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ON THE CLASSIFICATION OF COMPLEX LINDENSTRAUSS SPACES

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Abstract.

We prove the Lindenstrauss–Wulbert classification scheme for complex Banach spaces whose duals are L_1-spaces, and give some characterizations of the different classes by means of the unit ball in the dual space. The work leans heavily on [8] and the real theory. I am indebted to B. Hirsberg and A. Lazar for a preprint of [12]. Finally I would like to thank E. Alfsen and Å. Lima for making literature available and for helpful comments.

1. Preliminaries and notations.

Any unexplained notation in this paper will be standard or that of Alfsen's book [1]. Otherwise we will use the following notations:

T is the unit circle in C.
V is a complex Banach space.
K is the unit ball in V^* with the w^*-topology.
$M(K)$ is the Banach space of complex regular Borel measure on K with total-variation as norm.
$M_1(K)$ is the set of those measures in $M(K)$ with norm $\leqq 1$.
$M_1^+(K)$ is the set of probability measures on K.

When F is a convex set then $\partial_e F$ will denote the set of extreme points in F. If μ is a measure then $|\mu|$ is the total variation of μ. A measure μ is said to be maximal or a boundary measure if $|\mu|$ is maximal in Choquet's ordering. The set of maximal (probability-) measures on K is denoted by $M(\partial_e K)$ $(M_1^+(\partial_e K))$.

We shall now repeat some results and definitions from [8].

A function $f \in C_{\mathsf{C}}(K)$ is said to be T-*homogeneous* if $f(\alpha k) = \alpha f(k)$ for all $\alpha \in \mathsf{T}$, $k \in K$. The class of T-homogeneous functions in $C_{\mathsf{C}}(K)$ is denoted by $C_{\mathrm{hom}}(K)$. If $f \in C_{\mathsf{C}}(K)$, then the function

$$[\mathrm{hom}_{\mathsf{T}} f](k) = \int \alpha^{-1} f(\alpha k)\, d\alpha, \quad k \in K$$

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where $d\alpha$ is the unit Haar measure on T, is continuous and T-homogeneous. It is now verified that $\hom_{\mathsf{T}}$ is a norm-decreasing projection of $C_{\mathsf{C}}(K)$ onto $C_{\hom}(K)$. Taking the adjoint of this projection on $M(K)$

$$\hom_{\mathsf{T}}\mu = \mu \circ \hom_{\mathsf{T}} ,$$

we get a norm-decreasing w^*-continuous projection of $M(K)$ onto a linear subspace denoted by $M_{\hom}(K)$.

A measure $\mu \in M_{\hom}(K)$ is called T-homogeneous and satisfies $\sigma_\alpha \mu = \alpha\mu$ where $\sigma_\alpha : K \to K$ is the homeomorphism $k \mapsto \alpha k$, $\alpha \in \mathsf{T}$, $k \in K$.

Each $v \in V$ can in a canonical way be regarded as an affine T-homogeneous w^*-continuous function on K. Conversely, by a result of Banach–Dieudonné [1, corollary I.1.13], each affine T-homogeneous function can be extended to a w^*-continuous complex-linear functional on V^*, i.e. to an element of V. We may therefore identify V with the affine functions in $C_{\hom}(K)$. If $\mu \in M(K)$, then the *resultant* of μ is defined to be the unique point $r(\mu) \in V^*$ satisfying

$$r(\mu)(v) = \mu(v) \quad \text{for all } v \in V .$$

If $\mu \in M_1{}^+(K)$, then it can be proved that $r(\mu)$ coincides with the barycenter of μ. (See [8] for a proof.) Moreover, it is readily verified that $r : M(K) \to V^*$ is a w^*-continuous normdecreasing linear surjection.

Let X be a topological space and $\mu \in M^+(K)$. A function $f : K \to X$ is *μ-measureable* if for every $\varepsilon > 0$ there is a compact set $D \subseteq K$ such that $\mu(K \setminus D) < \varepsilon$ and $f|D$ is continuous. If $X = \mathsf{R}$ or C then this definition coincides with the customary one by virtue of Lusin's theorem. Let $\mu \in M(K)$. Then there is a complex $|\mu|$-measurable function φ on K with $|\varphi| = 1$ a.e. $|\mu|$ such that $\mu = \varphi|\mu|$, (that is, $\int f d\mu = \int f\varphi d|\mu|, f \in C_{\mathsf{C}}(K)$). This representation is called the polar decomposition for μ and is unique up to zero sets. Since $\varphi : K \to \mathsf{C}$ is $|\mu|$-measurable it follows that the map $\omega : K \to K$ defined by $\omega(p) = \varphi(p) \cdot p$ is also measurable. Hence, by Lusin's theorem, the measure $\omega(|\mu|)$ defined by

$$\omega(|\mu|)(f) = \int f \circ \omega \, d|\mu|, \quad f \in C_{\mathsf{C}}(K) ,$$

is a regular Borel measure. (This definition is due to Phelps.) Clearly $\|\omega(|\mu|)\| \leqq \|\mu\|$, and the other statements in the following lemma are proved in [8].

LEMMA 1. *Let $\mu \in M(K)$, then*

a) $r(\hom_{\mathsf{T}}\mu) = r(\mu)$
b) $r(\omega(|\mu|)) = r(\mu)$

c) $\|\omega(|\mu|)\| \leqq \|\mu\|$
d) $\mathrm{hom}_T \omega(|\mu|) = \mathrm{hom}_T \mu$
e) *If μ is maximal, then so are $\omega(|\mu|)$ and* $\mathrm{hom}_T \mu$.

LEMMA 2. *Let $\mu_1, \mu_2 \in M(K)$, and put $\mu = \mu_1 + \mu_2$. If $\|\mu\| = \|\mu_1\| + \|\mu_2\|$, then μ_1 and μ_2 admit the same polar decomposition, i.e., there is a complex measurable function φ on K with $|\varphi| = 1$* a.e. *$|\mu|$ such that*

$$\mu_1 = \varphi|\mu_1|, \quad \mu_2 = \varphi|\mu_2| .$$

PROOF. Since $\|\mu\| = \|\mu_1\| + \|\mu_2\|$, we easily get $|\mu| = |\mu_1| + |\mu_2|$. In particular $|\mu_1|, |\mu_2| \ll |\mu|$, so by the Radon–Nikodym theorem there are non-negative measurable function f_1, f_2 such that $|\mu_1| = f_1|\mu|$, $|\mu_2| = f_2|\mu|$. Let $\mu = \varphi|\mu|$, $\mu_1 = \varphi_1|\mu_1|$, $\mu_2 = \varphi_2|\mu_2|$ be the polar decompositions. Then

$$\begin{aligned} \varphi|\mu| &= \varphi_1|\mu_1| + \varphi_2|\mu_2| , \\ \varphi(f_1+f_2)|\mu| &= (\varphi_1 f_1)|\mu| + (\varphi_2 f_2)|\mu| , \\ \varphi(f_1+f_2) &= \varphi_1 f_1 + \varphi_2 f_2 \text{ a.e. } |\mu| , \\ \varphi &= \varphi_1 = \varphi_2 \text{ a.e. } |\mu| , \end{aligned}$$

which proves the lemma.

The above lemma immediately gives

COROLLARY 3. *Let $\mu_1, \mu_2 \in M(K)$ and put $\mu = \mu_1 + \mu_2$. If $\|\mu\| = \|\mu_1\| + \|\mu_2\|$, then*

$$\omega(|\mu|) = \omega(|\mu_1|) + \omega(|\mu_2|) .$$

2. Complex Lindenstrauss spaces and complex affine selections.

A complex Banach space W is called an L-space if $W \cong L_C^1(Q, \mathscr{B}, m)$ for some measure-space $(Q, \mathscr{B}, m)$.

A complex *Lindenstrauss space* is a complex Banach space whose dual is an L-space.

THEOREM 4. *If W is an L-space and $\pi\colon W \to W$ a projection with norm one, then $\pi(W)$ is an L-space.*

PROOF. See [8].

COROLLARY 5. *If V is a Lindenstrauss space and $\pi\colon V \to V$ a projection with norm one, then $\pi(V)$ is a Lindenstrauss space.*

PROOF. Let π^* be the adjoint projection. Then the restriction map $\gamma: V^* \to (\pi V)^*$ takes $\pi^*(V^*)$ isometrically onto (πV^*) and π^* is a projection with norm one.

In [8] Effros proved that

Complex Lindenstrauss spaces may be characterized by: If $\mu,\nu \in M_1^+(\partial_e K)$ and $r(\mu)=r(\nu)$, then $\hom_{\mathsf{T}}\mu = \hom_{\mathsf{T}}\nu$.

This theorem will be fundamental in the following, and we shall refer to it as *Effros' characterization.*

A map $\varphi: K \to M_1(K)$ is said to be a *complex affine selection* if φ is affine, $\varphi(\alpha k)=\alpha\varphi(k)$ and $r(\varphi(k))=k$; $k \in K$, $\alpha \in \mathsf{T}$. φ is called T-*homogeneous* if $\varphi(k)=\hom_{\mathsf{T}}\varphi(k)$, $k \in K$.

THEOREM 6. *V is a Lindenstrauss space if and only if there is a complex affine selection on K. Moreover, if a complex affine selection exists, then there is a unique T-homogeneous complex affine selection φ on K and $\varphi(k)$ is maximal for all $k \in K$.*

PROOF. *Necessity.* Put $\varphi(x)=\hom_{\mathsf{T}}\nu_x$, where ν_x is a maximal measure in $M_1^+(K)$ with $r(\nu_x)=x$. φ is well-defined by Effros' characterization, and it follows from his proof that φ is a complex affine selection.

Sufficiency. Assume that $\varphi: K \to M_1(K)$ is a complex affine selection. Let $\bar\varphi: V^* \to M(K)$ be defined by $\bar\varphi(k)=\|k\|\varphi(k/\|k\|)$. Then $\bar\varphi$ is complex linear and extends φ so $\|\bar\varphi\| \leqq 1$. Since r is a norm-decreasing projection, we get

$$\|k\| = \|r(\bar\varphi(k))\| \leqq \|\varphi(k)\| \leqq \|k\|, \quad k \in K .$$

Hence $\bar\varphi$ is an isometry. Let now $\pi: M(K) \to \bar\varphi(V^*)$ be defined by $\pi(\mu)=\bar\varphi(r(\mu))$. Then π is a projection onto $\bar\varphi(V^*)$ with norm one, and since $M(K)$ is an L-space it follows from theorem 4 that $\bar\varphi(V^*)$ is an L-space. Hence V^* is an L-space, which implies that V is a Lindenstrauss space.

Uniqueness. Let $x \in K$ with $\|x\|=1$. From Lemma 1 it follows that

$$1 = \|x\| = \|r(\omega(|\varphi(x)|))\| \leqq \|\omega(|\varphi(x)|)\| \leqq \|\varphi(x)\| \leqq 1 ,$$

so $\omega(|\varphi(x)|) \in M_1^+(K)$.

Let $\nu_x \in M_1^+(K)$ with $r(\nu_x)=x$, let $f: K \to \mathsf{R}$ be continuous and convex, and $\varepsilon > 0$. Choose a simple probability measure $\sum_{i=1}^n \alpha_i \varepsilon_{y_i}$ such that (by [1, proposition I.2.3])

$$(2.1) \qquad \nu_x(f) \leqq (\textstyle\sum_{i=1}^n \alpha_i \varepsilon_{y_i})(f)+\varepsilon, \quad \textstyle\sum_{i=1}^n \alpha_i y_i = x .$$

Since φ is affine, we get $\varphi(x)=\sum_{i=1}^n \alpha_i \varphi(y_i)$. Moreover

$$1 = \|\varphi(x)\| \leqq \textstyle\sum_{i=1}^n \alpha_i \|\varphi(y_i)\| \leqq \textstyle\sum_{i=1}^n \alpha_i = 1 ,$$

so by corollary 3

$$\omega(|\varphi(x)|) = \textstyle\sum_{i=1}^n \alpha_i \omega(|\varphi(y_i)|) .$$

Now by lemma 1

$$\textstyle\sum_{i=1}^n \alpha_i \varepsilon_{y_i} \prec \textstyle\sum_{i=1}^n \alpha_i \omega(|\varphi(y_i)|) .$$

Since f is convex, we get from (2.1):

$$\nu_x(f) \leqq [\textstyle\sum_{i=1}^n \alpha_i \omega(|\varphi(y_i)|)](f)+\varepsilon = [\omega(|\varphi(x)|)](f)+\varepsilon .$$

Hence $\omega(|\varphi(x)|)$ is maximal and it is *the only maximal probability measure with barycenter* x. By lemma 1, $\hom_{\mathsf{T}} \omega(|\varphi(x)|)$ is maximal. But if φ is T-homogeneous, we get from lemma 1

$$\hom_{\mathsf{T}} \omega(|\varphi(x)|) = \hom_{\mathsf{T}} \varphi(x) = \varphi(x) .$$

The theorem now follows from the relation

$$\varphi(x) = \|x\| \varphi(x/\|x\|), \quad x \in K .$$

The proof above also shows

Corollary 7. *If V is a Lindenstrauss space, then every $k \in K$ with norm one can be represented by a unique maximal probability measure.*

Now by [1, theorem II.3.6]

Corollary 8. *If V is a Lindenstrauss space and F is a w^*-closed face in K, then F is a compact simplex.*

Remark. The above corollary may of course be proved by a direct argument, since a face-cone in an L-space must be a lattice-cone.

Theorem 9. *The following statements are equivalent*

i) *V is a Lindenstrauss space such that $\partial_e K \cup \{0\}$ is w^*-closed.*
ii) *There exists a continuous complex affine selection $\varphi\colon K \to M_1(K)$.*
iii) *For each $f \in C_{\mathrm{hom}}(K)$ there exists $v \in V$ such that $f \mid \partial_e K = v \mid \partial_e K$.*

Proof. i) $\Rightarrow$ ii). Put $\varphi(x)=\hom_{\mathsf{T}} \mu_x$, where μ_x is a maximal probability measure with $r(\mu_x)=x$. Then, as in the proof of theorem 6, φ is a complex affine T-homogeneous selection. We first prove that $\varphi(K)$ is compact. Let $\{\mu_\gamma\} \subset \varphi(K)$ be a net which converges to $\mu \in M_1(K)$. Let $f \in C_{\mathsf{C}}(K)$. Then, since each μ_γ is T-homogeneous:

$$\begin{aligned}\mu(f) &= \lim \mu_\gamma(f) = \lim [\hom_{\mathsf{T}} \mu_\gamma](f) \\ &= \lim \mu_\gamma(\hom_{\mathsf{T}} f) = \mu(\hom_{\mathsf{T}} f) = \hom_{\mathsf{T}} \mu(f) ,\end{aligned}$$

which proves that μ is T-homogeneous. By lemma 1, each μ_γ is maximal, and since $\partial_e K \cup \{0\}$ is closed it follows from [1] that

$$\operatorname{supp}(\mu) \subseteq \partial_e K \cup \{0\} .$$

But since μ is T-homogeneous, $\mu(\{0\}) = 0$, hence μ is maximal (by [1, proposition I.4.5]). Let $k \in \partial_e K$. Then by lemma 1 the measure

$$\nu = \omega(|\mu|) + \tfrac{1}{2}(1 - \|\omega(|\mu|)\|)(\varepsilon_k + \varepsilon_{-k})$$

is a maximal probability measure. Since μ is T-homogeneous, we get by lemma 1

$$\varphi(r(\nu)) = \hom_{\mathsf{T}} \nu = \hom_{\mathsf{T}}(\omega(|\mu|)) = \hom_{\mathsf{T}} \mu = \mu .$$

Thus $\mu \in \varphi(K)$, which implies that $\varphi(K)$ is compact. The map $\mu \mapsto r(\mu)$ is $1-1$ and continuous from the compact set $\varphi(K)$ onto K, thus the inverse map is continuous, i.e. φ is continuous.

ii) $\Rightarrow$ iii). If φ is a complex affine continuous selection on K, then so is $\hom_{\mathsf{T}} \circ \varphi$. Hence we may assume that φ is T-homogeneous. By ii) the map $x \mapsto [\varphi(x)](f)$, $x \in K$, is continuous, affine and T-homogeneous for all $f \in C_{\mathsf{C}}(K)$. But if f is T-homogeneous, it follows from theorem 6, Effros' characterization and [1, corollary I.2.4]

$$f(x) = [\varphi(x)](f) \quad \text{for all } x \in \partial_e K .$$

iii) $\Rightarrow$ i). When $f \in C_{\hom}(K)$, then by iii) and Bauer's Maximum Principle [1, theorem I.5.3] there is a unique function v_f in V such that

$$f | \partial_e K = v_f | \partial_e K \quad \text{and} \quad \|f\| \geqq \|v_f\| . \tag{2.2}$$

Assume $\mu, \nu \in M_1{}^+(\partial_e K)$ with $r(\mu) = r(\nu) = k$. Let $f \in C_{\hom}(K)$. Then by (2.2)

$$\mu(f) = \mu(v_f) = v_f(k) = \nu(v_f) = \nu(f) .$$

Hence $\hom_{\mathsf{T}} \nu = \hom_{\mathsf{T}} \mu$, so by Effros' characterization, V is a Lindenstrauss space. It remains to prove that $\partial_e K \cup \{0\}$ is closed. By (2.2) it suffices to prove

$$\partial_e K \cup \{0\} = \bigcap_{f \in C_{\hom}(K)} \{x \in K \mid f(x) = v_f(x)\} \tag{2.3}$$

a) Assume $x \in K$ and $\|x\| < 1$. Let

$$g \colon \bigcup \{\alpha x \mid \alpha \in T\} \to \mathsf{C}$$

be defined by $g(\alpha x) = \alpha$. Then g is continuous. By Tietze's theorem, we can extend g to $\tilde{g} \colon K \to \mathsf{C}$ with $\|\tilde{g}\| = \|g\|$. Put $f = \hom_{\mathsf{T}} \tilde{g}$. Then $f(x) = 1$ and $\|f\| = 1$. Hence

$$f(x) = 1 = \|f\| \geqq \|v_f\| \geqq |v_f(x/\|x\|)\| \\ = \|x\|^{-1} |v_f(x)| > |v_f(x)| .$$

b) Assume $x \in K$ with $\|x\| = 1$ and that there is no $v \in V$ such that $\|v\| = 1$ and $v(x) = 1$. Construct f as above. Then $f(x) = 1 \neq v_f(x)$.

c) Assume $x \in K$, $\|x\| = 1$, $x \notin \partial_e K$ and that there is $v \in V$ such that $v(x) = 1 = \|v\|$. Then

$$F = \{y \in K \mid v(y) = 1\}$$

is a w^*-closed face in K. Since $x \notin \partial_e K$ there are $y, z \in F$ such that $x = \frac{1}{2}y + \frac{1}{2}z$, $y, z \neq x$. By the Hahn-Banach theorem, there is a real convex continuous function g_F on F such that

$$g_F(y) = g_F(z) = 1, \quad g_F(x) = 0 .$$

Define g on $\bigcup_{\alpha \in T} \alpha F$ by $g(\alpha k) = \alpha g_F(k)$, $\alpha \in T$, $k \in F$. The function g is well defined since F is a face. Extend g to $\tilde{g} \in C_{\mathsf{C}}(K)$ by Tietze's theorem with $\|\tilde{g}\| = \|g\|$ and put $f = \hom_{\mathsf{T}} \tilde{g}$. Then $f | F = g_F$. Let μ_x be a maximal probability measure on K with $r(\mu_x) = x$. Since F is a face, $\operatorname{supp}(\mu_x) \subseteqq F$ and μ_x is seen to be maximal on F. Hence

$$v_f(x) = \int_K v_f \, d\mu_x = \int_F v_f \, d\mu_x = \int_F g_F \, d\mu_x .$$

By corollary 8, F is a simplex so [1, theorem II.3.7] gives

$$v_f(x) = \int_F g_F \, d\mu_x = \hat{g}_F(x) = \tfrac{1}{2}(\hat{g}_F(y) + \hat{g}_F(z)) \\ \geqq 1 > 0 = f(x) .$$

($\hat{g}_F$ denotes the upper envelope of g_F, see [1, p. 4].) (2.3) now follows from a), b) and c) and the proof is complete.

NOTES. Theorem 6 was proved for simplexes by Namioka and Phelps, and for real Lindenstrauss spaces by Ka-Sing Lau [18] and independently by Lacey [25], and by Fakhoury in a weaker form [24]. However, as pointed out to us by Hirsberg, there exists a very simple proof in the simplex-case, and it is this idea we have used in the uniqueness-part. Ka-Sing Lau [18] also proved theorem 9 in the real case. We have proceeded in the same way, but the proof is somewhat simplified.

3. Complex C_σ-spaces.

A compact Hausdorff space X is called a T_σ-space if there exists a map $\sigma\colon \mathsf{T}\times X \to X$ such that

i) σ is continuous,
ii) $\sigma(\alpha,\sigma(\beta,x))=\sigma(\alpha\beta,x), \quad \alpha,\beta\in\mathsf{T},\ x\in X,$
iii) $\sigma(1,x)=x.$

Let X be a T_σ-space. Then each $\alpha\in\mathsf{T}$ defines a homeomorphism $\sigma_\alpha\colon X\to X$ where $\sigma_\alpha(x)=\sigma(\alpha,x)$, $x\in X$ (σ_α and $\sigma_{\alpha^{-1}}$ are continuous by i), and ii) and iii) imply that $\sigma_\alpha\circ\sigma_{\alpha^{-1}}$ is the identity on X). A function $f\in C_{\mathsf{C}}(x)$ is said to be *σ-homogeneous* if $f(\sigma_\alpha x)=\alpha f(_x)$ for all $\alpha\in\mathsf{T}$, $x\in X$. The class of σ-homogeneous function in $C_{\mathsf{C}}(X)$ is denoted by $C_\sigma(X)$.

A complex C_σ-*space* is a complex Banach space which is isometric to $C_\sigma(X)$ for some T_σ-space X. If $f\in C_{\mathsf{C}}(X)$, then the function

$$[\pi_\sigma f](p) = \int \alpha^{-1} f(\sigma_\alpha p)\,d\alpha, \quad p\in X\,, \tag{3.1}$$

where $d\alpha$ is the unit Haar measure, is seen to be continuous and σ-homogeneous. The operator π_σ is easily shown to be a normdecreasing projection of $C_{\mathsf{C}}(X)$ onto $C_\sigma(X)$. Hence, by corollary 5, complex C_σ-spaces are Lindenstrauss spaces.

When Y is a locally compact Hausdorff space, then $C_0(Y)$ shall denote the space of all continuous functions on Y vanishing at infinity.

PROPOSITION 10. *If Y is a locally-compact Hausdorff space, then $C_0(Y)$ is a C_σ-space.*

PROOF. Let $X=(\mathsf{T}\times Y)\cup\{\omega\}$ be the one point compactification of $\mathsf{T}\times Y$, and define $\sigma\colon\mathsf{T}\times X\to X$ by

$$\sigma(\alpha,x) = \begin{cases}(\alpha\alpha_0,y) & \text{if } x = (\alpha_0,y)\in\mathsf{T}\times Y\,,\\ \omega & \text{if } x = \omega\,.\end{cases}$$

i). σ is easily seen to be continuous.
ii). Let $x=(\alpha_0,y)\in\mathsf{T}\times Y$, $\beta\in\mathsf{T}$. Then

$$\begin{aligned}\sigma(\alpha,\sigma(\beta,x)) &= \sigma\big(\alpha,\sigma(\beta,(\alpha_0,y))\big) = \sigma(\alpha,(\beta\alpha_0,y))\\ &= \sigma(\alpha\beta\alpha_0,y) = \sigma(\alpha\beta,(\alpha_0,y)) = \sigma(\alpha\beta,x)\,.\end{aligned}$$

Moreover

$$\sigma(\alpha,\sigma(\beta,\omega)) = \sigma(\alpha,\omega) = \omega = \sigma(\alpha\beta,\omega)\,.$$

iii) is verified in a similar way as ii).

Hence X is a T_σ-space. Each $f \in C_0(Y)$ can in a canonical way be regarded as a continuous function on $(\{1\} \times Y) \cup \{\omega\}$ vanishing at ω. Extend f to $\tilde{f}$ on X by $\tilde{f}(\alpha,y) = \alpha f(y)$, $(\alpha,y) \in \mathsf{T} \times Y$. Then $\tilde{f}$ is continuous and σ-homogeneous. The map $f \mapsto \tilde{f}$ defined above is seen to be an isometry of $C_0(Y)$ into $C_\sigma(X)$. Since each $g \in C_\sigma(X)$ satisfies $g(\omega) = 0$, the above map is surjective, i.e. $C_0(Y)$ is a C_σ-space.

Let now X be a T_σ-space and $V = C_\sigma(X)$. A subset $Z \subset X$ is called *σ-symmetric* if $x \in Z$ implies $\sigma_\alpha(x) \in Z$ for all $\alpha \in T$. Observe that if Z is σ-symmetric, then $X \setminus Z$ is σ-symmetric as well. Let ϱ embed X into K in the canonical way. Then ϱ is continuous, and we have

LEMMA 11.

$$\partial_e K = \{\varrho(x) \mid \sigma_\alpha(x) \neq x \text{ for all } \alpha \in \mathsf{T} \setminus \{1\},\ x \in X\}$$

and $\varrho(X) \subseteq \partial_e K \cup \{0\}$.

PROOF. First we observe that $\alpha\varrho(x) = \varrho(\sigma_\alpha x)$ when $\alpha \in \mathsf{T}$, $x \in X$ and we note that $\varrho(x) = 0$ if $\sigma_\alpha(x) = x$ for some $\alpha \in \mathsf{T} \setminus \{1\}$. Hence by [5, p. 441 lemma 6]

$$\partial_e K \subseteq \{\varrho(x) \mid \sigma_\alpha(x) \neq x \text{ for all } \alpha \in \mathsf{T} \setminus \{1\},\ x \in X\}.$$

Let $x \in X$ and assume $\sigma_\alpha(x) \neq x$ for all $\alpha \in \mathsf{T} \setminus \{1\}$. We shall prove that $\varrho(x) \in \partial_e K$. We use a σ-symmetric modification of the argument given in [5, proof of Lemma 6]. Assume

$$\varrho(x) = \tfrac{1}{2}k_1 + \tfrac{1}{2}k_2, \quad k_1, k_2 \in K. \tag{3.2}$$

Let $f_0 \in C_\sigma(X)$ with $\|f\| \leqq 1$ and assume that f_0 vanishes on an open neighbourhood $N(x)$ of x. Since f_0 is σ-homogeneous, we may assume that $N(x)$ is σ-symmetric. Let

$$h\colon \{\sigma_\alpha(x) \mid \alpha \in \mathsf{T}\} \cup \{X \setminus N(x)\} \to \mathsf{C}$$

be defined by $h(\sigma_\alpha x) = \alpha$, $\alpha \in \mathsf{T}$ and $h(y) = 0$ if $y \in X \setminus N(x)$. Extend h by Tietze's theorem to $\tilde{h}$ on X with $\|\tilde{h}\| = \|h\|$ and put $g = \pi_\sigma(\tilde{h})$. Then

$$g(x) = 1,\ g(y) = 0 \text{ if } y \notin N(x) \text{ and } \|g\| \leqq 1.$$

Thus by (3.2)

$$\begin{aligned} 1 = g(x) &= \varrho(x)(g) \\ &= \tfrac{1}{2}\big(k_1(g) + k_2(g)\big) \leqq \tfrac{1}{2}\big(|k_2(g)| + |k_2(g)|\big) \leqq 1. \end{aligned}$$

Hence $k_1(g) = k_2(g) = 1$. Similarly we get $k_1(g + f_0) = k_2(g + f_0) = 1$. Hence

$$k_1(f_0) = k_2(f_0) = 0. \tag{3.3}$$

Let $f_1 \in C_\sigma(X)$ with $\|f\| \leqq 1$ and $f(x) = 0$. For each integer $n \geqq 2$ there is an open σ-symmetric neighbourhood $N_n(x)$ of x such that $|f_1(y)| \leqq 1/n$ if $y \in N_n(x)$.

Let $M_n(x)$ be an open set containing x such that

$$M_n(x) \subseteqq \bar{M}_n(x) \subseteqq N_n(x) .$$

Since $N_n(x)$ is σ-symmetric, we get

$$\bigcup_{\alpha \in \mathsf{T}} \sigma_\alpha(M_n(x)) \subseteqq \bigcup_{\alpha \in \mathsf{T}} \sigma_\alpha \bar{M}_n(x) \subseteqq N_n(x) ,$$

and note that

$$\bigcup_{\alpha \in \mathsf{T}} \sigma_\alpha \bar{M}_n(x) = \sigma(\mathsf{T} \times \bar{M}_n(x))$$

is closed. Thus as above we may construct $g_n \in C_\sigma^{\mathsf{T}}(X)$ such that $\|g_n\| \leqq 1/n$, $g_n(y) = 0$ if $y \notin N_n(x)$, and $g_n(y) = f_1(y)$ if $y \in \sigma(\mathsf{T} \times \bar{M}_n(x))$. Then $f_1 - g_n \to f_1$ uniformly and $\|f_1 - g_n\| \leqq 1$.

Now since $f_1 - g_n$ vanishes on $\sigma(\mathsf{T} \times M_n(x))$, we get by (3.3):

$$0 = \lim k_1(f_1 - g_n) = k_1(f_1)$$

$$0 = \lim k_2(f_1 - g_n) = k_2(f_1) .$$

Hence $\varrho(x)(f) = 0$ implies $k_1(f) = k_2(f) = 0$, $f \in C_\sigma(X)$. By [5, lemma 3.10] there are $\alpha_1, \alpha_2 \in \mathsf{C}$ such that $k_1 = \alpha_1 \varrho(x)$, $k_2 = \alpha_2 \varrho(x)$. But $\|k_1\|, \|k_2\| \leqq 1$, so $|\alpha_1|, |\alpha_2| \leqq 1$ and by (3.2) we get $\alpha_1 = \alpha_2 = 1$, that is, $\varrho(x) = k_1 = k_2$.

THEOREM 12. *V is a C_σ-space if and only if V is Lindenstrauss space and $\partial_e K \cup \{0\}$ is closed.*

PROOF. If V is a C_σ-space, then V is a Lindenstrauss space and $\partial_e K \cup \{0\}$ is closed by virtue of lemma 11. Conversely, assume that V is a Lindenstrauss space with $X = \partial_e K \cup \{0\}$ closed. X can be organized to be a T_σ-space by scalar multiplication. Then theorem 9 iii) completes the proof.

A complex C_Σ-space is a Banach space which is isometric to a $C_\sigma(X)$ for some T_σ-space X, where σ_α has no fixed points if $\alpha \in \mathsf{T} \setminus \{1\}$. Now as in the proof of proposition 10 we get

PROPOSITION 13. *If X is a compact Hausdorff-space, then $C_{\mathsf{C}}(X)$ is a C_Σ-space.*

The next theorem may be proved by a method similar to that used in proving theorem 12.

THEOREM 14. *V is a C_Σ space if and only if V is a Lindenstrauss space and $\partial_e K$ is closed.*

REMARK. Theorem 14 also proves proposition 13, just as theorem 12 proves proposition 10, by virtue of [5, p. 441 lemma 6].

NOTES. The real C_σ-spaces were introduced and studied by Jerison [16]. His results are presented in Day's book [4, p. 87–93]. The real version of theorem 12 was suggested by Effros [7], and proved by Fakhoury [9] and independently by Ka-Sing Lau [18]. Theorem 14 is due to Lindenstrauss and Wulbert. We have proceeded as in [18].

4. Complex simplex spaces.

Let $(Q,\mathscr{B},m)$ be a measure space and assume $V^* = L_{\mathrm{C}}^1(Q,\mathscr{B},m)$. Let $\varphi \in L_{\mathrm{C}}^\infty(Q,\mathscr{B},m)$ with $|\varphi| = 1$ a.e. m. Then

$$S = \{\varphi\cdot p \mid p \in K,\ p \geqq 0 \text{ a.e. } m,\ \|p\| = 1\} \tag{4.1}$$

is seen to be a maximal (with respect to inclusion) face in K. Conversely, since the norm must be additive on a face-cone [2], we get that all maximal faces in K are of the form given in (4.1). If $p \in \partial_e K$, then it is not hard to see that $p = a\chi_A$, where $a \in \mathrm{C}$ and χ_A is the characteristic function of an atom $A \in \mathscr{B}$. Thus if S is a maximal face in K and $p \in \partial_e K$, then $\alpha p \in S$ for some $\alpha \in T$. Hence

$$V \cong V|S\,. \tag{4.2}$$

A complex Lindenstrauss space V is called a *complex simplex-space* if there is a maximal face $S \subset K$ such that $\mathrm{conv}(S \cup \{0\})$ is w^*-closed. (Observe that this definition coincides with Effros' in the real case [6], see [9, théorème 18]).

We shall need the notion of split face which is defined in [1, p. 133]

LEMMA 15. *S is a split-face in* $\mathrm{conv}(S \cup -iS)$.

PROOF. Assume

$$\lambda_1 x_1 + (1-\lambda_1)(-ix_2) = \lambda_2 y_1 + (1-\lambda_2)(-iy_2)\,,$$

where $x_i, y_i \in S$, $0 \leqq \lambda_1 \leqq 1$, $i = 1,2$. Since S is a maximal face in K, there is $\varphi \in V^{**}$ such that $\varphi|S \equiv 1$. Thus $\lambda_1 = \lambda_2 = \lambda$. Let $\mu_i, \nu_i \in M_1^+(\partial_e K)$, $i = 1,2$, with

$$r(\mu_1) = x_1, \quad r(\mu_2) = -ix_2, \quad r(\nu_1) = y_1, \quad r(\nu_2) = -iy_2\,.$$

Since S is a face and $S_0 = S \cup \{0\}$ is w^*-compact, we get

$$(4.3) \qquad \operatorname{supp}(\mu_1),\ \operatorname{supp}(\nu_1) \subseteq S_0\ ,$$
$$\operatorname{supp}(\mu_2),\ \operatorname{supp}(\nu_2) \subseteq -iS_0\ .$$

Since the barycenter-map is norm decreasing, we also get

$$(4.4) \qquad \mu_i(\{0\}) = \nu_i(\{0\}) = 0, \quad i=1,2\ .$$

Let now $f \in C_{\mathsf{R}}(S_0)$ with $f(0)=0$. Extend f to a T-homogeneous function $\tilde{f}$ on K. By Effros' characterization we get

$$\lambda\mu_1(\tilde{f}) + (1-\lambda)\mu_2(\tilde{f}) = \lambda\nu_1(\tilde{f}) + (1-\lambda)\nu_2(\tilde{f})\ .$$

But $\tilde{f}$ is real on S_0 and imaginary on $-iS_0$, so by (4.3) $\mu_1(f) = \nu_1(f)$. But by (4.4) this holds for any $f \in C_{\mathsf{R}}(S_0)$. Hence $\nu_1 = \mu_1$, which gives $x_1 = y_1$ and the proof is complete.

Corollary 16. *Any $z \in Z_0 = \operatorname{conv}(S \cup -iS \cup \{0\})$ may be written uniquely in the form*

$$z = \alpha_1 x_1 + \alpha_2(-ix_2) + \alpha_3 \cdot 0$$

where $\alpha_i \geqq 0$, $i=1,2,3$, $\alpha_1+\alpha_2+\alpha_3 = 1$, $x_1, x_2 \in S$.

Lemma 17. *Let a be a real, affine, w^*-continuous function on $S_0 = \operatorname{conv}(S \cup \{0\})$ with $a(0)=0$. Then a may be extended to a real, affine, w^*-continuous function c on Z_0 such that $c \mid -iS_0 \equiv 0$.*

Proof. Let $c: Z_0 \to \mathsf{R}$ be defined by $c(z) = \alpha_1(x_1)$, $z \in Z_0$, where $z = \alpha_1 x_1 + \alpha_2(-ix_2) + \alpha_3 \cdot 0$ is the unique decomposition as in corollary 16. c is easily verified to be affine. To see that c is continuous, let $\{z^\gamma\} \subseteq Z_0$ be a net converging to $z \in Z_0$. Applying corollary 16, we get

$$z^\gamma = \alpha_1{}^\gamma x_1{}^\gamma + \alpha_2{}^\gamma(-ix_2{}^\alpha) + \alpha_3{}^\gamma 0\ ,$$
$$z = \alpha_1 x_1 + \alpha_2(-ix_2) + \alpha_3 0\ .$$

By compactness, we may assume that the nets $\{x_1{}^\gamma\}$, $\{x_2{}^\gamma\}$, $\{\alpha_1{}^\gamma\}$, $\{\alpha_2{}^\gamma\}$ are convergent. Let y_1, y_2, β_1, β_2 be the limitpoints. Then

$$z^\gamma \to \beta_1 y_1 + \beta_2(-iy_2) = \beta_1\|y_1\|(y_1/\|y_1\|) + \beta_2\|y_2\|(-iy_2/\|y_2\|) + \beta' \cdot 0$$

where $\beta' = 1 - (\|y_1\|\beta_1 + \|y_2\|\beta_2)$. (The case $\|y_1\| = 0$ or $\|y_2\| = 0$ can be treated similarly). Now, since the decomposition in corollary 16 is unique, we get

$$\alpha_1 = \beta_1\|y_1\|, \quad x_1 = y_1/\|y_1\|\ .$$

Hence

$$c(z^\gamma) = \alpha_1^\gamma a(x_1^\gamma) \to \beta_1 a(y_1)$$
$$= \beta_1 \|y_1\| a(y/\|y_1\|) = \alpha_1 a(x_1) = c(z) ,$$

which proves that c is continuous. Since c extends a and $c \mid -iS_0 \equiv 0$, the proof is complete.

When H is a compact convex set, $A(H)$ ($A_0(H)$) will denote the space of complex affine continuous functions on H (vanishing at a fixed extreme point x_0 in H).

THEOREM 18. *The following statements are equivalent.*

i) V *is a simplex-space.*

ii) $V \cong A_0(S_0)$ *for some simplex* S_0.

iii) $V \cong A$, *where* A *is a closed self-adjoint linear subspace of* $C_{\mathbf{C}}(X)$, *with* X *a compact Hausdorff space and where* $\operatorname{Re} A$ *is a real simplex space.*

PROOF. i) $\Rightarrow$ ii). Assume that V is a Lindenstrauss space with a maximal face $S \subset K$ such that $S_0 = \operatorname{conv}(S \cup \{0\})$ is w^*-compact. We have by (4.2)

$$V \cong V \mid S_0 \subseteq A_0(S_0), \quad (x_0 = 0) . \tag{4.5}$$

Let $a \in A_0(S_0)$ and put $b_1 = \operatorname{Re} a$, $b_2 = \operatorname{Im} a$. Then b_1, b_2 are real affine w^*-continuous functions on S_0 with $b_1(0) = b_2(0) = 0$, and may therefore, by corollary 17, be extended to affine w^*-continuous functions $\tilde{b}_1$, $\tilde{b}_2$ on Z_0 such that

$$\tilde{b}_1 \mid -iS_0 = 0, \quad b_2 \mid -iS_0 = 0 . \tag{4.6}$$

By [1, corollary I.1.5] there are sequences $\{b_1^n\}$, $\{b_2^n\}$ of w^*continuous real linear functionals on V^* such that $b_1^n \to b_1$, $b_2^n \to b_2$ uniformly on Z_0.

Let $a_1^n, a_2^n \in V$, $n = 1, 2, \ldots$, be defined by

$$a_1^n(x) = b_1^n(x) - i b_1^n(ix), \quad x \in V^* ,$$
$$a_2^n(x) = b_2^n(x) - i b_2^n(ix), \quad x \in V^* .$$

Then, by (4.2) and (4.6), $a_1^n + i a_2^n$ converges to an element $c \in V$ satisfying $c \mid S_0 = a$. The set S_0 is, by [1, theorem II.3.6] and corollary 7, a simplex, so the proof of ii) is complete.

ii) $\Rightarrow$ iii) is trivial.

iii) $\Rightarrow$ ii). Let $p \in (\operatorname{Re} A)^*$ and put

$$\tilde{p}(a) = p(\operatorname{Re} a) + i p(\operatorname{Im} a) .$$

Then $\tilde{p} \in A^*$ with $\|\tilde{p}\| = \|p\|$ and p has only this extension in A^*, so we may regard $(\mathrm{Re} A)^*$ as a subset of A^*. Let

$$S_0 = \{p \in A^* \mid \|p\| \leqq 1,\ p(a) \geqq 0 \text{ for all } a \in [\mathrm{Re} A]^+\}$$

and let $\psi\colon A \to A_0(S_0)$ be defined by

$$[\psi(a)](p) = p(a), \quad p \in S_0,\ a \in A .$$

Then ψ is an isometry since S_0 contains the evaluations. Theorem 2.2 in [6] implies that ψ is onto and that S_0 is a simplex.

ii) $\Rightarrow$ i). By Hirsberg's version of Hustad's theorem ([11] and [13]) each $p \in A(S_0)^*$ may be represented by a measure $\mu \in M(\partial_e S_0)$ such that $\|\mu\| = \|p\|$. Moreover, since S_0 is a simplex, this representation is unique. Hence

$$A(S_0)^* \cong M(\partial_e S_0) \tag{4.7}$$

and the latter is proved in [8, proof of theorem 4.3] to be an L-space. Let

$$S = \bigcup \{F \mid F \text{ a face in } S_0,\ F \cap \{x_0\} = \emptyset\} .$$

Then S is a G_δ set [1, proposition II.6.5]. Let $e\colon M(\partial_e S_0) \to M(\partial_e S_0)$ be defined by $e(\mu)(C) = \mu(C \cap S)$, C a Borel set in S_0. Then e is seen to be an L-projection in the sense of [2]. We shall prove

$$e[M(\partial_e S_0)] \cong A_0(S_0)^* , \tag{4.8}$$

which implies that $A_0(S_0)$ is a Lindenstrauss space. Let $p \in A_0(S_0)^*$ and let $\tilde{p} \in A(S_0)^*$ be a norm preserving extension of p. By (4.7) there is a unique measure $\mu \in M(\partial_e S_0)$ which represents $\tilde{p}$ and satisfies $\|\mu\| = \|\tilde{p}\|$. Let $\varepsilon > 0$. Choose $a \in A_0(S_0)$ with $\|a\| \leqq 1$ such that $|p(a)| > \|p\| - \varepsilon$. Then

$$\begin{aligned} \|\mu\| - \varepsilon &= \|\tilde{p}\| - \varepsilon = \|p\| - \varepsilon < |p(a)| \\ &= |\textstyle\int_{S_0} a\,d\mu| = |\textstyle\int_S a\,d\mu| \leqq |\mu|(S) \\ &\leqq |\mu|(S) + |\mu|(\{x_0\}) = |\mu|(S_0) = \|\mu\| . \end{aligned}$$

Hence $\mu \in e[M(\partial_e S_0)]$. Let χ denote the characteristic function to x_0 and assume that $\mu \in e[M(\partial_e S_0)]$ annihilates $A_0(S_0)$. Let $\{a_\alpha\}$ be a net of real affine w^*-continuous functions on S_0 such that

$$a_\alpha \nearrow 1 - \hat{\chi} .$$

(See [1, corollary I.1.4, theorem II.6.18 and II.6.22].)

Let $\varepsilon > 0$. By [1, (2.3)] we may choose α such that

$$|\mu((1-\hat{\chi}) - a_\alpha) < \tfrac{1}{2}\varepsilon \quad \text{and} \quad |a_\alpha(0)| < \tfrac{1}{2}\varepsilon\|\mu\| .$$

Then

$$|\mu(1)| = |\int \chi_s d\mu| = |\int (1-\hat{\chi})d\mu|$$
$$< |\int a_\alpha d\mu| + \tfrac{1}{2}\varepsilon \leqq |\int (a_\alpha - a_\alpha(0))d\mu| +$$
$$+ |\int a_\alpha(0) d\mu| + \tfrac{1}{2}\varepsilon \leqq \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon .$$

Hence μ annihilates $A(S_0)$, so by (4.7) $\mu \equiv 0$. This proves (4.8). Let $\varrho: S_0 \to e(M(\partial_e S_0))$ be the canonical map. Then by [1, lemma II.6.10]

$$\varrho(S) = \{\mu \in M_1{}^+(\partial_e S_0) \mid \mu(S) = 1\} .$$

The polar decomposition gives that $\|\mu + |\mu|\| = \|\mu\| + \|\mu\|$ implies $\mu \geqq 0$, so we may conclude that $\varrho(S)$ is a maximal face of the unit ball in $e(M(\partial_e S_0))$. Since $\varrho(S_0) = \operatorname{conv}(\varrho(S) \cup \{0\})$ is compact, the proof is complete.

Let V be a Lindenstrauss space and assume that *e is an extreme point of the unit ball in V*. Put

$$S = \{p \in V^* \mid p(e) = 1 = \|p\|\} .$$

Then S is w^*-compact. Let $\psi: V \to C_{\mathbf{C}}(S)$ be the canonical embedding. Then the following theorem is proved in [12]:

THEOREM 19. (Hirsberg–Lazar.) *The map ψ is an isometry such that $\psi(e) = 1_S$.*

As in the proof of i) $\Rightarrow$ ii) in theorem 18 we now get

COROLLARY 20. *If V is Lindenstrauss space and the unit ball of V admits an extreme point, then $V \cong A(S)$ where S is a compact simplex.*

REMARK. As in the last part of the proof of ii) $\Rightarrow$ i) in theorem 18, we see that the unit ball in *a Lindenstrauss space V admits an extreme point if and only if there is a maximal w^*-closed face in K*. For more information about such Lindenstrauss spaces see [12].

A complex Banach space V is called a *complex M-space* if it can be represented as follows: There is a compact Hausdorff space X and a set $\mathscr{A}$ of triples $(x_a, y_a, \lambda_a) \in X \times X \times [0,1]$ such that V is the subspace of $C_{\mathbf{C}}(X)$ satisfying

$$f(x_a) = \lambda_a f(y_a), \quad a \in \mathscr{A},\ f \in V .$$

Clearly V is self-adjoint, and by [17] $\operatorname{Re} V$ is a Kakutani M-space.

Moreover, each self-adjoint linear subspace of $C_{\mathsf{C}}(X)$ whose real part is a Kakutani M-space arises in this way. Now, by theorem 18, *a complex M-space is a complex simplex-space.*

NOTES. The real simplex-spaces were introduced and studied by Effros in [6]. Our results are based on the ideas of [12]. Furthermore, our lemma 17 is closely related to [1, proposition II.6.19].

5. Complex G-spaces.

Let X be a compact Hausdorff space. A linear subspace $V \subseteqq C_{\mathsf{C}}(X)$ is called a complex G-space, if V consists of those $f \in C_{\mathsf{C}}(X)$ satisfying a family $\mathscr{A}$ of relations:

$$f(x_a) = \lambda_a \alpha_a f(y_a); \quad x_a, y_a \in X,\ \alpha_a \in \mathsf{T},\ \lambda_a \in [0,1],\ a \in \mathscr{A} .$$

Complex G-spaces are complex Lindenstrauss spaces by corollary 5 and by the following:

PROPOSITION 21. *If V is a G-space, then there is an M-space A such that $V \cong P(A)$ where $P: A \to A$ is a projection with $\|P\| \leqq 1$.*

PROOF. We adopt the notation in the definition. Let $Y = \mathsf{T} \times X$ be organized to a T_σ-space in the canonical way. (See the proof of Proposition 10). Let A be the closed subspace of $C_{\mathsf{C}}(Y)$ satisfying

$$F(\beta, x_a) = \lambda_a F(\alpha_a \beta, y_a), \quad a \in \mathscr{A},\ \beta \in \mathsf{T} .$$

Then A is a complex M-space. The map $T: V \to A$ defined by

$$[Tf](\alpha, x) = \alpha f(x), \quad (\alpha, x) \in \mathsf{T} \times X ,$$

is seen to be an isometry of V onto a linear subspace of A, since

$$[Tf](\beta, x_a) = \beta f(x_a) = \lambda_a \beta \cdot \alpha_a f(y_a) = \lambda_a [Tf](\beta \alpha_a, y_a), \quad a \in \mathscr{A},\ \beta \in \mathsf{T} .$$

If $F \in A$ is σ-homogeneous, then

$$F(1, x_a) = \lambda_a F(\alpha_a, y_a) = \lambda_a \alpha_a F(1, y_a), \quad a \in \mathscr{A} .$$

Hence T takes V onto the σ-homogeneous functions in A. Now the projection $P = \pi_\alpha | A$ will do. In fact, let $F \in A$, then

$$\begin{aligned} P(F)(\beta, x_a) &= \int \alpha^{-1} F(\alpha\beta, x_a)\, d\alpha = \int \alpha^{-1} \lambda_a F((\alpha_a \beta)\alpha, y_a)\, d\alpha \\ &= \lambda_a P(F)(\alpha_a \beta, y_a), \quad a \in \mathscr{A},\ \beta \in \mathsf{T} . \end{aligned}$$

LEMMA 22. *Assume that V is a Lindenstrauss space and let $E \subseteq \partial_e K$ be compact with $E \cap \alpha E = \emptyset$ whenever $\alpha \in \mathsf{T} \setminus \{1\}$. Then $F = \overline{\mathrm{conv}}(E)$ is a w^*-closed face in K.*

PROOF. By Milman's theorem [1, p. 50] we have

$$F = \{r(\mu) \mid \mu \in M_1^+(E)\} .$$

Observe that a measure $\mu \in M_1^+(E)$ is maximal on K.

Assume $k_1, k_2 \in K$, $\lambda \in (0,1]$ such that

$$k = \lambda k_1 + (1-\lambda) k_2 \in F .$$

Let $\mu \in M_1^+(E)$ with $r(\mu) = k$, $\mu_1, \mu_2 \in M_1^+(\partial_e K)$ with $r(\mu_1) = k_1$, $r(\mu_2) = k_2$. Put $E' = \bigcup_{\alpha \in \mathsf{T}} \alpha E$, let $\varepsilon > 0$ and choose a compact set C such that

$$C \cap E' = \emptyset ,$$

$$\mu_1(E' \cup C) \geqq 1 - \varepsilon, \quad \mu_2(E' \cup C) \geqq 1 - \varepsilon .$$

Let f be a T-homogeneous function on K such that $f | E = 1$, $f | \bigcup_{\alpha \in \mathsf{T}} \alpha C = 0$, $\|f\| \leqq 1$. By Effros' characterization we get

$$\begin{aligned} 1 &= \mu(f) = \lambda \mu_1(f) + (1-\lambda)\mu_2(f) \\ &\leqq \lambda \textstyle\int_{E'} 1\, d\mu_1 + (1-\lambda) \int_{E'} 1\, d\mu_2 + 2\varepsilon \leqq 1 + 2\varepsilon . \end{aligned}$$

Hence $\mu_1(E') = \mu_2(E') = 1$. Assume now $\mu_1(E) \neq 1$.

Let f be a T-homogeneous function on K with $f | E = 1$ and $\|f\| \leqq 1$. Put $E' = \bigcup_{\alpha \in \mathsf{T} \setminus \{1\}} \alpha E$. By Effros' characterization we get

$$\begin{aligned} 1 &= \mu(\mathrm{Re} f) = \lambda \mu_1(\mathrm{Re} f) + (1-\lambda)\mu_2(\mathrm{Re} f) \\ &= \lambda \textstyle\int_E \mathrm{Re} f \, d\mu_1 + \lambda \int_{E'} \mathrm{Re} f \, d\mu_1 + (1-\lambda)\mu_2(\mathrm{Re} f) \\ &< \lambda \textstyle\int_E 1\, d\mu_1 + \lambda \int_{E'} 1\, d\mu_1 + (1-\lambda)\mu_2(1) = 1 , \end{aligned}$$

which is a contradiction. Hence $\mu_1(E) = 1$, which implies $k_1 \in F$ and the proof is complete.

Let $V \subseteq C_{\mathsf{C}}(X)$ be a G-space. Put

$$Z = \{x \in X \mid \exists (y, \lambda, \alpha) \in X \times [0, 1\rangle \times \mathsf{T} \text{ such that } f(x) = \lambda \alpha f(y) \text{ for all } f \in V\} .$$

Let $\delta : X \to K$ be the canonical map. Then we have

LEMMA 23.

$$\partial_e K = \bigcup_{\alpha \in \mathsf{T}} \alpha \delta(X \setminus Z) .$$

PROOF. We use the same notations as in the proof of proposition 21, and when W is a Banach space, then $B(W)$ will denote the unit ball. Clearly no point in $\delta(Z)$ is extreme, so by [5, p. 441 lemma 6] we have

$$\partial_e K \subseteqq \bigcup_{\alpha\in\mathsf{T}} \alpha\delta(X\setminus Z) .$$

To prove the converse inclusion, let $x_0 \in X\setminus Z$, $g \in A$. Then

$$\begin{aligned} P^*(\delta(1,x_0))(g) &= \delta(1,x_0)(P(g)) = \int \alpha^{-1} g(\alpha,x_0)\, d\alpha \\ &= \int \alpha^{-1}\delta(\alpha,x_0)(g)\, d\alpha . \end{aligned}$$

Hence

$$(5.1) \qquad P^*(\delta(1,x_0)) = \int \alpha^{-1}\delta(\alpha,x_0)(\cdot)\, d\alpha .$$

Let

$$S_0 = \overline{\operatorname{conv}}(\{\delta(\alpha,x) \mid \alpha\in\mathsf{T},\ x\in X\}\cup\{0\}) .$$

Then S_0 is a simplex and $A \cong A_0(S_0)$ (see section 4). By the real theory [7, Remark 8.2], $\bigcup_{\alpha\in\mathsf{T}}\delta(\alpha,x_0)\cup\{0\}$ is a w^*-closed subset of $\partial_e S_0$. Let

$$f_0 \colon \bigcup_{\alpha\in\mathsf{T}} \delta(\alpha,x_0) \cup \{0\} \to \mathsf{C}$$

be defined by

$$f_0(\delta(\alpha,x)) = \alpha, \qquad \alpha\in\mathsf{T},$$
$$f_0(0) = 0 .$$

By [3, corollary 4.6], f_0 can be extended to an element of $A_0(S_0)$ with norm one. Thus there is an $f\in A$ such that $f(\alpha,x_0)=\alpha$, $\alpha\in\mathsf{T}$ and $\|f\|=1$. Let

$$E_1 = \bigcup_{\alpha\in\mathsf{T}} \alpha^{-1}\delta(\alpha,x_0), \qquad E_2 = \bigcup_{\alpha,\beta\in\mathsf{T}} \beta\delta(\alpha,x_0) .$$

Then $E_1, E_2 \subseteqq \partial_e B(A^*)$ by the real theory. Put $F=\overline{\operatorname{conv}}(E_1)$, $H=\overline{\operatorname{conv}}(E_2)$. It follows from Milman's theorem that $\partial_e F = E_1$ and $\partial_e H = E_2$. Moreover,

$$f^{-1}(1)\cap E_2 = E_1 .$$

Hence $f^{-1}(1)\cap H = F$. Assume $P^*(\delta(1,x_0)) \notin F$. Then $P^*(\delta(1,x_0)) \notin H$ and since H is circled it follows from the Hahn–Banach theorem that there is $g\in A$ such that

$$|P^*\delta(1,x_0)(g)| > 1, \qquad |\delta(\alpha,x_0)(g)| < 1, \qquad \alpha\in\mathsf{T};$$

which by (5.1) gives a contradiction. Thus $P^*(\delta(1,x_0) \in F$.

Assume

$$\delta(x_0) = \lambda k_1 + (1-\lambda)k_2, \qquad k_1,k_2\in K .$$

Since $B(P^*(A^*))$ and K are affinely homeomorphic, there corresponds unique

$$\tilde{k}_1 = P^*T^{*-1}(k_1), \qquad \tilde{k}_2 = P^*T^{*-1}(k_2)$$

such that

$$P^*(\delta(1,x_0)) = \lambda\tilde{k}_1+(1-\lambda)\tilde{k}_2 .$$

But F is, by lemma 23, a face in $B(A^*)$. Hence $\tilde{k}_1,\tilde{k}_2 \in F$. Since each $g \in P(A)$ is constant on F, we get

$$\tilde{k}_1(g) = \tilde{k}_2(g) = P^*(\delta(1,x_0))(g), \quad \text{for all } g \in P(A) .$$

Thus $\delta(x_0)=k_1=k_2$, that is, $\delta(x_0)$ is an extreme point.

THEOREM 24. *Let $V \subseteq C_{\mathbf{C}}(x)$ be a Lindenstrauss space. Then the following statements are equivalent.*

i) *V is a G-space,*

ii) $\overline{\partial_e K} \subseteq [0,1]\partial_e K$.

PROOF. Let $x \in Z$. Put

$$\lambda_0 = \inf\{\lambda \in [0,1\rangle \mid \exists y_\lambda \in X,\ \alpha_\lambda \in \mathrm{T},\ f(x)=\lambda\alpha_\lambda f(y_\lambda) \text{ for all } f \in V\} .$$

By compactness, we may without loss of generality assume that $(\lambda, y_\lambda, \alpha_\lambda)$ converges to $(\lambda_0, y_0, \alpha_0) \in [0,1\rangle \times X \times \mathrm{T}$. By continuity

$$f(x) = \lambda_0\alpha_0 f(y_0) \quad \text{for all } f \in V .$$

If $\lambda_0=0$, then $\delta(x)=0 \in [0,1]\partial_e K$. If $\lambda_0 \neq 0$, then $\delta(y_0) \in \partial_e K$, which gives

$$\delta(x) = \lambda_0(\alpha_0\delta(y_0)) \in [0,1]\partial_e K .$$

In fact, if $\delta(y_0) \notin \partial_e K$, then by lemma 23 there is $(\lambda,y,\alpha) \in [0,1\rangle \times X \times \mathrm{T}$ such that $f(y_0)=\lambda\alpha f(y)$ for all $f \in V$. This implies that

$$f(x) = \lambda_0\alpha_0 f(y_0) = (\lambda_0\lambda)(\alpha_0\alpha)f(y) \quad \text{for all } f \in V ,$$

contradicting the definition of λ_0. i) $\Rightarrow$ ii) follows now easily from lemma 23.

ii) $\Rightarrow$ i) Let $A \subseteq C_{\mathbf{C}}(\overline{\partial_e K})$ be the space of T-homogeneous functions f satisfying

$$f(k) = \|k\| f(k/\|k\|), \quad k \in \overline{\partial_e K} . \tag{5.2}$$

Then A is a G-space. We shall prove $A \cong V$. It is enough to prove $A \subseteq V|\overline{\partial_e K}$.

Let $f \in A$. Then $\mathrm{Re} f$ satisfies (5.2), and since V is a Lindenstrauss space and f is T-homogeneous, we have

$$\nu_1(\mathrm{Re} f) = \nu_2(\mathrm{Re} f) \tag{5.3}$$

whenever $\nu_1,\nu_2 \in M_1{}^+(\partial_e K)$ with $r(\nu_1)=r(\nu_2)$. Assume $k \in \overline{\partial_e K}$. Then

$$\nu = \tfrac{1}{2}(1+\|k\|)\delta_{k/\|k\|} + \tfrac{1}{2}(1-\|k\|)\delta_{-k/\|k\|}$$

is a maximal probability-measure with $r(\nu)=k$ by ii), and $\nu(\mathrm{Re} f)=\mathrm{Re} f(k)$. By (5.3) this holds for any maximal probability measure with barycenter k. Hence by [7, theorem 2.3] $\mathrm{Re} f$ may be extended to an affine real w^*-continuous function g on K with $g(0)=0$. Let $F: K \to \mathbf{C}$ be defined by $F(x)=g(x)+ig(-ix)$, $x \in K$. Then $F \in V$ and $F|\overline{\partial_e K}=f$.

Remark. The G-spaces include the M-spaces and it is readily verified that a C_σ-space is a G-space.

Notes. The real G-spaces were introduced by Grothendieck in [10]. Proposition 21 was announced in [22] in the real case, but, as pointed out to us by Jan Raeburn, the proof is incomplete. However, the same idea can be used to give a correct proof. Theorem 24 was proved by Effros in the separable, real case [7] and in general by Fakhoury [9]. It is on his ideas that we have based the proof of lemma 23, and the other part of theorem 24 is proved as in [7]. Lemma 22 was proved by Lazar for real Lindenstrauss space [19].

6. The classification scheme.

Summarizing the foregoing we get the diagram

$$\begin{array}{ccccc}
\hat{C}_\Sigma & \leftarrow & \hat{C}(X) & \rightarrow & \hat{A}(S) \\
\downarrow & & \downarrow & & | \\
\hat{C}_\sigma & \leftarrow & \hat{C}_0(Y) & & | \\
\downarrow & & \downarrow & & \downarrow \\
\hat{G} & \leftarrow & \hat{M} & \rightarrow & \hat{A}_0(S) \\
\downarrow & & & & \downarrow \\
\multicolumn{5}{c}{\{V \mid V^* \cong L_{\mathbf{C}}{}^1(Q,\mathscr{B},m)\}}
\end{array}$$

where $\hat{A}(S)$ denotes the class of Lindenstrauss spaces with extreme points on the unit ball, $\hat{A}_0(S)$ denotes the simplex-spaces, and so on. The symbol $A \to B$ means that the class A is included in B.

It is also possible to derive the intersections between the classes. In fact:

$$\hat{G} \cap \hat{A}(S) = \hat{C}_\Sigma \cap \hat{A}_0(S) = \hat{C}(X)\,, \tag{6.1}$$

$$\hat{G} \cap \hat{A}_0(S) = \hat{M}\,, \tag{6.2}$$

$$C_\sigma \cap \hat{A}_0(S) = \hat{C}_0(Y)\,. \tag{6.3}$$

Proof. If V is a G-space with an extreme point on the unit ball, then there is a maximal w^*-closed face S in K with closed extreme-boundary. Hence S is a Bauer-simplex and the first equality in (6.1) follows from [1, theorem II.4.3]. If S is a maximal face in K such that $\operatorname{conv}(S \cup \{0\})$ is compact and $\partial_e K$ is closed, then $\partial_e S$ is closed. Hence S is closed and [1, theorem II.4.3] will do. If V is a simplex-space with

$$\overline{\partial_e K} \subseteqq [0,1]\partial_e K ,$$

then there is a maximal face S in K such that $\operatorname{conv}(S \cup \{0\})$ is w^*-compact and

$$\overline{\partial_e S} \subseteqq [0,1]\partial_e S .$$

Now [7, theorem 2.3] gives (6.2) as in the proof of theorem 24. If V is a simplex-space with $\partial_e K \cup \{0\}$ closed, then there is a maximal face S in K such that $\partial_e S \cup \{0\}$ is compact. Hence $S_0 = \operatorname{conv}(S \cup \{0\})$ is a Bauer simplex and by [1, theorem II.4.3] we get

$$C_0(\partial_e S) \cong A_0(S_0) \cong V .$$

NOTES. The classification scheme is essentially due to Lindenstrauss and Wulbert [22], but was later on modified in [20]. For more information about complex Lindenstrauss spaces, see Hustad's works [14] and [15], where he studies intersection properties of balls and extensions of compact operators. These topics are related to Lindenstrauss' results [21] in the real case.

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