MATH. SCAND. 35 (1974), 288–300

# THE CATEGORY OF GRADED MODULES

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Recently several studies have been made concerning Z-graded commutative rings and their graded modules. For example Iversen [6] has used the theory of graded modules and its relation to the theory of coherent sheaves in order to establish Serre duality on projective $n$-space $P^n$. Matijevic and Roberts [9, 10] and Nagata [11] have studied properties, like regularity and the Cohen-Macaulay property, for graded rings. And Fossum [2] has studied graded injective modules and graded completions. In this paper we expand on the theme: Let P be a property of commutative rings. If P holds for just graded objects, then P holds in general. A prime example of this phenomenon is the theorem of Matijevic: If a Z-graded ring has the ascending chain condition for homogeneous ideals, then it is noetherian. An outline of this paper follows. We suppose that the commutative ring $A$ is Z-graded and noetherian.

It is first shown that the maximal length of a chain of homogeneous prime ideals is at least one less than the Krull dimension of the ring. Then it is shown that the global dimension of $A$ and the global dimension of the category of graded $A$-modules differs by at most one. Finally it is shown that $\mathrm{id}_A M - 1 \leqq {}^*\mathrm{id}_A M \leqq \mathrm{id}_A M$ for all graded $A$-modules $M$, where $\mathrm{id}_A$ (respectively: ${}^*\mathrm{id}_A$) denotes the injective dimension of $M$ in the category of $A$-modules (respectively: category of graded $A$-modules).

## 1. The category ${}^*\mathbf{mod}_A$.

Throughout this paper $A = \coprod_{n \in \mathbf{Z}} A_n$ will be a Z-graded (or just graded) commutative ring with identity.

Let ${}^*\mathbf{mod}_A$ denote the category of Z-graded $A$-modules. An object in this category will be called an $A$-*module. If $M$ is an $A$-*module, then $M$ has a decomposition (as an abelian group) $M = \coprod_n M_n$ where each $M_n$ is an $A_0$-module and $A_n M_m \subseteq M_{n+m}$ for all pairs $n,m$ of integers. The set $\bigcup_n M_n$ will be denoted by $\mathrm{h}(M)$, the set of homogeneous elements in $M$. For a nonzero $x$ in $\mathrm{h}(M)$ we write $\deg x = n$ when $x \in M_n$. If $i$ is an inte-

Received June 6, 1974.

* Research partially support by the United States National Science Foundation.

ger, the *module $M(i)$ is the $A$-module $M$ with grading given by $M(i)_n = M_{i+n}$.

The group of morphisms from the *module $M$ to the *module $M'$ is denoted by $^*\mathrm{Hom}_0(M,M')$ and consists of all $A$-homomorphisms $f\colon M \to M'$ such that $f(M_n) \subseteq M'_n$ for all $n$. In general $^*\mathrm{Hom}_i(M,M')$ is the group of all homogeneous $A$-homomorphisms $f\colon M \to M'$ such that $f(M_n) \subseteq M'_{i+n}$, or in other words

$$^*\mathrm{Hom}_i(M,M') = {}^*\mathrm{Hom}_0(M(-i),M') = {}^*\mathrm{Hom}_0(M,M'(i)) .$$

The groups $^*\mathrm{Hom}_i(M,M')$ form a direct sum in $\mathrm{Hom}_A(M,M')$ and we let $^*\mathrm{Hom}_A(M,M')$ denote the $A$-submodule $\coprod_n {}^*\mathrm{Hom}_n(M,M')$ of $\mathrm{Hom}_A(M,M')$.

The tensor product $M \otimes_A M'$ of two *modules is also a graded module with $(M \otimes_A M')_n$ being generated by elements $x \otimes x'$ with $x \in M_i$, $x' \in M_j'$ where $i+j=n$.

Let $(M_\alpha)_{\alpha \in I}$ be a family of $A$-*modules. Then $\coprod_\alpha M_\alpha$ becomes a *module with $(\coprod_\alpha M_\alpha)_n = \coprod_\alpha (M_\alpha)_n$. The (direct) product also exists in ***mod**$_A$ with $(^*\prod_\alpha M_\alpha)_n = \prod_\alpha (M_\alpha)_n$. Thus $^*\prod_\alpha M_\alpha = \coprod_{n\in \mathbf{Z}} (\prod_{\alpha \in I} (M_\alpha)_n)$. Note that we have the bijections

$$^*\mathrm{Hom}(\textstyle\coprod_\alpha M_\alpha, -) \xrightarrow{\sim} {}^*\prod_\alpha {}^*\mathrm{Hom}(M_\alpha, -)$$

and

$$^*\mathrm{Hom}(-, {}^*\textstyle\prod_\alpha M_\alpha) \xrightarrow{\sim} {}^*\prod_\alpha {}^*\mathrm{Hom}(-, M_\alpha) .$$

Likewise limits and colimits exist in ***mod**$_A$ with

$$(^*\varinjlim M_\alpha)_n = \varinjlim (M_\alpha)_n$$

and

$$(^*\varprojlim M_\alpha)_n = \varprojlim (M_\alpha)_n$$

for direct and inverse systems respectively.

The category ***mod**$_A$ has enough projectives; in fact each *module is a homomorphic image (in ***mod**$_A$) of a *module of the form $\coprod_\alpha A(n_\alpha)$ with $(n_\alpha)_{\alpha \in I}$ a family of integers. From this it follows that a *module is projective in ***mod**$_A$ if and only if it is projective in **mod**$_A$.

There are enough injective objects in ***mod**$_A$ (See Grothendieck [5, 1.10] or Fossum [2]) and the injective envelope in ***mod**$_A$ of a *module $M$ will be denoted by $^*E_A(M)$ (or $^*E(M)$). Also the map $M \to {}^*E(M)$ is an essential injection in ***mod**$_A$ (short: the map $M \to {}^*E(M)$ is *essential). Thus it will follow from the lemma below that $^*E(M)$ is a submodule of the ordinary injective envelope, denoted by $E(M)$, of $M$. From the usual properties of essential extensions and injective objects, it follows that $*E(M)$ is maximal among the graded submodules of $E(M)$ which have $M$ as graded submodule.

LEMMA 1.1. *Let $M$ be a subobject of the $A$-*module $N$. If $M$ is *essential in $N$, then it is essential in $N$.*

PROOF. Suppose $M$ is *essential in $N$. Thus for each $x$ in $\mathrm{h}(N)-\{0\}$ there is an $a \in \mathrm{h}(A)$ such that $ax \in M-\{0\}$. Let $x$ be an element in $N$, say $x=x_r+\ldots+x_s \neq 0$ with each $x_i \in N_i$ and $r \leqq s$. We will prove, by induction on $s-r$, that there is an element $a \in \mathrm{h}(A)$ such that $ax \in M-\{0\}$, the case $s-r=0$ being already handled by the assumption. Suppose that $s-r>0$. Choose $a$ in $\mathrm{h}(A)$ such that $ax_r \in M-\{0\}$. Let $x'=x-x_r=x_{r+1}+\ldots+x_s$. If $ax'=0$, then $ax=ax_r \in M-\{0\}$ and we are done. Suppose that $ax' \neq 0$ and choose $b \in \mathrm{h}(A)$, by the induction hypothesis, such that $bax' \in M-\{0\}$. Then $bax=bax'+bax_r \in M$. It cannot be the zero element because either $bax_r=0$ and then $bax=bax' \neq 0$ or else $bax_r \neq 0$ and then $bax_r$ is the homogeneous component of least degree in $bax=bax_r+bax'$.

This chapter concludes with some remarks about our notation and graded localization.

The derived functors in $*\mathbf{mod}_A$ of *Hom are denoted by $*\mathrm{Ext}^i$. Note that $\mathrm{Tor}_i$ is the $i$th derived functor of $\otimes$ in both $\mathbf{mod}_A$ and $*\mathbf{mod}_A$. As we have already indicated, we will denote concepts in $*\mathbf{mod}_A$ in the same manner as corresponding concepts in $\mathbf{mod}_A$ except that we will place an *asterisk* (*) in front of the word. Examples: *module, *submodule, *ideal (= homogeneous ideal), *prime *ideal, *maximal *ideal (= maximal among *ideals), *local *ring (= graded ring with unique *maximal *ideal), *noetherian *ring (= graded ring with ascending chain condition on *ideals), *injective *module (= injective in the category $*\mathbf{mod}_A$), $*\dim A$ (= maximal length of a chain of *prime *ideals in $A$), etc.

If $M$ is a submodule (in $\mathbf{mod}_A$) of a *module $N$, then $*M$ denotes the *submodule in $N$ generated by the elements $\mathrm{h}(M)=M \cap \mathrm{h}(N)$, the set of homogeneous elements in $M$. If $\mathfrak{p}$ is an *ideal in $A$, then $\mathfrak{p}$ is a prime ideal if and only if $\mathrm{h}(A)-\mathrm{h}(\mathfrak{p})$ is a multiplicatively closed subset. Hence if $\mathfrak{p}$ is a prime ideal in $A$, then $*\mathfrak{p}$, the associated *ideal, is a *prime *ideal.

Suppose $\mathfrak{p}$ is a prime ideal. Let $S=\mathrm{h}(A)-\mathfrak{p}$. Then $S$ is a multiplicatively closed subset in $A$. We let $A_{(\mathfrak{p})}$ denote the ring $S^{-1}A$ (while $A_\mathfrak{p}$ denotes the ordinary localization at $\mathfrak{p}$). The ring $A_{(\mathfrak{p})}$ is graded, in a natural way, by

$$(A_{(\mathfrak{p})})_m = \{a/s : a \in \mathrm{h}(A),\ s \in S \text{ with } \deg a = \deg s + m\}.$$

The ring $A_{(\mathfrak{p})}$ is a *local *ring with *maximal *ideal $^*\mathfrak{p}A_{(\mathfrak{p})}$. If $\mathfrak{q}$ is a prime ideal in $A_{(\mathfrak{p})}$, then $\mathfrak{q}$ is of the form $\mathfrak{r}A_{(\mathfrak{p})}$ where $\mathfrak{r}$ is a prime ideal in $A$ with $^*\mathfrak{r} \leqq {}^*\mathfrak{p}$. If $\mathfrak{p}$ is itself homogeneous we denote by $^*k(\mathfrak{p})$ the residue class *ring $A_{(\mathfrak{p})}/\mathfrak{p}A_{(\mathfrak{p})}$ which is again a *local *ring (having a unique homogeneous ideal, namely 0).

## 2. Krull *dimension.

In this section we assume that $A$ is a *noetherian *ring — which is the same as to say that all *ideals are finitely generated. That such a ring is noetherian has already been remarked and the proof of the result is due to Matijevic.

LEMMA 2.1. *If $A$ is *noetherian, then it is noetherian.*

(For a proof see Matijevic [9] or Fossum [2].)

Thus we can assume, and indeed we do so, that $A$ is a noetherian ring in the remainder of this chapter.

THEOREM 2.2. *If $A$ has only the trivial *ideals, then either $A$ is a field* (and so $A = A_0$) *or $A$ is a* PID, *in which case* $A \cong A_0[T, T^{-1}]$.

PROOF. If $A$ is not a field, then there exist homogeneous elements of positive degree. Let $t$ be one of least positive degree, say $\deg t = d$. As the ideal $At$ is homogeneous, the element $t$ is invertible. Hence $A_d = A_0 t$ and, in general $A_{nd} = A_0 t^n$ for all $n \in \mathbf{Z}$. Hence the ring homomorphism $A_0[T, T^{-1}] \to A$ defined by $T \mapsto t$ (with $\deg T = d$ and $T$ an indeterminate) is an *isomorphism.

COROLLARY 2.3. *If $\mathfrak{p}$ is a prime ideal in $A$, then* $\operatorname{ht}\mathfrak{p} \leqq \operatorname{ht}{}^*\mathfrak{p} + 1$.

PROOF. We can compute $\operatorname{ht}\mathfrak{p}$ in the ring $A_{(\mathfrak{p})}$. Therefore we assume $A$ is *local with *maximal *ideal $^*\mathfrak{p}$. Now pick a prime ideal $\mathfrak{q}$ with $\mathfrak{q} < \mathfrak{p}$ and $\operatorname{ht}\mathfrak{q} = \operatorname{ht}\mathfrak{p} - 1$. (The case $\operatorname{ht}\mathfrak{p} = 0$ has been excluded, since in that case $\mathfrak{p} = {}^*\mathfrak{p}$.) Since $\mathfrak{q} \subseteqq \mathfrak{p}$, we have $^*\mathfrak{q} \subseteqq {}^*\mathfrak{p}$.

If $^*\mathfrak{q} = {}^*\mathfrak{p}$, we conclude that $\mathfrak{q} = {}^*\mathfrak{q}$ since

$$\dim(A/^*\mathfrak{q}) = \dim(A/^*\mathfrak{p}) \leqq 1$$

by the theorem, and hence $\operatorname{ht}\mathfrak{p} = \operatorname{ht}\mathfrak{q} + 1 = \operatorname{ht}{}^*\mathfrak{p} + 1$ as desired.

We proceed by induction on $\operatorname{ht}{}^*\mathfrak{p}$.

Suppose $\operatorname{ht}{}^*\mathfrak{p} = 0$. Then $^*\mathfrak{q} = {}^*\mathfrak{p}$ (since $^*\mathfrak{p}$ is the only prime *ideal in $A$) and we have shown our inequality in this case.

Suppose $\operatorname{ht}{}^*\mathfrak{p} > 0$ and that ${}^*\mathfrak{q} < {}^*\mathfrak{p}$. By the induction hypothesis, $\operatorname{ht}{}^*\mathfrak{q} + 1 \geqq \operatorname{ht}\mathfrak{q}$. But also $\operatorname{ht}{}^*\mathfrak{p} \geqq \operatorname{ht}{}^*\mathfrak{q} + 1$ and $\operatorname{ht}\mathfrak{q} = \operatorname{ht}\mathfrak{p} - 1$. Therefore $\operatorname{ht}{}^*\mathfrak{p} \geqq \operatorname{ht}\mathfrak{p} - 1$ which is our desired result.

PROPOSITION 2.4 (Matijevic). *If* $\mathfrak{p}$ *is a prime *ideal, then* ${}^*\operatorname{ht}\mathfrak{p} = \operatorname{ht}\mathfrak{p}$. *In other words: If* $h = \operatorname{ht}\mathfrak{p}$, *then there is a chain of homogeneous prime ideals*

$$\mathfrak{q}_1 < \mathfrak{q}_2 < \ldots < \mathfrak{q}_h < \mathfrak{p} .$$

PROOF. We go by induction on $\operatorname{ht}p$. If $\operatorname{ht}\mathfrak{p} = 0$ there is nothing to prove. Suppose the result is true for all prime *ideals $\mathfrak{q}$ with $0 \leqq \operatorname{ht}\mathfrak{q} < h$. Let $\operatorname{ht}\mathfrak{p} = h$ and let $\mathfrak{p}_1 < \ldots < \mathfrak{p}_h < \mathfrak{p}$ be a chain of prime ideals. Then $\mathfrak{p}_1 = {}^*\mathfrak{p}_1$ since $\operatorname{ht}\mathfrak{p}_1 = 0$. Pick an $a \in \mathrm{h}(\mathfrak{p}) - \mathfrak{p}_1$ and set

$$\bar{A} = A/(\mathfrak{p}_1 + Aa) \quad \text{and} \quad \bar{\mathfrak{p}} = \mathfrak{p}/(\mathfrak{p}_1 + aA) .$$

Since $\operatorname{ht}_{\bar{A}}\bar{\mathfrak{p}} = h - 1$, there exist, by the induction hypothesis, $h - 1$ prime *ideals $\mathfrak{q}_2, \ldots, \mathfrak{q}_h$ in $\bar{A}$ and thereby $\mathfrak{q}_2, \ldots, \mathfrak{q}_h$ prime *ideals in $A$ such that

$$\mathfrak{p}_1 + Aa \leqq \mathfrak{q}_2 < \ldots < \mathfrak{q}_h < \mathfrak{p} .$$

Hence we have the desired chain.

THEOREM 2.5. *If* $A$ *is a graded ring, then* $\dim A - 1 \leqq {}^*\dim A \leqq \dim A$.

PROOF. This follows directly from (2.3) and (2.4).

## 3. Global dimension.

We have seen already that a *module is *projective if and only if it is projective. Thus the next result is obvious.

PROPOSITION 3.1. *Let* $M$ *be an* $A$-**module. Then*

$${}^*\operatorname{pd}_A M = \operatorname{pd}_A M .$$

In fact there is a corresponding result for flat (or weak) dimension.

PROPOSITION 3.2. *Let* $M$ *be an* $A$-**module. Then*

$${}^*\operatorname{fd}_A M = \operatorname{fd}_A M .$$

PROOF. It is enough to prove that any *flat *module is flat. Suppose $M$ is a *flat *module. A careful revision of the proof of Théorème 2 in Lazard [8] shows that the *flat *module $M$ is the direct limit of a direct system of finitely generated free *modules (in the category $^*\mathbf{mod}_A$) and hence $M$ is flat.

Let $\mathfrak{a}$ be an *ideal in $A$. Then a basis for the $\mathfrak{a}$-*adic topology at zero is given by the family $\{\coprod_{i=r}^{s}(\mathfrak{a}^n)_i\}_{n,r,s}$ and the $\mathfrak{a}$-*adic completion of $A$ is

$$^*\hat{A} = {}^*\varprojlim_n A/\mathfrak{a}^n$$

which is a noetherian graded ring. (See Fossum [2].) As in the ungraded case it follows that $^*\hat{A}$ is a *flat $A$-*module.

COROLLARY 3.3. *The $\mathfrak{a}$-*adic completion $^*\hat{A}$ is a flat $A$-module.*

Suppose we assign $\deg T = 1$ to the indeterminate $T$ over $A$. The $(T)$-*adic completion of $A[T]$ is denoted by $A[[^*T]]$ and is a subring of the ring of formal power series $A[[T]]$ defined by

$$(A[[^*T]])_n = \{\textstyle\sum_{i=n}^{-\infty} a_i T^{n-i} : a_i \in A_i\} .$$

(Note that in case $A_i = 0$ for $i < 0$, then $A[[^*T]] = A[T]$.)

COROLLARY 3.4. *The base change $A \to A[[^*T]]$ is flat.*

Just as in the ungraded case, the base changes $A \to {}^*\hat{A}$ and $A \to A[[^*T]]$ preserve, for example, finite global dimension.

Let $\mathrm{gl}\,^*\mathrm{dim}\,A$ denote the global dimension of $^*\mathbf{mod}_A$.

THEOREM 3.5. *For the graded ring $A$, the following inequalities hold:*

$$\mathrm{gl\,dim}\,A - 1 \leqq \mathrm{gl}\,^*\mathrm{dim}\,A \leqq \mathrm{gl\,dim}\,A .$$

PROOF. The inequality on the right follows from (3.1). As for the inequality on the left, recall that

$$\mathrm{gl\,dim}\,A = \sup_{\mathfrak{m}} \mathrm{pd}_A A/\mathfrak{m} ,$$

the supremum taken over all maximal ideals $\mathfrak{m}$ in $A$. Let $\mathfrak{m}$ be one of these maximal ideals. If $\mathfrak{m} = {}^*\mathfrak{m}$, then

$$\mathrm{pd}_A A/\mathfrak{m} = {}^*\mathrm{pd}_A A/\mathfrak{m} \leqq \mathrm{gl}\,^*\mathrm{dim}\,A .$$

Suppose, on the other hand, that there is a proper inclusion $\mathfrak{m} \supsetneqq {}^*\mathfrak{m}$. Let $B$ denote the *local ring $A_{(\mathfrak{m})} = A_{(^*\mathfrak{m})}$ with a maximal ideal $\mathfrak{m}B$ and a *maximal *ideal $^*\mathfrak{m}B$, which we denote by $\mathfrak{n}$. Then $A/\mathfrak{m} = B/\mathfrak{m}B$ and $\mathrm{pd}_A A/\mathfrak{m} = \mathrm{pd}_B B/\mathfrak{m}B$. Also $\mathrm{pd}_B B/\mathfrak{n} = {}^*\mathrm{pd}_B B/\mathfrak{n} \leqq \mathrm{gl}\,^*\mathrm{dim}\,A$. Since $(B, \mathfrak{n})$

is a *local ring, the residue class ring $B/\mathfrak{n}$ is a PID (see (2.2)) and hence there is an $a \in B$ such that $\mathfrak{m}B = \mathfrak{n} + Ba$. Then the sequence

$$0 \to B/\mathfrak{n} \xrightarrow{a} B/\mathfrak{n} \to B/\mathfrak{m}B \to 0$$

is exact, showing that

$$\mathrm{pd}_B B/\mathfrak{m}B \leqq \mathrm{pd}_B B/\mathfrak{n} + 1 .$$

Hence $\mathrm{pd}_A A/\mathfrak{m} \leqq \mathrm{gl}^*\mathrm{dim} A + 1$.

## 4. *Injective *modules.

Using almost the same proof as in the ungraded case, we get this next result.

LEMMA 4.1. *An $A$-*module $E$ is *injective if and only if the canonical homomorphism $E \to {}^*\mathrm{Hom}(\mathfrak{a}, E)$ is surjective for all *ideals $\mathfrak{a}$.*

In the rest of this section we assume that $A$ is *noetherian.*

LEMMA 4.2. *Let $M$ and $N$ be *modules with $M$ finitely generated. Then*

$$^*\mathrm{Hom}(M,N) = \mathrm{Hom}_A(M,N) .$$

*(In other words: Each homomorphism $M \to N$ is a sum of homogeneous homomorphisms.) Furthermore for each $i \geqq 0$, we have*

$$^*\mathrm{Ext}_A{}^i(M,N) = \mathrm{Ext}_A{}^i(M,N) .$$

PROOF. Since $A$ is noetherian, the module $M$ is finitely presented by free *modules

$$F_1 \to F_0 \to M \to 0 .$$

Both $^*\mathrm{Hom}(-,N)$ and $\mathrm{Hom}_A(-,N)$ are left exact. So it is sufficient to show that $^*\mathrm{Hom}(A(n),N) = \mathrm{Hom}(A,N)$. But this is obvious.

EXAMPLE. Assume that the grading is not finite, so there is an element $(a_n)_{n\in Z}$ in $\prod_{n\in Z} A_n$ not in $\coprod A_n$. Let $M = \coprod_{n \in Z} A$ be the direct sum of countably many copies of $A$ and let $f\colon M \to A$ be defined by

$$f((x_i)_{i\in Z}) = \textstyle\sum_{i\in Z} a_i x_i .$$

This $f$ is not the sum of finitely many homogeneous homomorphisms $M \to A$ and thus

$$^*\mathrm{Hom}(M,A) \neq \mathrm{Hom}(M,A) .$$

COROLLARY 4.3. *An $A$-*module $E$ is *injective if and only if*

$$\mathrm{Ext}_A{}^i(A/\mathfrak{a},E) = 0$$

*for all *ideals $\mathfrak{a}$ in $A$ and for all $i > 0$.*

COROLLARY 4.4. *Let $S \subseteqq \mathrm{h}(A)$ be a multiplicatively closed set and let $M$ be an $A$-*module. Then*

$$^*\mathrm{id}_{S^{-1}A}\, S^{-1}M \leqq {}^*\mathrm{id}_A M\ .$$

The next two results follow as in the ungraded case.

LEMMA 4.5. *When $\mathfrak{p}$ is a prime *ideal, then*

$$^*E_{A_{(\mathfrak{p})}}(^*k(\mathfrak{p})) \cong {}^*E_A(A/\mathfrak{p})_{(\mathfrak{p})} \cong {}^*E_A(A/\mathfrak{p})\ .$$

LEMMA 4.6. *Let $f\colon A \to B$ be a *ring homomorphism and assume that $B$ is finitely generated as an $A$-*module. Let $M$ be an $A$-*module. Then*

$$^*E_B(\mathrm{Hom}_A(B,M)) \cong \mathrm{Hom}_A(B, {}^*E_A(M))\ .$$

LEMMA 4.7. *Let $\mathfrak{p}$ be a prime *ideal. Then*

$$\mathrm{Hom}_A(A/\mathfrak{p}, {}^*E_A(A/\mathfrak{p}))_{(\mathfrak{p})} \cong {}^*k(\mathfrak{p})\ .$$

PROOF. By (4.5) and (4.6), the left hand side is just $^*E_{^*k(\mathfrak{p})}(^*k(\mathfrak{p}))$, but $^*k(\mathfrak{p})$ has only the trivial *ideals. Hence, by (4.3), all $^*k(\mathfrak{p})$-*modules are $^*k(\mathfrak{p})$-*injective. In particular

$$^*E_{^*k(\mathfrak{p})}(^*k(\mathfrak{p})) \cong {}^*k(\mathfrak{p})\ .$$

THEOREM 4.8. *Each *injective *module is a unique sum of indecomposable *injective *modules and each of these has the form $^*E(^*k(\mathfrak{p}))$ where $\mathfrak{p}$ is a prime *ideal.*

PROOF. This follows from Gabriel [4, Chapitre IV, Théorème 2].

The structure of $^*E(^*k(\mathfrak{p}))$ is discussed in Fossum [2].

Let $M$ be a *module and let

$$0 \to M \to {}^*I^0 \to {}^*I^1 \to \cdots$$

and

$$0 \to M \to I^0 \to I^1 \to \cdots$$

be minimal injective resolutions of $M$ in ***mod**$_A$ and **mod**$_A$ respectively. For the prime *ideal $\mathfrak{p}$ let $^*\mu^n(\mathfrak{p},M)$ and $\mu^n(\mathfrak{p},M)$ be the number of copies of $^*E(A/\mathfrak{p})$ in $^*I^n$ and of $E(A/\mathfrak{p})$ in $I^n$ respectively.

COROLLARY 4.9. *Let $M$ be a *module and $\mathfrak{p}$ a prime *ideal.*

(a) *The group* $\operatorname{Ext}_A{}^n(A/\mathfrak{p},M)_{(\mathfrak{p})}$ *is a free* $*k(\mathfrak{p})$-**module of rank* $*\mu^n(\mathfrak{p},M)$.
(b) $*\mu^n(\mathfrak{p},M)=\mu^n(\mathfrak{p},M)$.
(c) $\operatorname{id}_{A_\mathfrak{p}} M_\mathfrak{p} = *\operatorname{id}_{A_{(\mathfrak{p})}} M_{(\mathfrak{p})}$.

PROOF. As in the ungraded case (see Bass [1]) it follows that

$$\operatorname{Ext}_A{}^n(A/\mathfrak{p},M)_{(\mathfrak{p})} \cong \operatorname{Hom}(A/\mathfrak{p},*I^n)_{(\mathfrak{p})} .$$

So (a) follows from (4.7) since, for $\mathfrak{q} \neq \mathfrak{p}$, the group

$$\operatorname{Hom}(A/\mathfrak{p},*E(A/\mathfrak{q}))_{(\mathfrak{p})} = 0 .$$

If we localize $\operatorname{Ext}_A{}^n(A/\mathfrak{p},M)_{(\mathfrak{p})}$ at $\mathfrak{p}$ we get a vector space over $k(\mathfrak{p})$ (which is just $*k(\mathfrak{p})_\mathfrak{p}$) of rank $*\mu^n(\mathfrak{p},M)$. Thus (b) follows. And (c) is a direct consequence of (b).

Now we will compare the injective dimension of an object in $*\mathbf{mod}_A$ with its injective dimension in $\mathbf{mod}_A$.

THEOREM 4.10. *Let $E$ be an *injective *module. Then*

$$\operatorname{id}_A E \leqq 1 .$$

*In fact the minimal injective resolution for* $*E(A/\mathfrak{p})$ *is*

$$0 \to *E(A/\mathfrak{p}) \to E(A/\mathfrak{p}) \to \textstyle\coprod_\mathfrak{p} E(A/\mathfrak{q}) \to 0$$

*where the sum is taken over all prime ideals* $\mathfrak{q} \neq \mathfrak{p}$ *with* $*\mathfrak{q}=\mathfrak{p}$.

PROOF. According to (4.8) we may assume that $E=E^*(A/\mathfrak{p})$ where $\mathfrak{p}$ is a prime *ideal. We may assume that $(A,\mathfrak{p})$ is *local because $E=E_{(\mathfrak{p})}$ by (4.5). Furthermore if

$$0 \to E \to I^0 \to I^1 \to 0$$

is a (minimal) $A_{(\mathfrak{p})}$-injective resolution for $E$, then it is a (minimal) $A$-injective resolution for $E$.

Let now $\mathfrak{m}$ be a prime ideal containing $\mathfrak{p}$. If $\mathfrak{m}=\mathfrak{p}$, then

$$\operatorname{Ext}^i(A/\mathfrak{m},E) = 0 \quad \text{for } i>0$$

by (4.3). So assume $\mathfrak{m} \neq \mathfrak{p}$. Then there is an $a$ in $\mathfrak{m}-\mathfrak{p}$ such that $\mathfrak{m}=\mathfrak{p}+Aa$ and consequently an exact sequence

$$\text{(4.11)} \quad \xrightarrow{a} \operatorname{Ext}_A{}^i(A/\mathfrak{p},E) \to \operatorname{Ext}_A{}^{i+1}(A/\mathfrak{m},E) \to \operatorname{Ext}_A{}^{i+1}(A/\mathfrak{p},E) \xrightarrow{a}$$

from which it follows that $\mathrm{Ext}^i(A/\mathfrak{m},E)=0$ for $i>1$. Suppose that $\mathrm{id}_A E \geqq 2$ and choose (e.g. by Foxby [3]) a prime ideal $\mathfrak{q}$ such that

$$\mathrm{Ext}_A{}^2(A/\mathfrak{q},E)_\mathfrak{q} \neq 0 .$$

Then $\mathfrak{q}$ must be in the support of $E$ which is $V(\mathfrak{p})$. Therefore $\mathfrak{q}$ is one of the above maximal ideals. We have a contradiction, so $\mathrm{id}_A E \leqq 1$.

To establish the second statement, observe that the exact sequence (4.11) above begins like

$$0 \to A/\mathfrak{p} \xrightarrow{a} A/\mathfrak{p} \to \mathrm{Ext}^1(A/\mathfrak{m},E) \to 0$$

since $\mathrm{Hom}_A(A/\mathfrak{p},E) \cong A/\mathfrak{p}$ by (4.7). Hence $\mathrm{Ext}^1(A/\mathfrak{m},E)$ is cyclic. So if

$$0 \to {}^*E(A/\mathfrak{p}) \to E(A/\mathfrak{p}) \to I^1 \to 0$$

is the minimal injective resolution for ${}^*E(A/\mathfrak{p})$, then $I^1$ contains exactly one copy of $E(A/\mathfrak{m})$ and we are done.

Corollary 4.12. *If $M$ is an $A$-*module, then*

$$\mathrm{id}_A M - 1 \leqq {}^*\mathrm{id}_A M \leqq \mathrm{id}_A M .$$

Matijevic has demonstrated the equivalence of the first two statements in the next corollary.

Corollary 4.13. *The following statements are equivalent.*

(a) *The *ring $A$ is Gorenstein.*

(b) *The local rings $A_\mathfrak{p}$ are Gorenstein for all graded prime ideals $\mathfrak{p}$.*

(c) *The ${}^*\mathrm{id}_{A_{(\mathfrak{p})}} A_{(\mathfrak{p})} < \infty$ for all prime ideals $\mathfrak{p}$ (respectively graded prime ideals).*

Proof. Suppose (a). Note first that $\mathrm{id}_{A_{(\mathfrak{p})}} A_{(\mathfrak{p})} = \sup_\mathfrak{q} \mathrm{ht}\,\mathfrak{q}$ where the supremum is taken over all prime ideals $\mathfrak{q}$ with ${}^*\mathfrak{q}=\mathfrak{p}$. From (2.3) it follows that

$${}^*\mathrm{id}_{A_{(\mathfrak{p})}} A_{(\mathfrak{p})} \leqq \mathrm{id}_{A_{(\mathfrak{p})}} A_{(\mathfrak{p})} \leqq \mathrm{ht}\,\mathfrak{p} + 1 < \infty .$$

Therefore (a) implies (c).

Since $\mathrm{id}_{A_\mathfrak{q}} A_\mathfrak{q} \leqq \mathrm{id}_{A_{(\mathfrak{q})}} A_{(\mathfrak{q})} \leqq {}^*\mathrm{id}_{A_{(\mathfrak{q})}} A_{(\mathfrak{q})} + 1$ for all prime ideals $q$, according to the theorem, we get that (c) implies (b).

Now (b) implies (a) by (4.9).

Note that for a prime *ideal $\mathfrak{p}$ for which $A_{(\mathfrak{p})}$ is Gorenstein we have

$${}^*\mathrm{id}_{A_{(\mathfrak{p})}} A_{(\mathfrak{p})} = \mathrm{ht}\,\mathfrak{p} .$$

Similarly we have the following result.

COROLLARY 4.14. *The following statements are equivalent.*

(a) $A$ *is a* (locally) *regular ring.*
(b) $A_{\mathfrak{p}}$ *is a regular local ring for all graded prime ideals* $\mathfrak{p}$.
(c) $\mathrm{gl}^*\dim A_{(\mathfrak{p})} < \infty$ *for all prime ideals* $\mathfrak{p}$.

PROOF. The proof is the same as for (4.13) except we use (3.5) instead of (4.9).

In [11] Nagata raised the following conjecture: If (the non-negatively graded ring) $A$ is Cohen–Macaulay at all the homogeneous maximal ideals, then $A$ is Cohen–Macaulay. Matijevic and Roberts have (independently) solved the conjecture in the affirmative, see [10]. For completion we have included a proof (in the $\mathbf{Z}$-graded case).

PROPOSITION 4.15. (Matijevic and Roberts). *If* $A_{\mathfrak{p}}$ *is a Cohen–Macaulay ring for all graded prime ideals* $\mathfrak{p}$ *in* $A$, *then* $A$ *is a Cohen–Macaulay ring.*

PROOF. Let $\mathfrak{m}$ be any maximal ideal in $A$ and put $\mathfrak{p} = {}^*\mathfrak{m}$ and $\mathfrak{n} = \mathfrak{m}A_{(\mathfrak{p})}$. We want to show that $A_{\mathfrak{m}}$ is Cohen–Macaulay. But $A_{\mathfrak{m}} = (A_{(\mathfrak{p})})_{\mathfrak{n}}$. Therefore it suffices to assume that $(A, \mathfrak{p})$ is *local with $A_{\mathfrak{p}}$ Cohen–Macaulay, and we are then required to show that $A_{\mathfrak{m}}$ is Cohen–Macaulay for all maximal ideals $\mathfrak{m}$ containing $\mathfrak{p}$ strictly.

Put $d = \operatorname{depth} A_{\mathfrak{p}} = \operatorname{ht}\mathfrak{p}$. By (4.9) we have that $\operatorname{Ext}^d(A/\mathfrak{p}, A)$ is a free $A/\mathfrak{p}$-*module and that $\operatorname{Ext}^i(A/\mathfrak{p}, A) = 0$ for $i < d$. Choose an $a$ in $A$ such that $\mathfrak{m} = \mathfrak{p} + (a)$, cf. (2.2), and consider the long-exact sequence:

$$\ldots \to \operatorname{Ext}^{i-1}(A/\mathfrak{p}, A) \to \operatorname{Ext}^i(A/\mathfrak{m}, A) \to \operatorname{Ext}^i(A/\mathfrak{p}, A) \to \ldots .$$

It follows that $\operatorname{Ext}^i(A/\mathfrak{m}, A) = 0$ for $i < d$ and that $\operatorname{Ext}^d(A/\mathfrak{m}, A)$ is a submodule of $\operatorname{Ext}^d(A/\mathfrak{p}, A)$. Since $\operatorname{Ext}^d(A/\mathfrak{p}, A)$ is a free $A/\mathfrak{p}$-module we obtain $\operatorname{Ext}^d(A/\mathfrak{m}, A) = 0$, and hence

$$\operatorname{depth} A_{\mathfrak{m}} \geqq d + 1 = \operatorname{ht}\mathfrak{p} + 1 \geqq \operatorname{ht}\mathfrak{m}$$

(cf. 2.3) as desired.

REMARK. If $A$ is a homomorphic image (in the graded sense) of a graded Gorenstein ring, say $R$, then the above result follows directly from the fact that the modules of dualizing differentials $\operatorname{Ext}_R{}^i(A, R)$ are graded $A$-modules (and therefore — if non-zero — have graded prime ideals in their supports).

## 5. Complete graded ring of quotients.

Finally we define the complete *ring of quotients. But first we give an example.

EXAMPLE. Let $k$ be a field and $X$ and $Y$ indeterminates over $k$ with $\deg X = 1$ and $\deg Y = -1$. Set $A = k[X,Y]/(XY)$. Then $^*\mathrm{id}_A A = 1$ and all *homogeneous* regular elements are units. Thus $S^{-1}A$ is not self-injective, where $S$ is the set of regular homogeneous elements in $A$.

Let $A$ be arbitrary and denote by $Z$ the set of regular elements in $A$. Set $Q = Z^{-1}A$, so $Q$ is the classical ring of quotients of $A$. When $A$ is noetherian it has only finitely many associated prime ideals and then $Q$ is also the complete ring of quotients of $A$ defined in the sense of Utumi with "dense ideals". (See Lambek [7].) We will copy this idea, using graded regular ideals, i.e. ideals containing a regular element, for our set of dense ideals.

Let $^*Q_n$ denote the set of fractions $a/z$ in $Q$ such that there is a regular *ideal $\mathfrak{a}$ such that $(a/z)\mathfrak{a}_i \subseteqq A_{i+n}$ for all $i$. It is not hard to show that

$$^*Q_n {}^*Q_m \subseteqq {}^*Q_{n+m}$$

and that the sum $\sum_{n\in \mathbf{Z}} {}^*Q_n$ is direct in $Q$. Thus $^*Q = \coprod {}^*Q_n$ becomes a graded ring — the *complete graded ring of quotients of* $A$.

Let $\mathfrak{a}$ be an *ideal which meets $Z$ in $z$. There is an injection

$$\mathrm{Hom}(\mathfrak{a}, A) \to Q$$

defined by $f \to f(z)/z$. Note that then

$$^*Q = \bigcup_{\mathfrak{a}} \mathrm{Hom}(\mathfrak{a}, A) \quad \text{and} \quad {}^*Q_n = \bigcup_{\mathfrak{a}} {}^*\mathrm{Hom}_n(\mathfrak{a}, A)\,,$$

the union taken over all ideals $\mathfrak{a}$ with $\mathfrak{a} \cap Z \neq \emptyset$. We obtain the following inclusions:

$$\begin{array}{ccccccc} A & \to & S^{-1}A & \to & {}^*Q & \to & {}^*E(A) \\ & & & & \downarrow & & \downarrow \\ & & Z^{-1}A & = & Q & \to & E(A) \end{array}$$

where $S = Z \cap \mathrm{h}(A)$ and the maps on the top are in $^*\mathbf{mod}_A$.

PROPOSITION 5.1. *The complete graded ring of quotients $^*Q$ of $A$ is *injective if and only if $Q$ is injective.*

PROOF. Suppose that $^*Q$ is injective. Since $Q = Q \otimes_A {}^*Q$ we have

$$\mathrm{id}_Q Q \leqq \mathrm{id}_A {}^*Q \leqq {}^*\mathrm{id}_A {}^*Q + 1\,.$$

Hence $Q$ is injective (cf. [1]).

If, on the other hand, $Q$ is injective, then $Q=E(A)$. Choose $e \in \mathrm{h}(*E(A))$. Let $\bar{e}$ be its image in $*E(A)/A$, a submodule of $E(A)/A$. Set $\mathfrak{a}=\mathrm{Ann}_A \bar{e}$. Then $\mathfrak{a}$ is a graded ideal (namely the annihilator of a homogeneous element in a *module) and $\mathfrak{a} \cap Z \neq \emptyset$ since $\bar{e} \in Q/A$. Now $e\mathfrak{a}_n \subseteq A_{d+n}$ where $d=\deg e$, so, in fact, we have that $e \in *Q$. Hence the homogeneous elements of $*E(A)$ are in $*Q$, and so $*Q=*E(A)$.

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