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ON THE DIFFERENTIABILITY POINTS OF A FUNCTION OF TWO REAL VARIABLES ADMITTING PARTIAL DERIVATIVES

J. HOFFMANN-JØRGENSEN

1. Introduction.

Let f be a map from R^2 into R. We define the partial derivatives, $D_1(f,x)$ and $D_2(f,x)$, in the usual way:

$$D_j(f,x) = \lim_{h\to 0}\frac{f(x+he_j)-f(x)}{h}$$

where $\{e_1,e_2\}$ is the unit vector basis in R^2. We shall say that f is *partially differentiable* on $A \subseteq \mathsf{R}^2$, if $D_1(f,x)$ and $D_2(f,x)$ exists and are finite for all $x \in A$.

We shall use the term "differentiable" in the sense of Stolz. That is, f is *differentiable* at x with *differential* $D \in \mathsf{R}^2$, if

$$\lim_{h\to 0}\frac{|f(x+h)-f(x)-\langle h,D\rangle|}{\|h\|} = 0$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product in R^2.

Stepanoff has shown in [3] that if f is continuous on R^2 and partially differentiable on a continuum K (i.e. a compact connected subset of R^2) then the Lipshitzian $L(f,x)$ is finite at every point $x \in D$, for some dense subset D of K. Here the *Lipshitzian* is defined by

$$L(f,x) = \limsup_{h\to 0}\frac{|f(x+h)-f(x)|}{\|h\|} \quad \forall x \in \mathsf{R}^2 .$$

In this note we shall show that, if f is continuous and partially differentiable on a differentiable curve $\Gamma \subseteq \mathsf{R}^2$, then f is differentiable at x for all x in a dense G_δ-subset of Γ.

In [3] Stepanoff gives 3 important examples. The first example of Stepanoff is a continuous function f, which is partially differentiable almost everywhere in R^2, but nowhere differentiable. Stepanoff's second

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example is a continuous function f which is partially differentiable on all of R^2, but the set of differentiability points has Lebesgue measure smaller than any prescribed positive number ε. The last examply of Stepanoff is a continuous function f which is partially differentiable on all of R^2, so that there exists a continuum K with $\{x \in K \mid L(f,x) = \infty\}$ of second category in K.

2. Differentiability on a curve.

In this section we shall present a proof of the result announced in the introduction, but under essentially weaker conditions. In order to state the theorem we shall need the following definition: If $A \subseteqq \mathsf{R}^2$, then $\theta(A)$ is defined to be the set of points $x \in A$, such that there exist $\beta(x) = \beta > 0$ and $\delta(x) = \delta > 0$ with the property

(2.1) $\forall z \in b(x,\delta)$, $\exists y \in A$ so that $\|y-z\| \leqq \beta\|z-x\|$ and either $p_1(y) = p_1(z)$ or $p_2(y) = p_2(z)$,

where $b(x,\delta)$ denotes the closed ball with center at x and radius δ, and p_j is the projection on e_j. Now we can state the main theorem:

THEOREM 2.1. *Let f be a map*: $\mathsf{R}^2 \rightharpoonup \mathsf{R}$ *and Γ a subset of* R^2 *satisfying*

(2.1.1) *Γ is a G_δ-set,*
(2.1.2) *$f(\cdot + ae_j) \mid \Gamma$ is continuous for all $a \in \mathsf{R}$ and $j = 1,2$,*
(2.1.3) *f is partially differentiable on Γ,*
(2.1.4) *$\theta(\Gamma)$ contains a G_δ-set Γ_0 which is dense in Γ.*

Then the set

$$\Delta = \{x \in \Gamma \mid f \text{ is differentiable at } x\}$$

contains a G_δ-set which is dense in Γ.

PROOF. Let $\Gamma_j^+(\varepsilon)$ be the set of $x \in \Gamma$ so that there exists a neighborhood U of x, relatively in Γ, and a $\delta > 0$ so that

$$|(f(y+te_j) - f(y))/t - D_j(y)| \leqq \varepsilon \quad \forall y \in U,\ \forall 0 < t \leqq \delta\,.$$

Let $\Gamma_j^-(\varepsilon)$ be the set of $x \in \Gamma$ so that there exists a neighborhood U of x, relatively to Γ, and a $\delta > 0$ so that

$$|(f(y+te_j) - f(y))/t - D_j(y)| \leqq \varepsilon \quad \forall y \in U,\ \forall -\delta \leqq t < 0\,,$$

where D_1 and D_2 are the partial derivatives of f in the directions e_1 and e_2. Then we have:

(2.2) $\Gamma_1^+(\varepsilon)$, $\Gamma_1^-(\varepsilon)$, $\Gamma_2^+(\varepsilon)$ and $\Gamma_2^-(\varepsilon)$ are open and dense relatively in Γ for all $\varepsilon > 0$.

Let us consider $\Gamma_1^+(\varepsilon)$. It is obvious that $\Gamma_1^+(\varepsilon)$ is open relatively in Γ. Since Γ is a G_δ-set in R^2 we can find a complete metric $\varrho(x,y)$ on Γ which generates the topology of Γ. Now suppose that $\Gamma_1^+(\varepsilon)$ is not dense in Γ. Then we can find $x_0 \in \Gamma$ and $0 < r_0 \leqq 1$ so that

$$B(x_0,r_0) \cap \Gamma_1^+(\varepsilon) = \emptyset$$

where we define

$$B(x,r) = \{y \in \Gamma \mid \varrho(x,y) \leqq r\} ,$$
$$B^0(x,r) = \{y \in \Gamma \mid \varrho(x,y) < r\}$$

for $x \in \Gamma$ and $r > 0$. Now $x_0 \notin \Gamma_1^+(\varepsilon)$ and $B^0(x_0,r_0)$ is a neighborhood of x_0, so there exist $0 < t_1 < \frac{1}{2}$ and $x_1 \in B^0(x_0,r_0)$ with

$$|(f(x_1+t_1e_1)-f(x_1))/t_0 - D_1(x_1)| > \varepsilon .$$

Then we can find $0 < s_1 < \frac{1}{2}$ with

$$\left|\frac{f(x_1+t_1e_1)-f(x_1)}{t_1} - \frac{f(x_1+s_1e_1)-f(x_1)}{s_1}\right| > \varepsilon ,$$

since f is partially differentiable at x_1 by (2.1.3). Now by (2.1.2) we can find $0 < r_1 \leqq \frac{1}{2}$ so that $B(x_1,r_1) \subseteqq B(x_0,r_0)$ and

$$\left|\frac{f(x+t_1e_1)-f(x_1)}{t_1} - \frac{f(x+s_1e_1)-f(x)}{s_1}\right| > \varepsilon$$

for all $x \in B(x_1,r_1)$. Continuing in this way we can inductively define $x_n \in \Gamma$, t_n, s_n and r_n in $(0,2^{-n}]$ so that

(i) $$B(x_{n+1},r_{n+1}) \subseteqq B^0(x_n,r_n) \quad \forall n \geqq 0 ,$$

(ii) $$\left|\frac{f(x+t_ne_1)-f(x)}{t_n} - \frac{f(x+s_ne_1)-f(x)}{s_n}\right| > \varepsilon$$

for all $x \in B(x_n,r_n)$ and all $n \geqq 1$.

From (i) it follows that we can find $\hat{x}$ with $\hat{x} \in B(x_n,r_n)$ for all $n \geqq 1$, since the metric ϱ is complete. Hence by (ii) we have

$$\left|\frac{f(\hat{x}+t_ne_1)-f(\hat{x})}{t_n} - \frac{f(\hat{x}+s_ne_1)-f(\hat{x})}{s_n}\right| > \varepsilon$$

for all $n \geqq 1$. Now f is partially differentiable at $\hat{x}$ and $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n = 0$, so for $n \to \infty$ we find

$$|D_1(\hat{x}) - D_1(\hat{x})| \geqq \varepsilon ,$$

which is impossible. Hence $\Gamma_1^+(\varepsilon)$ is dense in Γ. Similarly one may prove that $\Gamma_1^-(\varepsilon)$, $\Gamma_2^+(\varepsilon)$ and $\Gamma_2^-(\varepsilon)$ are open and dense relatively in Γ, and so (2.2) is proved.

Now let

$$\Gamma(\varepsilon) = \Gamma_1^+(\varepsilon) \cap \Gamma_1^-(\varepsilon) \cap \Gamma_2^+(\varepsilon) \cap \Gamma_2^-(\varepsilon) .$$

Then we obviously have:

(2.3) $x \in \Gamma(\varepsilon)$ if and only if there exists a $\delta(x) = \delta > 0$ so that for all $y \in \Gamma \cap b(x,\delta)$, all $0 < |t| \leqq \delta$ and $j = 1$ or 2,

$$|(f(y+te_j) - f(y))/t - D_j(y)| \leqq \varepsilon .$$

Since Γ is a Baire space it follows from (2.2) that

(2.4) $\Gamma(\varepsilon)$ is open and dense relatively in Γ for all $\varepsilon > 0$.

Now we shall prove:

(2.5) $\forall x \in \Gamma(\varepsilon)$, $\exists \delta > 0$ such that

$$|D_j(x) - D_j(y)| \leqq 3\varepsilon \quad \forall y \in \Gamma \cap b(x,\delta), \quad \forall j = 1,2 .$$

First we choose $\delta_0 > 0$ so that the inequalities in (2.4) are satisfied. Then we choose $0 < \delta \leqq \delta_0$ so that

$$|f(x) - f(y)| \leqq \tfrac{1}{2}\varepsilon\delta_0 ,$$
$$|f(x+\delta_0 e_j) - f(y+\delta_0 e_j)| \leqq \tfrac{1}{2}\varepsilon\delta_0$$

for all $y \in \Gamma \cap b(x,\delta)$ and for $j = 1,2$, which is possible by (2.1.2). Then we have for $y \in \Gamma \cap b(x,\delta)$ and for $j = 1$ or 2:

$$\begin{aligned}|D_j(x) - D_j(y)| &\leqq \left| D_j(x) - \frac{f(x+\delta_0 e_j) - f(x)}{\delta_0} \right| + \left| \frac{f(y+\delta_0 e_j) - f(y)}{\delta_0} - D_j(y) \right| + \\ &\quad + \delta_0^{-1}|f(x) - f(y)| + \delta_0^{-1}|f(x+\delta_0 e_j) - f(y+\delta_0 e_j)| \\ &\leqq 3\varepsilon ,\end{aligned}$$

and so (2.5) is proved.

Now let Γ_0 be the dense G_δ-set from (2.1.4), and put

$$\Gamma_1 = \Gamma_0 \cap \bigcap_{n=1}^{\infty} \Gamma(1/n) .$$

Then Γ_1 is a G_δ-set which is dense in Γ, since Γ is a Baire space and (2.4) holds. We shall now prove that f is differentiable at all points of Γ_1. So let $x \in \Gamma_1$ and let $\varepsilon > 0$ be given. Since $x \in \theta(\Gamma)$ we can find $\beta > 0$ and $\delta > 0$, so that (2.1) holds.

Now we choose $k \geqq 1$ so large that $k \geqq \varepsilon^{-1}(2\beta+5)$. Since $x \in \Gamma(1/k)$, we can find $0 < \delta_1 \leqq \delta$ so that

$$\text{(iii)} \qquad |(f(y+te_j)-f(y))/t - D_j(y)| < 1/k\,,$$

$$\text{(iv)} \qquad |D_j(x)-D_j(y)| < 3/k$$

for all $y \in \Gamma \cap B(x,\delta_1)$, all $0<|t|\leqq \delta_1$, and $j=1,2$. Now let $\delta_2=(\beta+1)^{-1}\delta_1$. Then we shall show that

$$\text{(2.6)} \qquad |f(z)-f(x)-\langle z-x, D(x)\rangle| \leqq \varepsilon\|z-x\| \quad \forall z \in B(x,\delta_2)\,.$$

So let $z \in B(x,\delta_2)$, and put $r=\|z-x\|$. Now $\|z-x\|=r\leqq\delta_2\leqq\delta$. Hence by (2.1) we can find $y \in \Gamma$ so that $\|y-z\|\leqq\beta r$ and either $p_1(y)=p_1(z)$ or $p_2(y)=p_2(z)$. Let us assume that the first case occurs. Then we put $x'=(p_1(z),p_2(x))$, and we have:

$$\begin{aligned} z &= y+te_2 \quad &&\text{with } |t|=\|y-z\|\leqq\beta r\leqq\delta_1\,,\\ x' &= y+se_2 \quad &&\text{with } |s|=\|x'-y\|\leqq(\beta+1)r\leqq\delta_1\,,\\ x' &= x+ue_1 \quad &&\text{with } |u|=\|x'-x\|\leqq r\leqq\delta_1\,. \end{aligned}$$

So by (iii) and (iv),

$$\begin{aligned} &|f(z)-f(x)-\langle z-x,D(x)\rangle|\\ &\leqq |f(z)-f(y)-\langle z-y,D(y)\rangle| + |f(y)-f(x')-\langle y-x',D(y)\rangle| +\\ &\quad + |f(x')-f(x)-\langle x'-x,D(x)\rangle| + |\langle z-x',D(y)-D(x)\rangle|\\ &\leqq k^{-1}|t|+k^{-1}|s|+k^{-1}|u|+\|z-x'\|\,|D_2(y)-D_2(x)|\\ &\leqq k^{-1}(\beta r+(\beta+1)r+r+3r) = k^{-1}(2\beta+5)r\\ &\leqq r\varepsilon\,. \end{aligned}$$

Hence (2.6) is proved, and so f is differentiable at all points of the G_δ-set Γ_1, and Γ_1 is dense in Γ.

PROPOSITION 2.2. *Let γ be a differentiable non-constant map from* $[0,1]$ *into* R^2 *satisfying*:

(2.2.1) *There exist an F_σ-set $T_0 \subseteqq [0,1]$ so that $S_0 \subseteqq T_0$ and $T_0 \setminus S_0$ is a Lebesgue-nullset*,

where $S_0=\{t \mid \gamma'(t)=0\}$. Then the curve $\Gamma=\gamma([0,1])$ satisfies (2.1.1) *and* (2.1.4) *in Theorem* 2.1.

REMARK. If γ' only has finitely many discontinuities, then it is easily checked that (2.2.1) holds for $T_0 = S_0$. If γ' only has countably many zeros, then obviously (2.2.1) holds with $T_0 = S_0$.

PROOF. Let

$$S_+ = \{t \mid 0 < t < 1 \text{ and } \gamma'(t) \neq 0\} ,$$
$$\Gamma_+ = \gamma(S_+) .$$

We shall then show that

$$\Gamma_+ \subseteqq \theta(\Gamma) . \tag{2.7}$$

So let $x_0 = \gamma(t_0)$ for some $t_0 \in S_+$. Then one of the following four cases must occur: (i) $\gamma_1'(t_0) > 0$, (ii) $\gamma_1'(t_0) < 0$, (iii) $\gamma_2'(t_0) > 0$, or (iv) $\gamma_2'(t_0) < 0$. If the first case occurs we can find $r_0 > 0$ so that $[t_0 - r_0, t_0 + r_0] \subseteqq [0,1]$ and

$$\begin{aligned} \gamma_1(t_0+r) - \gamma_1(t_0) &\geqq ar && \forall\, 0 \leqq r \leqq r_0 , \\ \gamma_1(t_0+r) - \gamma_1(t_0) &\leqq ar && \forall\, -r_0 \leqq r \leqq 0 , \\ |\gamma_2(t_0+r) - \gamma_2(t_0)| &\leqq A|r| && \forall\, -r_0 \leqq r \leqq r_0 , \end{aligned}$$

where $a = \frac{1}{2}\gamma_1'(t_0)$ and $A = 1 + |\gamma_2'(t_0)|$. Let $\beta = a^{-1}A + 1$ and $\delta = ar_0$. If $z \in B(x_0, \delta)$ we have

$$\gamma_1(t_0 - r_0) \leqq p_1(x_0) - ar_0 \leqq p_1(z) \leqq p_1(x_0) + ar_0 \leqq \gamma_1(t_0 + r_0) .$$

So there exist r_1 with $|r_1| \leqq r_0$ and $p_1(z) = \gamma_1(t_0 + r_1)$. Let $y = \gamma(t_0 + r_1)$. Then $y \in \Gamma$ and $p_1(y) = p_1(z)$. Moreover,

$$\begin{aligned} \|y - z\| &= |p_2(y) - p_2(z)| \leqq |p_2(y) - p_2(x_0)| + |p_2(x_0) - p_2(z)| \\ &\leqq \|x_0 - z\| + |\gamma_2(t_0 + r) - \gamma_2(t_0)| \\ &\leqq \|x_0 - z\| + A|r_1| \\ &\leqq \|x_0 - z\| + a^{-1}A|\gamma_1(t_0 + r_1) - \gamma_1(t_0)| \\ &\leqq \beta\|x_0 - z\| . \end{aligned}$$

This shows that $x_0 \in \theta(\Gamma)$, and since the three remaining cases may be proved similarly, we have proved (2.7).

We may of course assume that $0 \in T_0$ and $1 \in T_0$. Then $T_0 \cup S_+ = [0,1]$, and so $\Gamma_0 \cup \Gamma_+ = \Gamma$ where $\Gamma_0 = \gamma(T_0)$. Moreover, Γ_0 is an F_σ-set, since T_0 is σ-compact. By Theorem 3.2.3 in [1] we have

$$\int_{T_0} \|\gamma'(t)\| dt = \int_{\mathbb{R}^2} \#\{\gamma^{-1}(y) \cap T_0\} H^1(dy) ,$$

where H^1 is the 1-dimensional Hausdorff measure in $\mathbb{R}^2$. Now the left hand side is 0 and

$$\#\{\gamma^{-1}(y) \cap T_0\} \geqq 1 \quad \forall y \in \Gamma_0 .$$

So we find that $H^1(\Gamma_0) = 0$. Moreover, Γ is connected and locally connected and contains at least 2 points, since γ is continuous and non-constant. Hence, if U is open relatively in Γ and $U \neq \emptyset$, then U contains a connected set with at least 2 points. So by Corollary 2.10.12 in [1] we have

$$H^1(U) > 0$$

for all non empty sets U which are relatively open in Γ. This implies that the interior of Γ_0 relatively in Γ is empty. Hence $\Gamma_1 = \Gamma \setminus \Gamma_0$ is a G_δ-set which is dense in Γ (note that Γ is compact and so a fortiori a G_δ-set), and $\Gamma_1 \subseteqq \Gamma_+ \subseteqq \theta(\Gamma)$. So (2.1.1) and (2.1.4) holds.

3. Differentiability along a curve.

In this section we shall prove a result supplementary to the result in section 2. The result is based on the following simple lemma:

LEMMA 3.1. *Let f be a map from $\mathbb{R}^2$ in $\mathbb{R}$ whose partial derivatives $D_1(f,x_0)$ and $D_2(f,x_0)$ exist at the point x_0. If one of the partial derivatives exists and is bounded in the neighborhood of x_0, then the Lipshitzian $L(f,x_0)$ of f is finite at x_0.*

PROOF. Suppose that $|D_1(f,x)| \leqq M$ for all $x \in B(x_0,\delta)$, and suppose that $\delta > 0$ is chosen so small that

$$|f(x_0+h_2e_2)-f(x_0)| \leqq K|h_2| \quad \forall\, |h_2| \leqq \delta ,$$

where $K = |D_2(f,x_0)|+1$. Then we have for all $h = (h_1,h_2) \in B(x_0,\delta)$, by the Mean Value Theorem:

$$\begin{aligned} |f(x_0+h)-f(x_0)| &\leqq |f(x_0+h_1e_1+h_2e_2)-f(x_0+h_2e_2)| + |f(x_0+h_2e_2)-f(x_0)| \\ &\leqq |h_1|\,|D_1(f,x_0+\theta h_1e_1+h_2e_2)| + K|h_2| \end{aligned}$$

where $0 < \theta < 1$. So we find $L(f,x_0) \leqq K+M$.

THEOREM 3.2. *Let γ be a map from $[0,1]$ into $\mathbb{R}^2$ which is differentiable at almost all points in $[0,1]$. Suppose that f maps $\mathbb{R}^2$ into $\mathbb{R}$ so that:*

(3.2.1) *For almost all points $t \in [0,1]$ the partial derivatives, $D_1(f,\gamma(t))$ and $D_2(f,\gamma(t))$, exists at the point $\gamma(t)$.*

(3.2.2) *For almost all points $t \in [0,1]$, the one of the partial derivatives of f exists and is bounded in a neighborhood of $\gamma(t)$.*

Then $f \circ \gamma$ is differentiable almost everywhere.

PROOF. From Lemma 3.1 it follows that there exists a nullset $N \subseteqq [0,1]$ so that

$$L(f,\gamma(t)) < \infty \quad \forall t \notin N ,$$
$$L(\gamma,t) < \infty \quad \forall t \notin N .$$

Let $g = f \circ \gamma$ and let $t \in [0,1] \setminus N$. Then there exists a $\delta > 0$ so that

$$|f(\gamma(t)+h) - f(\gamma(t))| \leqq K\|h\| \quad \forall \|h\| \leqq \delta ,$$

where $K = L(f,\gamma(t)) + 1$. Then we choose $r > 0$ so that

$$\|\gamma(t+s) - \gamma(t)\| \leqq M|s| \quad \forall |s| \leqq r ,$$

where $M = L(\gamma,t) + 1$. We may of course assume that $Mr \leqq \delta$, and so for $|s| \leqq r$,

$$|g(t+s) - g(t)| \leqq K\|\gamma(t+s) - \gamma(s)\| \leqq KM|s| .$$

Hence $L(g,t) < \infty$ for almost all t, and so g is differentiable for almost all t by Denjoy's theorem (see Theorem (4.2) p. 270 in [2]).

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UNIVERSITY OF AARHUS, DENMARK