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# ON FOURIER STIELTJES TRANSFORMS OF DISCRETE MEASURES

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It is well-known that if a set of integers $E$ is sufficiently lacunary then every restriction of a Fourier transform of an $L^1$ function to $E$ can be interpolated by a transform of a discrete measure (see [5], for example). In this note we show that a diophantine condition used in our previous work (e.g. [2]) is sufficient to imply the above property (Corollary 1). As usual, we deduce the abundance of non-Sidon sets with this property, and, as another application, we obtain a short proof of the existence of sets $E_1$ and $E_2$ so that

$$c_0(E_1) \hat{\otimes} c_0(E_2) \approx A(E_1+E_2)$$

(see Theorem 3.1 of [7]). We conclude with an open question.

Let $\Gamma$ be any countable discrete abelian group, $G$ its dual, and let the Bohr compactification of $\Gamma$ $(=(G_d)\hat{\ })$ be denoted by $\bar{\Gamma}$. We refer to [3] and [6] for basic notation and facts.

Let $D$ be a dense countable subgroup of $G$, and $\varphi_D \equiv \varphi : \Gamma \to \hat{D}$ be the injective canonical map: $(\varphi(\gamma),d)=(\gamma,d)$. We set

$$A(E,\Gamma) = L^1(G)\hat{\ }/\{f \in L^1(G)\hat{\ } : f=0 \text{ on } E\},$$

and

$$A\big((\varphi_D(E))^-,\hat{D}\big) = l^1(D)\hat{\ }/\{f \in l^1(D)\hat{\ } : f=0 \text{ on } \varphi_D(E)\},$$

where $(\varphi_D(E))^-$ denotes the closure of $\varphi_D(E)$ in $\hat{D}$. To simplify notation, we shall refer to $A(E)$ and $A\big((\varphi(E))^-\big)$, respectively. Let $E \subset \Gamma$ satisfy the following condition:

(*) $(\varphi(E))^-$ is a countable set so that $\partial(\varphi(E))^- \cap \varphi(E) = \emptyset$ $\big(\partial(\varphi(E))^-$ denotes the derived set of $(\varphi(E))^-\big)$.

We set

$$A_0\big((\varphi(E))^-\big) = \{f \in A\big((\varphi(E))^-\big) : f=0 \text{ on } \partial(\varphi(E))^-\}.$$

We need the following lemma, a similar version of which appears in [2] (Lemma 2.2); for the sake of completeness we prove it here:

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LEMMA. *Let $E \subset \Gamma$ satisfy* (*). *Then, $A(E)$ and $A_0\big((\varphi(E))^-\big)$ are isometric in the natural way*:

$$A(E) \ni \lambda \leftrightarrow \lambda \circ \varphi^{-1} \in A_0\big((\varphi(E))^-\big) .$$

PROOF. We first note that finitely supported functions are norm dense in $A(E)$. Similarly, since $\partial(\varphi(E))^-$ obeys synthesis in $\tilde{D}$ (7.2.4 of [6]), and since for any open set $\mathcal{O}$ containing $\partial(\varphi(E))^-$, $\varphi(E) \setminus \big(\mathcal{O} \cap \partial(\varphi(E))^-\big)$ is finite, it follows that finitely supported functions are norm dense in $A_0\big((\varphi(E))^-\big)$ as well. Therefore, it suffices to prove that

$$\|h\|_{A(E)} = \|h \circ \varphi^{-1}\|_{A((\varphi(E))^-)} ,$$

where $h$ is a finitely supported function on $E$. But, by the way that the above norms are computed, it suffices to check that if $\{a_i\}_{i=1}^N$ is any finite set of complex numbers, and $\{\gamma\}_{i=1}^N$ is any finite subset of $E$, then

$$\sup_{g \in G} |\textstyle\sum_{i=1}^N a_i(\gamma_i, g)| = \sup_{y \in \tilde{D}} |\textstyle\sum_{i=1}^N a_i(\varphi(\gamma_i), y)| .$$

The above equality follows from the density of $D$ in both $\tilde{D}$ and $G$.

We now let

$$B_d(E) = l^1(G)^\wedge / \{h \in l^1(G)^\wedge : h = 0 \text{ on } E\} .$$

That is, $B_d(E)$ is the algebra of restrictions to $E$ of Fourier Stieltjes transforms of discrete measures on $G$. It is clear from the definition of $\varphi$ that $A\big((\varphi(E))^-\big)$ can be canonically identified with a closed subalgebra of $B_d(E)$: If $h$ is the restriction of a function in $l^1(D)^\wedge$ to $\varphi(E)$, then $h \circ \varphi^{-1}$ is the restriction of the same function in $l^1(D)^\wedge \subset l^1(G)^\wedge$ to $E$. We therefore obtain

COROLLARY 1. *Let $E \subset \Gamma$ satisfy* (*) *with respect to some $D \subset G$; then, $A(E)$ is a closed subalgebra of $B_d(E)$.*

As usual, the following is a consequence of Lemma 2.3 in [1] (see also Lemma 1.1 in [2]):

COROLLARY 2. *Let $E \subset \Gamma$ be a non-Sidon set. Then, there exists a non-Sidon set $F \subset E$ such that $A(F)$ is a closed subalgebra of $B_d(F)$.*

AN APPLICATION TO TENSOR PRODUCTS. We recall that if $E_1$ and $E_2$ are countable sets, then

$$c_0(E_1) \hat{\otimes} c_0(E_2) = \{\varphi \in c_0(E_1 \times E_2) : \varphi = \textstyle\sum f_j g_j, \text{ where } f_j \in c_0(E_1) , \\ g_j \in c_0(E_2), \text{ and } \sum \|f_j\|_\infty \|g_j\|_\infty < \infty\} .$$

We set

$$\|\varphi\|_{\hat{\otimes}} = \inf\{\sum \|f_j\|_\infty \|g_j\|_\infty : \varphi = \sum f_j g_j\} .$$

The reader is referred to [7] for a detailed study of tensor algebras in the context of discrete abelian groups. We give here a short proof of the existence of sets $E_1$ and $E_2$ in $\Gamma$ so that $c_0(E_1)\hat{\otimes}c_0(E_2)\approx A(E_1+E_2)$ (see Theorem 3.1 of [7]). Fix a $D$, a dense countable subgroup of $G$, and let $E_1$ and $E_2$ in $\Gamma$ be any two sets so that

(1) $E_1 \cap E_2 = \emptyset$, $0 \notin E_1 \cup E_2$, and $E_1 \cup E_2$ is $Z_3$-independent, i.e., the relation $\sum_{j=1}^N \omega_j \lambda_j = 0$, where $\omega_j = -1$, 1, or 0, and $\{\lambda_j\}_{j=1}^N \subset E_1 \cup E_2$, can hold only if $\omega_j = 0$ for all $j$.

(2) $\partial(\varphi(E_1 \cup E_2))^{-} = \{x_0\} \subset \hat{D}$, and without loss of generality we assume that $x_0 = 0$.

It is clear from the independence assumption that functions on $E_1 \times E_2$ can be freely identified with functions on $E_1 + E_2$. In fact, if $f \in c_0(E_1)$ and $g \in c_0(E_2)$, we think of $f \cdot g$ as a function $h$ on $E_1 + E_2$:

$$h(\lambda+\nu) = f(\lambda)\cdot g(\nu), \quad \text{where } \lambda \in E_1 \ \nu \in E_2 .$$

Also, it follows from (2) that $\partial(\varphi(E_1+E_2))^{-} = E_1 \cup E_2 \cup \{0\}$. Therefore, via Corollary 1, we conclude that $A(E_1+E_2)$ is a closed subalgebra of $B_d(E_1+E_2)$, and in particular

$$A(E_1+E_2) \subseteq c_0(E_1) \hat{\otimes} c_0(E_2) .$$

To prove the reverse inclusion, it suffices to check that if $f \in c_0(E_1)$ and $g \in c_0(E_2)$, where $\|f\|_\infty$ and $\|g\|_\infty \leqq 1$, then $f \cdot g$ (as a function on $E_1+E_2$) can be interpolated by $\hat{\mu} \in M(G)^{\wedge}$, where $\|\mu\|_M \leqq 1$. This follows easily by considering a Riesz product whose transform equals $f$ on $E_1$, and $g$ on $E_2$.

Another consequence of our lemma is that if a Sidon set $E \subset \Gamma$ satisfies (*) then $\bar{E}$ (closure in $\bar{\Gamma}$) is a Helson set in $\bar{\Gamma}$.

Open question. Is a Sidon set in $\Gamma$ a finite union of (Sidon) sets that satisfy (*) ((*) may be satisfied with respect to different $D$'s)?

Recalling that a Sidon set in $\oplus Z_p$ ($p$ a prime) is a finite union of independent sets (see [4]), we easily answer affirmatively the above question in this setting: Let $E = \{\chi_j\}$ be an (infinite) independent set in $\oplus Z_p$.

We think of $D = Gp(\chi_j)$ as sitting in $\oplus Z_p$, (note that $D$ is isomorphic to $\oplus Z_p$) and map, as before, $\oplus Z_p$ into $Gp(\chi_j)^\wedge$. It follows that the closure of the image of $E$ in $Gp(\chi_j)^\wedge$ is a countable set with 0 in $Gp(\chi_j)^\wedge$ as its only accumulation point.

## REFERENCES


1. R. C. Blei, *On trigonometric series associated with separable, translation invariant subspaces of $L^\infty(G)$*, Trans. Amer. Math. Soc. 173 (1972), 491–499.
2. R. C. Blei, *A simple diophantine condition in harmonic analysis*, Studia Math. (to appear).
3. J.-P. Kahane, *Séries de Fourier absolument convergentes* (Ergebnisse der Mathematik 50), Springer-Verlag, Berlin, New York, 1970.
4. P. Malliavin, *Ensembles de Helson dans les groupes profinis*, C. R. Acad. Sci. Paris Sér. A 267 (1968), 285–286.
5. J. F. Méla, *Approximation diophantienne et ensembles lacunaires*, Bull. Soc. Math. France, 19 (1969), 26–54.
6. W. Rudin, *Fourier analysis on groups* (Interscience Tracts in Pure and Applied Mathematics 12), Interscience Publ. New York, London, 1962.
7. N. Varopoulos, *Tensor algebras over discrete spaces*, J. Functional Analysis 3 (1969), 321–335.


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