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MAXIMAL p-SYSTEMS AND REALCOMPLETENESS

DON A. MATTSON

1. Introduction.

Let X be a non-compact, completely regular Hausdorff space, and let $C(X)$ be the ring of continuous real-valued functions defined on X. The subalgebras of functions in $C(X)$ that vanish at infinity or which have compact support are denoted by $C_\infty(X)$ and $C_K(X)$, respectively.

Each (Hausdorff) compactification δX of X may be viewed as the Smirnov compactification of the proximity space (X,δ), where subsets A and B of X satisfy $A\delta B$ if and only if the closures of A and B in δX have non-empty intersection. Let $P(X)$ be the collection of proximity mappings of (X,δ) into the real numbers R, where the proximity on R is induced by the usual metric. The rudimentary algebraic structure of $P(X)$ has been observed in [1].

If $P^*(X)$ is the algebra of bounded members of $P(X)$, then $P^*(X)$ is also the algebra of real-valued uniformly continuous functions relative to the uniformity on R associated with the standard metric and the unique totally bounded uniformity in the proximity class of δ.

In this paper we show that for any compatible proximity on X, $C_\infty(X)$ is the intersection of all the free maximal p-systems in $P(X)$, and that $C_K(X)$ is always the intersection of all free ideals in $P^*(X)$.

A member f of $C(X)$ is constant at infinity if $f-r \in C_\infty(X)$, for some $r \in \mathsf{R}$. The collection of functions constant at infinity is characterized as the collection of functions uniformly continuous with respect to every admissible uniformity on X.

Realcompleteness for (X,δ) is characterized by means of clusters and p-stable families of closed subsets of X. From this several characterizations of realcompactness are obtained. When $\delta=\beta$, the proximity associated with the Stone–Čech compactification βX of X, it is shown that X is realcompact if and only if no free maximal p-system is an ideal.

2. Proximity spaces and $C_K(X)$, $C_\infty(X)$.

The following results concerning $C_K(X)$ and $C_\infty(X)$ may be found in [2].

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(2.1) $C_K(X)$ is the intersection of all free ideals in $C^*(X)$.

(2.2) $C_K(X)$ is the intersection of all free ideals in $C(X)$.

(2.3) $C_\infty(X)$ is the intersection of all free maximal ideals in $C^*(X)$.

(2.4) $C_K(X) \subseteqq C_\infty(X)$, and the inclusion is proper if X is locally compact, σ-compact and non-compact.

Notation and general background for $C(X)$ may be found in [2]. The definition of a p-system in $P(X)$ and properties of p-systems are developed by Njåstad in [6] and [7]. For $f \in P(X)$, let f^δ denote the Smirnov extension of f from δX into the Smirnov compactification of R. The distinct maximal p-systems in $P(X)$ can be characterized by

$$I^x = \{f \in P(X) : f^\delta(x) = 0\},$$

where $x \in \delta X$. (See [5], [6].)

Let $Z(f)$ denote the zero-set of a member f of $C(X)$. For $f \in C_\infty(X)$ and each positive integer n, set

$$F_n = \{x \in X : |f(x)| \geqq 1/n\}.$$

By definition, each F_n is a compact subset of X.

PROPOSITION 2.5. *For each proximity space (X, δ), if $f \in C_\infty(X)$, then $f \in P^*(X)$.*

PROOF. Take $A \delta B$ in X and f in $C_\infty(X)$. If $A \subseteqq F_n$, for some n, then $\mathrm{Cl}_X A$ is compact so that $\mathrm{Cl}_X B$ meets $\mathrm{Cl}_X A$. The continuity of f now provides that $f[A]$ is close to $f[B]$ in R.

Next, suppose that for every n, neither A nor B is contained in F_n. Take $\varepsilon > 0$ and choose n such that $2/n < \varepsilon$. If

$$a \in (X - F_n) \cap A \quad \text{and} \quad b \in (X - F_n) \cap B,$$

then $|f(a) - f(b)| < \varepsilon$, and it follows that $f[A]$ is close to $f[B]$.

Thus, in either case $f \in P^*(X)$, and the proof is complete.

For $\delta = \beta$, we note that $P(X) = C(X)$ and $P^*(X) = C^*(X)$.

The following theorem shows that (2.1) is true for all such "P^*-algebras".

THEOREM 2.6. *For each compatible proximity δ for X, $C_K(X)$ is the intersection of all free ideals in $P^*(X)$.*

PROOF. Let $f \in C_K(X)$ and let I^* be any free ideal in $P^*(X)$. Since $\mathrm{Cl}_X(X - Z(f))$ is compact, there exists $g \in I^*$ for which

$$Z(g) \cap \mathrm{Cl}_X(X - Z(f)) = \varnothing .$$

Thus $Z(f)$ is a neighborhood of $Z(g)$ and $f = g \cdot h$ where $Z(h) = Z(f)$. (See I.D. of [2].) By Proposition 2.5, $h \in P^*(X)$, hence $f \in I^*$. It now follows that $C_K(X)$ is contained in every free ideal in $P^*(X)$.

Conversely, assume that f is a member of every free ideal in $P^*(X)$. For $x \in \delta X - X$, let $\mathscr{F}^x$ be the unique maximal round filter in (X, δ) which converges to x. Set

$$J^{*x} = \{g \in P^*(X) : Z(g) \in \mathscr{F}^x\} ,$$

so that each J^{*x} is a free ideal for $x \in \delta X - X$. If $g \in J^{*x}$, $\mathrm{Cl}_{\delta X} Z(g)$ is a neighborhood of x in δX. Since $f \in J^{*x}$, for all $x \in \delta X - X$, $\mathrm{Cl}_{\delta X} Z(f)$ is a neighborhood of $\delta X - X$. Let $Z_{\delta X}(f^\delta)$ be the zero-set of f^δ in δX. Then $\delta X - Z_{\delta X}(f^\delta) = X - Z(f)$, so that

$$\mathrm{Cl}_{\delta X}(\delta X - Z_{\delta X}(f^\delta)) = \mathrm{Cl}_{\delta X}(X - Z(f))$$

is a compact subset of X. Hence $\mathrm{Cl}_X(X - Z(f))$ is compact and $f \in C_K(X)$. This completes the proof.

In contrast to (2.3) we have the following theorem and example.

THEOREM 2.7. *For each compatible proximity δ for X,*

$$C_\infty(X) = \bigcap \{I^x : x \in \delta X - X\} .$$

PROOF. Take $f \in C_\infty(X)$ and $x \in \delta X - X$. Now $f \in P^*(X)$ implies that the Smirnov extension f^δ of f takes real values. Since the maximal round filter $\mathscr{F}^x$ corresponding to I^x is free, no $F_n \in \mathscr{F}^x$. Thus, for each n, there exists $E_n \in \mathscr{F}^x$ such that $E_n \subseteqq X - F_n$. Since $|f| \leqq 1/n$ on E_n, it follows that $f^\delta(x) = 0$, hence $f \in I^x$.

Conversely, if $f \in I^x$, for all $x \in \delta X - X$, then $f^\delta(x) = 0$ on $\delta X - X$. Thus, $f^\delta[\delta X]$ is a compact subset of R, and the set

$$\{x \in \delta X : |f^\delta(x)| \geqq 1/n\} = \{x \in X : |f(x)| \geqq 1/n\}$$

is a compact subset of X. Therefore $f \in C_\infty(X)$, and the proof is complete.

EXAMPLE 2.8. Let $X = N$, the positive integers with the discrete topology, and take $\delta = \beta$. If $j(x) = x^{-1}$, for $x \in X$, it is well-known (see 4.7 of [2]) that j is a member of every free maximal ideal of $C^*(X)$. Since j is a unit of $C(X)$, j belongs to no maximal ideal of $C(X)$. Yet $j \in C_\infty(X)$, so that

$$j \in \bigcap \{I^x : x \in \beta X - X\} .$$

Thus, even when $P(X)=C(X)$, maximal p-systems are, in general, distinct from maximal ideals.

3. Functions constant at infinity.

A member f of $C(X)$ is called *constant at infinity* if $f-r \in C_\infty(X)$, for some $r \in \mathsf{R}$.

THEOREM 3.1. *For $f \in C(X)$, the following are equivalent:*

(A) *f is constant at infinity.*
(B) *f can be extended continuously (with real values) to every compactification of X.*
(C) *f is uniformly continuous with respect to every admissible uniformity on X.*
(D) *$f \in P^*(X)$ for every compatible proximity on X.*

PROOF. The equivalences of (B), (C) and (D) follow readily from basic properties of proximity and uniform spaces.

(A) implies (B). If $f-r \in C_\infty(X)$, it follows from Theorem 2.7 that

$$f-r \in \bigcap \{I^x :\ x \in \delta X - X\},$$

where δ is any compatible proximity for X. Then $(f-r)^\delta(x)=0$ implies $f^\delta(x)-r=0$, for all $x \in \delta X-X$, so that f^δ carries δX into R. Thus the Smirnov extension of f takes real values. Since every (Hausdorff) compactification of X can be viewed as the Smirnov compactification of its associated proximity space, we now have established (B).

(B) implies (A). Let βX be the Stone–Čech compactification of X. If $\beta X-X$ consists of a single point, then (A) follows trivially. Hence we assume that card $(\beta X-X)$ is greater than one. Suppose that f satisfies (B), but there exist $x,y \in \beta X-X$ such that $f^\beta(x) \neq f^\beta(y)$. We can assume that $f^\beta(x)=0$ and $f^\beta(y)=1$. In βX choose disjoint neighborhoods N_x and N_y of x and y, respectively, such that $f^\beta[N_x]$ is remote from $f^\beta[N_y]$. Let $\delta(X)=\{\beta X-\{x,y\}\} \cup \{z\}$, where $z \notin \beta X$, and define a mapping τ of βX onto δX by $\tau(p)=p$, if $p \neq x$, y, and $\tau(x)=\tau(y)=z$. Let δX have the largest topology rendering τ continuous. Since the restriction of τ to X is the identity, δX is a compactification of X. Let δ be the proximity on X associated with δX.

Set $A=N_x \cap X$ and $B=N_y \cap X$. For any neighborhood N of z in δX, $\tau^{-1}(N)$ is a neighborhood of both x and y. But if $a \in \tau^{-1}(N) \cap A$ and $b \in \tau^{-1}(N) \cap B$, then $\tau(a)=a \in N$ and $\tau(b)=b \in N$. Thus

$$z \in \mathrm{Cl}_{\delta X} A \cap \mathrm{Cl}_{\delta X} B ,$$

so that $A\delta B$ in (X,δ). But f separates A and B, hence $f \notin P^*(X)$ and f does not have a Smirnov extension to δX. This contradicts (B).

Thus, there exists $r \in R$ such that $f^\beta(x)=r$, for all $x \in \beta X - X$. Now by Theorem 2.7, $f-r \in C_\infty(X)$, and the proof is complete.

From Theorem 3.1 it is clear that the collection of functions constant at infinity is precisely $\bigcap P^*(X)$, where the intersection is taken over all compatible proximities for X, or equivalently, over all admissible totally bounded uniformities for X.

4. Realcompact and realcomplete spaces.

We recall that a proximity space (X,δ) is realcomplete if there is no point of $\delta X - X$ to which every member of $P(X)$ can be proximity-extended with real values. (See [6]). Let R^* be the one-point compactification of R, and let f^* be the extension of a member f of $C(X)$ mapping βX into R^*.

For $x \in \beta X$, the maximal ideals M^x in $C(X)$ are characterized by

$$M^x = \{f \in C(X) : \ x \in \mathrm{Cl}_{\beta X} Z(f)\}\,.$$

(See Theorem 7.3 of [2]).

THEOREM 4.1. *For $\delta=\beta$ and $x \in \beta X$, the following are equivalent.*

(A) *M^x is real.*

(B) *$M^x = I^x$.*

(C) *I^x is an ideal of $C(X)$.*

PROOF. For $x \in X$, the equivalences are clear. Thus we assume that $x \in \beta X - X$. If δ_1 is the usual metric proximity for R and δ^* is the proximity for R associated with R^*, then the identity mapping τ_0 of (R,δ_1) onto (R,δ^*) is a p-mapping. Let τ be the continuous extension of τ_0 mapping $\delta_1\mathsf{R}$ onto R^*. Take $f \in C(X)$, and let f^β be the Smirnov extension of f mapping βX into $\delta_1\mathsf{R}$. Evidently, $f^* = \tau \circ f^\beta$. Now the statement $f \in M^x$ if and only if $f^*(x)=0$ holds precisely when M^x is real, by Theorem 7.6 of [2]. But $f^*(x)=0$ implies that $f^\beta(x)=0$, since τ carries $\delta_1\mathsf{R} - \mathsf{R}$ onto the ideal point of R^*. Thus, when M^x is real, $f \in M^x$ if and only if $f \in I^x$. Hence (A) implies (B).

That (B) implies (C) is obvious.

Next, assume that I^x is an ideal. Now $f \in M^x$ implies $f^*(x)=0$. Thus $f^\beta(x)=0$ and $f \in I^x$. Since now $M^x \subseteq I^x$ and M^x is maximal, we have $M^x = I^x$. Thus (C) implies (A), and the proof is complete.

Corollary 4.2. *X is realcompact if and only if no free maximal p-system in $C(X)$ is an ideal.*

By 7.9 of [2], for $x \in \beta X - X$, I^x fails to be an ideal precisely when I^x contains a unit of $C(X)$. Thus X is realcompact if and only if every free maximal p-system in $C(X)$ contains a unit. It follows immediately from Theorem 2.7 that if $C(X)$ contains a unit which vanishes at infinity, then X is realcompact. The converse is false, since the space Q of rationals is realcompact, but $C_\infty(\mathrm{Q}) = \{0\}$. (See 7.F.5 of [2].) Theorem 4.1 and Theorem 5.8 of [2] also show that X is pseudocompact if and only if every maximal p-system is $C(X)$ is an ideal.

The following definition extends that of Mandelker in [4] to proximity spaces. For the case $\delta = \beta$, the definitions coincide.

Definition. A family $\mathscr{M} = \{F_\alpha : \alpha \in A\}$ of subsets of a proximity space (X, δ) is p-stable if, for each $f \in P(X)$, there exists $F_\alpha \in \mathscr{M}$ such that f is bounded on F_α.

The theory of clusters in proximity spaces is developed by Leader in [3]. In particular, it is shown that (X, δ) is compact if and only if each cluster contains a point. The following theorem now provides a characterization of realcompleteness in terms of clusters and p-stable families of closed sets.

Theorem 4.3. *For a proximity space (X, δ), the following are equivalent:*

(A) *(X, δ) is realcomplete.*

(B) *Every p-stable cluster in (X, δ) contains a point.*

(C) *Every p-stable family of closed subsets of X having the finite intersection property has nonempty intersection.*

Proof. (A) implies (B). Let $\mathscr{C}$ be a p-stable cluster in (X, δ), and suppose that $\mathscr{C}$ does not contain a point. By Theorems 2 and 3 of [3], we can choose $p \in \delta X - X$ such that

$$p \in \bigcap \{\mathrm{Cl}_{\delta X} A : A \in \mathscr{C}\}.$$

If $\mathscr{F}^p$ is the unique maximal round filter in (X, δ) which converges to p, then $\mathscr{F}^p$ is not real. By Theorem 2.2 of [5], for each positive integer n, there exist $f \in P(X)$ and sets G_n in $\mathscr{F}^p$ such that $|f| \geqq n$ on G_n.

Take $A \in \mathscr{C}$. Now $\mathrm{Cl}_{\delta X} G_n$ is a neighborhood of p in δX, hence each G_n meets A. But $|f| \geqq n$ on $G_n \cap A$, so that f is unbounded on A, which contradicts the assumption on $\mathscr{C}$.

(B) implies (C). Let $\mathscr{M}$ be a p-stable family of closed subsets of X having the finite intersection property. Then the collection $\{\mathrm{Cl}_{\delta X} F : F \in \mathscr{M}\}$ has the finite intersection property, and there exists p in δX satisfying

$$p \in \bigcap \{\mathrm{Cl}_{\delta X} F :\ F \in \mathscr{M}\} .$$

Thus every member of $\mathscr{M}$ is also a member of the cluster $\mathscr{C}_p$ in (X,δ) consisting of all subsets A of X satisfying $p \in \mathrm{Cl}_{\delta X} A$. Now $\mathscr{M}$ is p-stable implies $\mathscr{C}_p$ is p-stable. By (B), $\mathscr{C}_p$ contains p. Thus p belongs to every member of $\mathscr{M}$, and $\mathscr{M}$ has non-empty intersection.

(C) implies (A). Assume that (X,δ) is not realcomplete. Choose $p \in \delta X - X$ such that $\mathscr{F}^p$ is real. For $f \in P(X)$, it follows from Theorem 2.2 of [5] that there exists $F \in \mathscr{F}^p$ such that f is bounded on F, hence on $\mathrm{Cl}_X F$. Thus the family $\{\mathrm{Cl}_X F : F \in \mathscr{F}^p\}$ is p-stable and has the finite intersection property. But

$$\bigcap \{\mathrm{Cl}_X F :\ F \in \mathscr{F}^p\} = \varnothing ,$$

contradicting (C).

This completes the proof.

For $\delta = \beta$, the equivalence of (A) and (C) in Theorem 4.3 is Theorem 5.1 of [4]. In this case, if $\mathscr{O}^x$ is the ideal in $C(X)$ consisting of all $f \in C(X)$ with the property that $\mathrm{Cl}_{\beta X} Z(f)$ is a neighborhood of x, then the z-filter $Z[\mathscr{O}^x]$ is a base for the maximal round filter $\mathscr{F}^x$ in (X,β).

The following corollary provides several characterizations of realcompactness. That (C) implies (A) is Theorem 4 of [6].

COROLLARY 4.4. *For a completely regular Hausdorff space X, the following are equivalent*:

(A) *X is realcompact.*
(B) *(X,β) is realcomplete.*
(C) *X admits a compatible proximity δ for which (X,δ) is realcomplete.*
(D) *Every stable cluster in X contains a point.*
(E) *X admits a compatible proximity δ such that every p-stable family of closed sets with the finite intersection property has non-empty intersection.*
(F) *For $x \in \beta X - X$, there exists $f \in C(X)$ such that f is unbounded on every member of the z-filter $Z[\mathscr{O}^x]$.*
(G) *X admits a compatible proximity δ such that if $x \in \delta X - X$, there exists $f \in P(X)$ such that f is unbounded on every member of $\mathscr{F}^x$.*

PROOF. All implications except (C) implies (A) follow from the previous theorems and Theorem 2.2 of [5]. For completeness, we provide a new proof that (C) implies (A). Let I^x be a real maximal p-system in $C(X)$, and let τ_0 be the canonical injection of X into δX. Then τ_0 has a continuous extension τ mapping βX onto δX. For $y = \tau(x)$ in δX and $f \in P(X)$, we have $f^\delta(y) = (f^\delta \circ \tau)(x) = f^\beta(x)$, which is real. Thus, the maximal p-system I_δ^y in $P(X)$ is real, so that $y \in X$. But τ carries $\beta X - X$ into $\delta X - X$, hence $x \in X$. Thus X is realcompact, and the proof is complete.

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MOORHEAD STATE COLLEGE
MOORHEAD, MINNESOTA