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ON NORM-CONTINUITY AND COMPACTNESS OF SPECTRUM

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Introduction.

In [5] R. R. Kallman proved that a unitary representation of a locally compact abelian group is norm-continuous, if and only if, the support of its associated spectral measure is compact. For the case of a one-parameter group of invertible isometries of a Banach space, it is well-known that norm-continuity is equivalent to boundedness of the infinitesimal generator.

In section 2 we generalize these statements using the concept of spectrum of a representation introduced by W. Arveson in [2] (see 1.4(i) below). Specifically, we prove that a representation of a locally compact abelian group as invertible isometries on a Banach space is norm-continuous if and only if it has a compact spectrum.

In section 3 we show that when two representations (U,X) and (V,Y) of the group G give rise to a suitably continuous representation Φ on $B(X,Y)$,

$$\Phi_g(A) = V_g A U_{-g} \quad \forall g \in G,\ \forall A \in B(X,Y)\,,$$

then the spectrum of Φ is the closure of the difference between the spectrum of V and the spectrum of U. This generalizes a result in [6] concerning the spectra of a derivation of $B(B(X,Y))$ and its generators in $B(X)$ and $B(Y)$, in the case where the operators are Hermitian.

In [7] J. Moffat shows that a norm-continuous representation of a connected abelian group as automorphisms on a von Neumann algebra is inner, i.e. implemented by a unitary group in the algebra. His proof uses the result by Kadison and Ringrose [4] that the connected component of the identity automorphism consists of inner automorphisms. In section 4 we use the compactness of spectrum to construct a unitary group with minimal positive spectrum which implements the norm-continuous automorphism group, in the case where the group is locally compact abelian and connected. The structure theorem for locally compact abelian groups is used to reduce this problem to that of constructing

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mutually commuting unitary implementations for the reals and a compact connected group. As shown in [8], this method yields a proof of the result that every derivation of an AW*-algebra is inner (thus a new proof that the same holds in a von Neumann algebra, see also [2, theorem 4.1]).

In section 5 we show that if a norm-continuous locally compact abelian connected group of automorphisms acts on a simple C*-algebra A, the ,,minimal'' unitary implementing group in the enveloping von Neumann algebra of A actually multiplies A, thus belongs to A when A has a unit. This result generalizes the result by Sakai [13] that every derivation of a simple C*-algebra is inner (see also [9, 2.4 and 2.5]). In case the group is compact, the spectral projections of this unitary group multiply A.

The idea of investigating the action of a compact connected group on a simple C*-algebra arose in a conversation with G. K. Pedersen.

1. Notation and preliminaries.

1.1. In the following, G denotes a locally compact abelian group, $\hat{G}$ its dual group, X a Banach space, X_* a subspace of the dual space of X satisfying that

(i) $\|x\| = \sup\{|\varrho(x)| \mid \varrho \in X_*,\ \|\varrho\| \leq 1\}$ for every x in X;
(ii) the $\sigma(X,X_*)$-closed convex hull of a $\sigma(X,X_*)$-compact subset of X is $\sigma(X,X_*)$-compact.

For example, X_* may be the whole dual of X, or it may be a predual. We say that

Definition 1.2. (U,X) is a *representation* of G on X, if U is a homomorphism of G into the group of invertible isometries in $B(X)$ which is $\sigma(X,X_*)$-continuous, i.e.

$$\varrho(U_g x - x) \to 0 \text{ as } g \to 0 \quad \forall x \in X,\ \forall \varrho \in X_* .$$

1.3. If (U,X) is a representation of G and μ a complex measure of finite variation, then the linear functional on X_* defined by

$$\varrho \to \int \varrho(U_g x)\, d\mu(g) = \varrho(y)$$

gives an element y in X, thus the map $x \mapsto y$ defines an operator $U(\mu)$ on X. The map

$$\mu \mapsto U(\mu)$$

is a homomorphism of the convolution algebra of bounded measures into $B(X)$, which satisfies that $\|U(\mu)\| \leqq \|\mu\|$.

With I the identity on X we say that

(i) U is norm-continuous when

$$\|U_g - I\| \to 0 \quad \text{as } g \to 0$$

(ii) U is strongly continuous when

$$\|U_g x - x\| \to 0 \quad \text{as } g \to 0 \quad \forall x \in X .$$

(Note that (ii) is equivalent to $\sigma(X,X')$-continuity.)

Definition 1.4. Let (U,X) be a representation of G.

(i) *the spectrum* of U is

$$\operatorname{sp} U = \operatorname{hull}\{f \in L^1(G) \mid U(f) = 0\}$$

(ii) *the spectrum* of x in X with respect to U is

$$\operatorname{sp}_U x = \operatorname{hull}\{f \in L^1(G) \mid U(f)x = 0\}$$

(iii) for every closed subset E of $\hat{G}$ the *spectral subspace* is

$$M^U(E) = \{x \in X \mid \operatorname{sp}_U x \subset E\}$$

Proposition 1.5. *Let μ_1 and μ_2 be measures of finite variation such that $\hat{\mu}_1 = \hat{\mu}_2$ on a neighbourhood of* $\operatorname{sp} U$. *Then*

$$U(\mu_1)x = U(\mu_2)x \quad \forall x \in X .$$

Proof (see also [3, lemma 2.1.3(k)]). Let $(f_\lambda)_{\lambda \in \Lambda}$ be an approximate unit for $L^1(G)$, then

$$U(f_\lambda)U(\mu_1 - \mu_2)x \to U(\mu_1 - \mu_2)x$$

and each function $f_\lambda * (\mu_1 - \mu_2)$ will by the generalized Wiener Tauberian theorem belong to the ideal

$$I = \{f \in L^1(G) \mid U(f) = 0\}$$

since its transform vanishes on a neighbourhood of $\operatorname{hull} I$.

Proposition 1.6. *Let (U,X) be a representation of G, let Y be the norm-closure in $B(X)$ of the image algebra $U(L^1(G))$. The spectrum of Y, Δ_Y, is equal to the spectrum of U.*

Proposition 1.7. *Let* (U,X) *and* (V,Y) *be representations of* G, *let* $\Phi = V\cdot U^{-1}$ (*that is,* $\Phi_g(A) = V_g A U_g{}^{-1}$ *for every* A *in* $B(X,Y)$) *be a representation on* $B(X,Y)$. *For* E_1 *and* E_2 *two closed subsets of* $\hat{G}$, *let* E_3 *be the closure of* E_1+E_2. *Then*

$$M^{\Phi}(E_1)M^{U}(E_2) \subseteq M^{V}(E_3)\ .$$

For the proofs of these statements, the reader is referred to [2], [3] or [10].

2. On norm-continuity and compactness of spectrum.

Proposition 2.1. *Let* (U,X) *be a norm-continuous representation of* G. *Then* $\operatorname{sp} U$ *is compact.*

Proof. Take $(f_\lambda)_\Lambda$ to be an approximate unit for $L^1(G)$. Under the assumption of norm-continuity of U we see that

$$\|U(f_\lambda)-I\| \to 0 \quad \text{along } \Lambda\ ,$$

where I denotes the unit element in $B(X)$. Thus $Y = U(L^1(G))^-$ is a commutative Banach algebra with unit, from which it follows that its spectrum $\Delta_Y = \operatorname{sp} U$ is compact.

Proposition 2.2. *Let* (U,X) *be a representation of* G *with compact spectrum. Then* U *is norm-continuous.*

Proof. Choose $f \in L^1(G)$ such that $\hat{f}$ is identically 1 on a neighbourhood of $\operatorname{sp} U$ and has compact support. Then by 1.5 above we have that

$$U(f)x = x \quad \forall x \in X\ ,$$

that is, $U(f) = I$. But then with $f_g = \varepsilon_g * f$ we see that

$$U_g - I = U(f_g - f)$$

thus

$$\|U_g - I\| = \|U(f_g - f)\| \leqq \|f_g - f\|$$

from which the desired continuity property is evident.

Lemma 2.3. *Let* (U,X) *be a representation with compact spectrum. Let* $\sigma(U_g)$ *denote the spectrum of the operator* U_g *as an element of* $B(X)$. *Then*

$$\sigma(U_g) = \{(g,\gamma) \mid \gamma \in \operatorname{sp} U\} \quad \forall g \in G\ .$$

PROOF. As seen above, $U_g = U(f_g)$, where f is an element of $L^1(G)$, whose transform $\hat{f}$ has compact support and is identically 1 on a neighbourhood of $\operatorname{sp} U$. Now

$$\begin{aligned}\sigma_Y(U(f_g)) &= \{\hat{f}_g(\gamma) \mid \gamma \in \operatorname{sp} U\} \\ &= \{\hat{f}(\gamma)(g,\gamma) \mid \gamma \in \operatorname{sp} U\} \\ &= \{(g,\gamma) \mid \gamma \in \operatorname{sp} U\},\end{aligned}$$

and since the spectrum is a subset of the unit circle, thus equal to its boundary, we have that

$$\sigma_{B(X)}(U_g) = \sigma_Y(U_g) = \{(g,\gamma) \mid \gamma \in \operatorname{sp} U\}.$$

EXAMPLE 2.4. Let (U,H) be a strongly continuous unitary group. Then by Stone's theorem we know that

$$U_g = \int_{\hat{G}} (g,\gamma)\, dP(\gamma)$$

with P the associated spectral measure. It is not hard to see that $\operatorname{sp} U$ is the support of P on $\hat{G}$ (for a more detailed account, see [10, 2.5]).

EXAMPLE 2.5. Let D be a Hermitian operator on X, that is, $U_t = e^{itD}$ is a one-parameter group of isometries on X. It follows from the above considerations that

$$\sigma(D) = \operatorname{sp} U.$$

EXAMPLE 2.6. Let (U,X) be a representation of the compact abelian group G. Then

$$\gamma \in \operatorname{sp} U \Leftrightarrow U(-\gamma) \neq 0.$$

This is seen as follows:

Each γ in G is in $L^1(G)$ with

$$\hat{\gamma}(-\omega) = \int (g,\gamma)\overline{(g,\omega)}\, dg = \int (g,\gamma-\omega)\, dg = \begin{cases} 0 & \gamma \neq \omega \\ 1 & \gamma = \omega \end{cases}$$

Thus $-\gamma \notin \operatorname{sp} U$ implies that the support of $\hat{\gamma}$ is in the complement of $\operatorname{sp} U$, that is, $U(-\gamma) = 0$.

If $\gamma \in \operatorname{sp} U$ then by definition $U(-\gamma) \neq 0$.

Note that for γ and ω in $\hat{G}$

$$U(\gamma)U(\omega) = U(\gamma * \omega) = \begin{cases} U(\gamma) & \gamma = \omega \\ 0 & \gamma \neq \omega \end{cases}$$

EXAMPLE 2.7. Let (U,X) be as in 2.6, and assume that $\operatorname{sp} U$ is compact, that is, $\operatorname{sp} U = \{\gamma_0, \ldots, \gamma_n\}$. Then

$$U_g = \sum_{i=0}^{n} U(-\gamma_i)(g,\gamma_i) \quad \forall g \in G.$$

To see this, let $x \in X$, $\varrho \in X_*$. Denote by f the function $g \mapsto \varrho(U_g x)$. We have

$$\hat{f}(\gamma) = \int \varrho(U_g x)(g,\gamma)dg = \varrho\big((U(\gamma))x\big)$$

so $\hat{f}$ has support in $\{-\gamma_0,\ldots,-\gamma_n\}$. It follows that

$$f(g) = \varrho(U_g x) = \textstyle\sum_{i=0}^n \varrho(U(-\gamma_i)x)(g,\gamma_i)$$

and so we have the conclusion

$$U_g = \textstyle\sum_{i=0}^n U(-\gamma_i)(g,\gamma_i)\,.$$

(That U is then norm-continuous as claimed in proposition 2.2 is now immediate.)

3. On the spectrum of composed groups.

3.1. Let (U,X) and (V,Y) be strongly continuous representations of G and let Φ be the homomorphism defined by

$$\Phi_g(A) = V_g A U_g^{-1} \quad \forall A \in B(X,Y)\,.$$

Then Φ is a representation on $B(X,Y)$ in the sense of 1.2, when we take $B(X,Y)_*$ to be the closed linear span of elements $\varrho \otimes x$ for $\varrho \in Y'$, $x \in X$, where $(\varrho \otimes x)(A) = \varrho(Ax)$.

The integral

$$\Phi(\mu)A = \int \Phi_g(A)\,d\mu(g)$$

defines a $\sigma(B(X,Y),B(X,Y)_*)$-continuous linear operator $\Phi(\mu)$. For the proof of this, see [2, section 1].

Proposition 3.2. *Let (U,X) and (V,Y) be strongly continuous representations, and let $\Phi = V \cdot U^{-1}$ be as in* 3.1. *Then* $\operatorname{sp}\Phi$ *is the closure in $\hat{G}$ of* $\{\operatorname{sp} V - \operatorname{sp} U\}$.

Proof. Let $\gamma \in \hat{G}$ which is not in the closure of $\{\operatorname{sp} V - \operatorname{sp} U\}$. Let V_0 be a compact neighbourhood of γ, which is disjoint from $\{\operatorname{sp} V - \operatorname{sp} U\}$. Then

$$(V_0 + \operatorname{sp} U) \cap \operatorname{sp} V = \emptyset$$

from which it follows that

$$M^{\Phi}(V_0)M^U(\operatorname{sp} U) \subseteq M^V(V_0 + \operatorname{sp} U) = \{0\}$$

since $M^V(\operatorname{sp} V) = Y$ and

$$M^V(V_0 + \operatorname{sp} U) \cap M^V(\operatorname{sp} V) = M^V((V_0 + \operatorname{sp} U) \cap \operatorname{sp} V) = M^V(\emptyset)\,.$$

Since $M^U(\operatorname{sp} U) = X$ this means that $M^\Phi(V_0) = \{0\}$, thus $\Phi(f) = 0$ for every f in $L^1(G)$ whose transform has support in V_0. This shows that $\gamma \notin \operatorname{sp}\Phi$. So we have obtained that

$$\operatorname{sp}\Phi \subseteq \{\operatorname{sp} V - \operatorname{sp} U\}^- .$$

We also want to show that

$$\operatorname{sp} V - \operatorname{sp} U \subseteq \operatorname{sp}\Phi .$$

Let $\lambda \in \operatorname{sp} V$, $\mu \in \operatorname{sp} U$. Let V_λ, respectively V_μ, be compact neighbourhoods of these elements. Then

$$M^V(V_\lambda) \neq \{0\} \quad \text{and} \quad M^U(V_\mu) \neq \{0\} .$$

Pick x_0 in $M^U(V_\mu)$, with $\|x_0\| = 1$, and y_0 in $M^V(V_\lambda)$, with $\|y_0\| = 1$. Choose φ in X' such that $\varphi(x_0) = 1$, $\|\varphi\| = 1$. The operator $A : x \mapsto y_0\varphi(x)$ in $B(X, Y)$ has norm 1. Thus

$$M^V(V_\lambda) \subseteq B(X,Y)M^U(V_\mu) = M^\Phi(\operatorname{sp}\Phi)M^U(V_\mu) \subseteq M^V(\operatorname{sp}\Phi + V_\mu) .$$

It follows that $\lambda \in \operatorname{sp}\Phi + V_\mu$, since otherwise we could choose a neighbourhood V_λ^0 of λ disjoint from $\operatorname{sp}\Phi + V_\mu$, and

$$M^V(V_\lambda^0 \cap V_\lambda) \subseteq M^V(V_\lambda^0) \cap M^V(\operatorname{sp}\Phi + V_\mu) = \{0\}$$

in contradiction with $\lambda \in \operatorname{sp} V$. Thus we obtain

$$\lambda \in \textstyle\bigcap_{\text{all } V_\mu} (\operatorname{sp}\Phi + V_\mu) = \operatorname{sp}\Phi + \{\mu\} .$$

COROLLARY 3.3. *Let (U,X), (V,Y) and $(\Phi, B(X,Y))$ be as in* 3.2. *Then Φ is norm-continuous if and only if U and V are both norm-continuous.*

PROOF. Follows directly from 2.1, 2.2 and 3.2.

COROLLARY 3.4. *Let α be the element of $B(B(X,Y))$ defined by*

$$\alpha(X) = AX - XB$$

where A and B are Hermitian elements of the algebras $B(Y)$ and $B(X)$ respectively.

Then $\sigma(\alpha) = \sigma(A) - \sigma(B)$.

PROOF. Both e^{itA} and e^{itB} are norm-continuous representations of $\mathbf{R}$ with spectra equal to those of the generators (compare with 2.5).

4. Connected groups acting on von Neumann algebras.

Let A be a von Neumann algebra acting on a Hilbert space H. Let G be a locally compact abelian group, and assume (α, A) to be a σ-weakly continuous representation of G as $*$-automorphisms on A. For every closed subset E of $\hat{G}$ $p(E)$ denotes the largest left-annihilating projection of $M^\alpha(E)$ in A. Let $Z(M^\alpha\{0\})$ denote the center of the fixed-point algebra $M^\alpha\{0\}$.

Lemma 4.1. $$p(E) \in Z(M^\alpha\{0\}).$$

Proof. That $p(E) \in M^\alpha\{0\}$ follows from the invariance of $M^\alpha(E)$ under the group. Assume x to be a self-adjoint element of $M^\alpha\{0\}$. Since, by taking $U = V = \alpha$ and noting that for every closed F in $\hat{G}$, $M^\Phi(F)$ contains the left multiplication opperators defined by elements in $M^\alpha(F)$, we see from 1.7 that

$$M^\alpha\{0\}M^\alpha(E) \subseteq M^\alpha(E) .$$

We conclude that $p(E)x \in Ap(E)$ and thus

$$p(E)x = p(E)xp(E) = (p(E)xp(E))^* = xp(E) .$$

From now on, assume that G is also *connected.* By the well-known structure theorem (see [12, 2.4.1]) G is the direct sum of R^n for some n in N and the compact group K. Since K is a continuous image of G, K is also connected, thus the dual group $\hat{K}$ can be ordered (see [12, 8.1.2 (a) and 2.5.6 (c)]). Let $\hat{S}$ be a semigroup in $\hat{K}$ which satisfies that

$$\hat{S} \cap (-\hat{S}) = \{0\}, \quad \hat{S} \cup (-\hat{S}) = \hat{K} .$$

Let $\hat{V}$ denote the product $[0, \infty)^n$.

Theorem 4.2. *Let (α, A) be a norm-continuous representation of $G = \mathsf{R}^n \otimes K$.*

There exists a unitary group (u, H) in A with $\operatorname{sp} u \subset \hat{V} \times \hat{S}$ *such that*

$$\alpha_g(x) = u_g x u_g^*$$

for g in G and x in A.

If (v, H) is another unitary group implementing α with $\operatorname{sp} v \subset \hat{V} \times \hat{S}$, *then (vu^*, H) is a unitary group satisfying*

$$\operatorname{sp} vu^* \subset \hat{V} \times \hat{S} .$$

Proof. Let (α^i, A) denote the subgroup $\alpha_{(0,\ldots,0,t,0,\ldots,0)}$, where t is the ith coordinate. This one-parameter group has a self-adjoint derivation δ^i as its infinitesimal generator, that is, $\alpha^i{}_t = e^{t\delta i}$ for every t in R.

Let $p_s = p[s, \infty)$ be the largest left-annihilating projection of $M^{\alpha^i}[s,\infty)$. Then

$$s \mapsto p_s$$

is an increasing projection valued map with $p_0 = 0$ and

$$p_{\|\delta^i\|+\varepsilon} = 1, \quad \forall \varepsilon > 0 .$$

It is proved in [8, theorem 2] that the unitary group in A defined by

$$u^i{}_t = \lim_{\varepsilon \to 0} \int_0^{\|\delta^i\|+\varepsilon} e^{its} dp(s)$$

(where the integration is carried out in the Riemann–Stieltjes sense) satisfies that

$$\alpha^i{}_t(x) = u^i{}_t x (u^i{}_t)^* \quad \forall x \in A, \ \forall t \in \mathsf{R} .$$

Whenever $v^i{}_t$ implements α^i each $v^i{}_t \in A$, and $\operatorname{sp} v^i \subset [0,\infty)$, we have by [8, proposition 3] that

$$\operatorname{sp} v^i (u^i)^* \subset [0, \infty) .$$

Since $p_s \in Z(M^{\alpha^i}\{0\})$ by 4.1, the families all commute, i.e.

$$u^i u^j = u^j u^i \quad \forall i,j .$$

Now let (β, A) denote the compact subgroup $\beta_g = \alpha_{(0,\ldots,0,g)}$. Here we define

$$(1-p_\gamma)H = [M^\beta\{\gamma + \hat{S}\}H]$$

(that is, $p_\gamma = p(\gamma + \hat{S})$ with notation as in 4.1). The map $\gamma \mapsto p_\gamma$ is monotone increasing, and is piecewise constant, taking on only a finite number of values (compare with examples 2.6 and 2.7). Let

$$\operatorname{sp}\beta = \{-\gamma_n, \ldots, -\gamma_1, 0, \gamma_1, \ldots, \gamma_n\} .$$

(That the spectrum is symmetric follows from the biimplication

$$\beta_g(x) = (g, \gamma_k)x \Leftrightarrow \beta_g(x^*) = \overline{(g,\gamma_k)}x^* .$$

Define

$$u_g = \textstyle\sum_{i=0}^n (g, \gamma_i)(p_{\gamma_{i+1}} - p_{\gamma_i}) .$$

Then

$$M^u\{\gamma + \hat{S}\} = [M^\beta\{\gamma + \hat{S}\}H]$$

and so for all ω and γ in $\hat{G}$

$$M^\beta\{\omega + \hat{S}\} M^u\{\gamma + \hat{S}\} \subseteq M^u\{\omega + \gamma + \hat{S}\} .$$

Thus by [2, theorem 2.3]

$$M^\beta\{\omega + \hat{S}\} \subseteq M^{u \cdot u*}\{\omega + \hat{S}\}$$

for every ω in $\hat{G}$, which shows that

$$M^{\beta}\{\omega\} = M^{\beta}\{\omega+\hat{S}\} \cap M^{\beta}\{\omega-\hat{S}\} = M^{\beta}\{\omega+\hat{S}\} \cap (M^{\beta}\{-\omega+S\})^* \subseteq M^{u\cdot u^*}\{\omega+\hat{S}\} \cap (M^{u\cdot u^*}\{-\omega+\hat{S}\})^* = M^{u\cdot u^*}\{\omega\}.$$

Thus inspection of 2.7 reveals that $\beta = u\cdot u^*$.

Whenever v_g is an implementing group for β with $\operatorname{sp} v \subset \hat{S}$, that is, $M^v(\hat{S}) = H$ we have that

$$M^{\beta}\{\gamma+\hat{S}\}H = M^{\beta}\{\gamma+\hat{S}\}M^v(\hat{S}) \subseteq M^v\{\gamma+\hat{S}\}$$

thus

$$M^u\{\gamma+\hat{S}\} \subseteq M^v\{\gamma+\hat{S}\} \quad \forall \gamma \in \hat{G},$$

from which we see that $\operatorname{sp} vu^* \subset \hat{S}$.

Using 4.1 to see that u_g commutes with each $u^i{}_t$, $i=1,\ldots,n$, we have that the unitary group

$$(t_1,\ldots,t_n,g) \mapsto u^1{}_{t_1} u^2{}_{t_2} \ldots u^n{}_{t_n} u_g$$

implements α and has positive spectrum.

The spectral minimality claimed in the theorem follows from the minimality property of each subgroup.

5. Connected groups acting on simple C^*-algebras.

Let A be a C*-algebra and denote by A'' the enveloping von Neumann algebra of A, isomorphic with the second dual of A. For any set B in the self-adjoint part A''_{sa} of A'', let B^- denote the norm-closure and B^m the set of operators in A''_{sa}, which can be obtained as strong limits of increasing nets from B. Likewise $B_m = -(-B)^m$. The class $((A_{sa})^m)^-$ consists of the so-called lower semi-continuous elements of A''_{sa}. If $\tilde{A}$ denotes the C*-algebra obtained by adjoining the unit 1 of A'' to A, then

$$((A_{sa})^m)^- + \mathsf{R}1 = (\tilde{A}_{sa})^m.$$

([1, Proposition 2.5]).

If $M(A)$ denotes the C*-algebra in A'' of elements x such that $xA \subset A$ and $Ax \subset A$ then

$$M(A) = (\tilde{A}_{sa})^m \cap (\tilde{A}_{sa})_m,$$

([11, theorem 2.5]).

Assume (α, A) to be a norm-continuous representation of G on A. Then (α'', A'') is a norm-continuous representation of G on A'' and this implies that the results of the preceding section are applicable, so we have the following

PROPOSITION 5.1. *Let (α, A) be a norm-continuous representation of the locally compact abelian and connected group G. Then there is a unitary group (u, H_u) in A'' with spectrum in the positive cone of $\hat{G}$, which implements the action of α'' on A'', and which satisfies that whenever (v, H_u) has spectrum in the positive cone of $\hat{G}$ and implements α'', then the spectrum of vu^* is positive in $\hat{G}$.*

Here we want to sharpen this result in the special case where A is a simple C*-algebra, that is, A has no closed ideals except $\{0\}$ and A. As a first step we prove the following, which may be of independent interest. Let $\hat{A}$ denote the spectrum of A.

PROPOSITION 5.2. *Let G be a compact connected abelian group, let (α, A) be a norm-continuous representation of G. Assume $\hat{G}$ to be ordered by the semi-group $\hat{S}$. For each π in $\hat{A}$, let γ_π denote the maximal element in the spectrum of $\pi \cdot \alpha$. The map $\pi \mapsto \gamma_\pi$ is continuous on $\hat{A}$ if and only if the largest left-annihilating projection p_γ and the largest right-annihilating projection q_γ of $M^\alpha\{\gamma + \hat{S}\}$ in A'' both belong to $M(A)$ for every γ in $\hat{G}$.*

PROOF. Let $\mathrm{sp}\,\alpha = \{-\gamma_n, \ldots, -\gamma_1, 0, \gamma_1, \ldots, \gamma_n\}$, with $\gamma_i < \gamma_{i+1}$ for each $i = 0, \ldots, n$. (γ_0 denotes 0). Let m denote the map $\pi \mapsto \gamma_\pi$. If m is continuous, the inverse image $m^{-1}(\{\gamma_k\})$ is both closed and open for every $k = 0, \ldots, n$. Thus A is decomposed as a direct sum of two-sided ideals

$$A = I_0 \oplus \ldots \oplus I_n$$

with $\hat{I}_k = m^{-1}(\{\gamma_k\})$. As the action of α leaves each I_k invariant we may restrict attention to the case where $\pi \mapsto \gamma_\pi$ is the constant γ_n. Now the unitary group with minimal positive spectrum which implements α is

$$u_g = \textstyle\sum_{i=0}^{n} (g, \gamma_i)(p_{\gamma_{i+1}} - p_{\gamma_i}),$$

$p_{\gamma_{n+1}}$ denoting 1. The group with minimal positive spectrum implementing α^{-1} is

$$v_g = \textstyle\sum_{i=0}^{n} (g, \gamma_i)(q_{\gamma_{i+1}} - q_{\gamma_i}),$$

$q_{\gamma_{n+1}}$ denoting 1.

The group $(g, \gamma_n)v_{-g}$ implements α and has positive spectrum, thus by 5.1

$$\mathrm{sp}((g, \gamma_n)v_{-g}u_{-g}) \subset \hat{S}$$

that is,

$$\mathrm{sp}(uv) \subset \gamma_n - \hat{S}.$$

We know from our assumption that α and then also $(\alpha)^{-1}$ have γ_n in the spectrum, and so this must also hold for the groups u_g and v_g implementing these automorphism groups. The spectral projections corresponding to γ_n are $1-p_{\gamma_n}$ and $1-q_{\gamma_n}$, and both are thus different from 0. We see, however, that

$$(1-p_{\gamma_n})(1-q_{\gamma_n}) = 0 .$$

If not, $\gamma_n+\gamma_n \in \operatorname{sp}(uv)$, a contradiction to γ_n being an upper bound on this set. Analogously

$$(1-p_{\gamma_n})(q_{\gamma_{i+1}}-q_{\gamma_i}) = 0 \quad \forall i=1,\ldots,n$$

and so we have by summation

$$(1-p_{\gamma_n})(1-q_{\gamma_1}) = 0$$

that is,

$$(1-p_{\gamma_n})q_{\gamma_1} = 1-p_{\gamma_n} .$$

(Note that this observation formed a part of the proof of 4.2.) Thus $1-p_{\gamma_n} \neq 0$ implies that $(1-p_{\gamma_n})q_{\gamma_1} \neq 0$ and since this projection is a subprojection of the spectral projection for $u_g v_g$ corresponding to γ_n we have that $\gamma_n \in \operatorname{sp}(uv)$.

Analogously we can show that the assumption $\gamma_n \in \operatorname{sp}(\pi \cdot \alpha)$ for every irreducible representation π implies that $\gamma_n \in \operatorname{sp}(\pi(u)\pi(v))$. Since for every g $u_g v_g$ implements the identity automorphism, it is a central element. Thus its spectral projections are all central, and therefore their images under π are either 0 or 1. The fact that $\gamma_n \in \operatorname{sp}(\pi(u)\pi(v))$ thus implies that

$$\pi(u_g v_g) = (g,\gamma_n)1$$

for every g in G.

Using that

$$1-p_\gamma \in (A_+)^m, \quad (1-q_\gamma) \in (A_+)^m$$

we have that u_g and v_g are universally measurable in the sense of [11] and as the atomic representation $\pi_a = \oplus_{\pi \in \hat{A}} \pi$ is faithful on this subset of A'' (see [11, theorem 3.8]) we get that

$$u_g = (g,\gamma_n)v_g{}^* \quad \forall g \in G .$$

This shows that

$$\operatorname{sp} u = \{0,\ldots,\gamma_n\} = \gamma_n + \operatorname{sp} v^*$$

and as $\gamma_i < \gamma_{i+1}$ for $i=0,\ldots,n$ we have the identification

$$\gamma_i = \gamma_n - \gamma_{n-i}$$

of the eigenvalues as well as the identification

$$p_{\gamma_{i+1}} - p_{\gamma_i} = q_{\gamma_{n-i+1}} - q_{\gamma_{n-i}}$$

of the spectral projections.

$$1-p_{\gamma n} = q_{\gamma 1}\,,$$

$$p_{\gamma n}-p_{\gamma n-1} = q_{\gamma 2}-q_{\gamma 1} = q_{\gamma 2}-1+p_{\gamma n}\,,$$

and so on. Thus in general

$$1-p_{\gamma i} = q_{\gamma n+1-i}\,.$$

As $(1-p_{\gamma i}) \in (A_+)^m$ this shows that $q_{\gamma n+1-i} \in (A_+)^m \cap (\tilde{A})_m$.

Thus each $q_\gamma \in M(A)$ by [11, theorem 2.5] and so this also holds for each p_γ, γ in $\hat{G}$.

Conversely, if we assume that $p_\gamma \in M(A)$ and $q_\gamma \in M(A)$ for every γ in $\hat{G}$, then the group

$$u_g{}' = u_g v_g = v_g u_g$$

is central and multiplies A. So its spectral projections

$$p_k = \int \overline{(g,\gamma_k)}\, u'_g\, dg$$

are central and multiply A, which means that the function $e\colon \pi \mapsto \|\pi(p_k)\|$ takes on the values 0 and 1 only and is continuous. Thus

$$\{\pi \mid \gamma_\pi = \gamma_k\} = \{\pi \mid \pi(p_k) = 1\} = e^{-1}(\{1\}) = e^{-1}((\tfrac{1}{2}, \tfrac{3}{2}))$$

is both open and closed, that is, $\pi \mapsto \gamma_\pi$ is continuous.

THEOREM 5.3. *Let A be a simple C*-algebra, let G be a locally compact abelian and connected group. Let (α, A) be a norm-continuous representation of G. Then α is implemented by a unitary group in $M(A)$.*

PROOF. For $G = \mathsf{R}$, the proof of this is given in [9, 2.4 and 2.5]. For G compact connected and abelian it is an immediate corollary of 5.2. As in the proof of 4.2 we note that the unitary groups in $M(A)$ we have constructed, which implement the subgroups

$$\alpha^i{}_t = \alpha_{(0,\ldots,t,\ldots,0)}, \quad \beta_{k_{\alpha''}} = \alpha_{(0,\ldots 0,k)}$$

all belong to $Z(M\{0\})$, thus are mutually commuting, and so their product is a unitary group in $M(A)$ implementing the total action of α.

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