ be a left-invariant Fourier integral distribution and let $A_e \in I_\varrho^{m+n/4}(G, \Lambda_e)$ correspond to A. One can then choose principal symbols σ_A for A and σ_{A_e} for A_e such that the following diagram commutes:*

$$\begin{array}{ccc} \Omega_{\frac{1}{2}}(\Lambda) \otimes L & \xrightarrow{I \circ (\Omega_{\frac{1}{2}}(S) \otimes s^L)} & \Omega_{\frac{1}{2}}(G) \boxtimes (\Omega_{\frac{1}{2}}(\Lambda_e) \otimes L_e) \\ \sigma_A \uparrow & & \uparrow (2\pi)^{n/4}\sqrt{d\mu} \boxtimes \sigma_{A_e} \\ \Lambda & \xleftarrow[S]{} & G \times \Lambda_e \end{array}$$

Here I denotes the line bundle isomorphism of Corollary 3.6.

PROOF. Since $A = s_{\frac{1}{2}}(\sqrt{d\mu} \otimes A_e)$ (by Theorem 5.2), the result is an immediate consequence of Remark 3.7 and Theorem 3.8.

COROLLARY 5.5. *If $A \in I_\varrho^m(G \times G, \Lambda)$ is left-invariant then its principal symbol σ_A may be chosen left-invariant in the sense that*

$$\left(\Omega^{\frac{1}{2}}(L(g)^{*-1}) \otimes L(g)^L\right) \circ \sigma_A = \sigma_A \quad \text{for every } g \in G .$$

PROOF. Choose any principal symbol σ_{A_e} of A_e and define σ_A so that the diagram of Theorem 5.4 is commutative. Since

$$L(g)^* = S \circ (L(g)^* \times \mathrm{id}) \circ S^{-1} \quad \text{for every } g \in G ,$$

that diagram together with the left-invariance of $d\mu$ implies that

$$\begin{aligned} &\left(\Omega^{\frac{1}{2}}(L(g)^{*-1}) \otimes L(g)^L\right) \circ \sigma_A \\ &= (\Omega^{\frac{1}{2}}(S^{-1}) \otimes (s^{-1})^L) \circ I^{-1} \circ \left(\Omega^{\frac{1}{2}}(L(g)^{*-1}) \times \mathrm{id}\right) \circ I \circ (\Omega^{\frac{1}{2}}(S) \otimes s^L) \circ \sigma_A \\ &= (\Omega^{\frac{1}{2}}(S^{-1}) \otimes (s^{-1})^L) \circ I^{-1} \circ \left(\Omega^{\frac{1}{2}}(L(g)^{*-1}) \times \mathrm{id}\right) \circ ((2\pi)^{n/4}\sqrt{d\mu} \boxtimes \sigma_{A_e}) \circ S^{-1} \\ &= (\Omega^{\frac{1}{2}}(S^{-1}) \otimes (s^{-1})^L) \circ I^{-1} \circ ((2\pi)^{n/4}\sqrt{d\mu} \boxtimes \sigma_{A_e}) \circ S^{-1} \\ &= \sigma_A . \end{aligned}$$

6. Bi-invariant Fourier integral operators.

Let G be a Lie group with left Haar measure μ. Let Λ be a conic, closed, left-invariant Lagrangean submanifold of $T^*(G\times G)\setminus 0$. The following theorem shows that the bi-invariance of $A\in I^m(G\times G,\Lambda)$ can be described by the properties of $A_e\in I^{m+n/4}(G,\Lambda_e)$.

THEOREM 6.1. *Let $A\in I^m(G\times G,\Lambda)$ be the left-invariant Fourier integral distribution corresponding to $A_e\in I^{m+n/4}(G,\Lambda_e)$. Let Δ be the modular function of G.*

Then A is bi-invariant if and only if

$$\mathrm{Ad}(g)_{\frac12}(A_e) = \sqrt{\Delta(g^{-1})}A_e \quad \textit{for all } g\in G\,, \tag{1}$$

where $\mathrm{Ad}(g)(h) := ghg^{-1}$ *for all* $g,h\in G$.

PROOF. By Theorem 5.2 we have

$$\begin{aligned}
R(g)_{\frac12}A &= R(g)_{\frac12}s_{\frac12}(\sqrt{d\mu}\otimes A_e)\\
&= (R(g)\circ s)_{\frac12}(\sqrt{d\mu}\otimes A_e)\\
&= \big(s\circ(R(g)\times \mathrm{Ad}(g^{-1}))\big)_{\frac12}(\sqrt{d\mu}\otimes A_e)\\
&= s_{\frac12}(R(g)_{\frac12}\sqrt{d\mu}\otimes \mathrm{Ad}(g^{-1})_{\frac12}A_e)\\
&= \sqrt{\Delta(g^{-1})}\,s_{\frac12}(\sqrt{d\mu}\otimes \mathrm{Ad}(g^{-1})_{\frac12}A_e)\,,
\end{aligned}$$

so A is right-invariant, that is, $R(g)_{\frac12}A=A$ if and only if

$$\mathrm{Ad}(g)_{\frac12}A_e = (\Delta(g))^{-\frac12}A_e \quad \text{for all } g\in G\,.$$

Earlier investigations have shown that in many cases the only bi-invariant pseudo-differential operators on a Lie group are differential operators and integral operators with smooth kernel. See Melin [4], Preiss Rotschild [5] and Stetkær [6]. That is not the case for Fourier integral operators: For example, translation by an element of the center of G is a bi-invariant Fourier integral operator which is not a differential operator. Here is a more interesting example:

EXAMPLE 6.2. Let G be the unimodular, 3-dimensional Lie group $\mathrm{SL}(2,\mathsf{R})$, and let c be a real number, $c\neq\pm 2$. The trace of the matrix $g\in G$ will be denoted $\mathrm{tr}(g)$.

The function $\varphi\colon G\times\mathsf{R}^1\to\mathsf{R}$ defined by

$$\varphi(g,\theta) := (\mathrm{tr}(g)-c)\theta \quad \text{for } g\in G,\ \theta\in\mathsf{R}^1$$

is a non-degenerate phase function on G because it is linear in θ and the map $g\to\mathrm{tr}(g)$ is regular on

$$M := \{g \in G \mid \operatorname{tr}(g) = c\}$$

as is easily seen. The corresponding Lagrangean submanifold Λ_e of T^*G is the normal bundle of M.

M is an (imbedded) submanifold of G. It is well-known that G acts transitively on M by inner automorphisms. The isotropy groups are 1-dimensional since $\dim M = 2$, so they are unimodular. By a general theorem (Helgason [2, p. 369]) there is a G-invariant measure $\nu > 0$ on M, and this measure is unique up to a constant factor.

The Fourier integral distribution $A_e \in I^{-\frac{1}{4}}(G, \Lambda_e)$ that we want to study, is given by

$$\langle A_e, u\rangle = \int_{\mathsf{R}} \int_G e^{i(\operatorname{tr}(g)-c)\theta} \frac{u(g)}{\sqrt{d\mu(g)}}\, d\mu(g)\, d\theta \quad \text{for } u \in C_0^\infty(G, \Omega_{\frac{1}{2}}) .$$

It is clearly invariant under inner automorphisms, since tr is.

As is well-known we have for $x \in \mathsf{R}$ that

$$\int_{\mathsf{R}} e^{ixt}\, dt = 2\pi\delta(x)$$

where δ is Dirac measure on R. Hence

$$\langle A_e, u\rangle = 2\pi \int_G \delta(\operatorname{tr}(g) - c) \frac{u(g)}{\sqrt{d\mu(g)}}\, d\mu(g) .$$

Since $g \to \operatorname{tr}(g)$ is regular on M we see that $\langle A_e, u\rangle$ is the integral of the function $u(g)/\sqrt{d\mu(g)}$ over M with respect to some measure on M. Since A_e and $d\mu$ are invariant under inner automorphisms the same holds for the measure. But then it is proportional to the G-invariant measure ν on M.

Modulo a constant factor which we will disregard, we therefore have

$$\langle A_e, u\rangle = \int_{\{g \in G \mid \operatorname{tr}(g) = c\}} \frac{u}{\sqrt{d\mu}}\, d\nu \quad \text{for } u \in C_0^\infty(G, \Omega_{\frac{1}{2}}) .$$

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UNIVERSITY OF AARHUS
DENMARK