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PFISTER'S DIMENSION AND THE LEVEL OF FIELDS

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1.

Let K be a field. For every integer $n \geqq 1$ let $[n]_K$ denote the set of sums of n squares of elements of K, let $[\infty]_K$ denote the set of sums of squares of elements of K.

DEFINITION 1. The *Pfister dimension of* K is ∞ or the smallest integer $n \geqq 1$ such that $[\infty]_K = [n]_K$. We denote it by $\mathrm{Pf}(K)$.

Thus $\mathrm{Pf}(K) = 1$ if and only if every sum of squares of elements of K is the square of an element of K. In this case, K is called a Pythagorean field. For example, if K has characteristic 2, or if K is an algebraically closed field, or a real closed field, or an F-closed field (where F is a formally positive subset) (see [5, pages 145 to 153]), then $\mathrm{Pf}(K) = 1$.

Lagrange's theorem states that $\mathrm{Pf}(\mathsf{Q}) = 4$. Pfister and Cassels have shown that if K is a real closed field then

$$n+1 \leqq \mathrm{Pf}(K(X_1,\ldots,X_n)) \leqq 2^n$$

for every $n \geqq 1$ (see [5, pages 205 and 211]).

Landau has shown that $\mathrm{Pf}(\mathsf{Q}(X)) \leqq 8$ (see [3]) and recently Pourchet proved that $\mathrm{Pf}(\mathsf{Q}(X)) = 5$ (see [4]).

DEFINITION 2. If K is an orderable field, its *level* is infinity. If K is not orderable, the level of K is the smallest integer m such that $-1 \in [m]_K$. We denote by $\lambda(K)$ the level of K.

Pfister has shown that the level of any non-orderable field is a power of 2 (see [5, page 191]).

Hilbert has stated and Siegel published a proof (see [5]) of the fact that the level of any totally imaginary algebraic number field K is at most 4. Connell has indicated the necessary and sufficient condition in order that $\lambda(K) = 2$ (see [1]).

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2.

We state some properties relating the level and the Pfister dimension of a field.

(a) *If* $K \subseteqq L$ *then* $\lambda(K) \geqq \lambda(L)$.

PROOF. This is obvious.

(b) $\lambda(K) = \lambda(K(X)) = \ldots = \lambda(K(X_1, \ldots, X_n))$.

PROOF. If K is orderable, so is $K(X)$ and both fields have infinite level. We assume that K is not orderable, hence $K(X)$ is also not orderable. Let $\lambda(K) = 2^n$, $\lambda(K(X)) = 2^s$ hence $2^m \geqq 2^s$. Since $-1 \in [2^s]_K$, by eliminating denominators, we have

$$-g^2 = \sum_{i=1}^{2^s} h_i^2 \quad \text{with } g, h_i \in K[X] .$$

Let $g = X^r(a_r + a_{r+1}X + \ldots)$ with $r \geqq 0$, $a_r \neq 0$, and let

$$h_i = X^t(b_{i,t} + b_{i,t+1}X + \ldots)$$

with $t \geqq 0$ and $b_{i_0,t} \neq 0$ for at least one index i_0, $1 \leqq i_0 \leqq 2^s$. If $t < r$ we would have

$$0 = \sum_{i=1}^{2^s} b_{i,t}^2$$

and dividing by $b_{i_0,t}^2$ we have

$$-1 = \sum_{i \neq i_0} (b_{i,t}/b_{i_0,t})^2 \in [2^s - 1]_K .$$

Hence $2^m \leqq 2^s - 1 < 2^s$, which is impossible. Similarly, if $r < t$ then $a_r^2 = 0$, which is impossible. So $r = t$ and therefore

$$-a_r^2 = \sum_{i=1}^{2^s} b_{i,r}^2 ,$$

hence

$$-1 = \sum_{i=1}^{2^s} (b_{i,r}/a_r)^2 \in [2^s]_K .$$

Therefore $2^m \leqq 2^s$ and we have the equality $\lambda(K(X)) = \lambda(K)$.

(c) $\mathrm{Pf}(K) \leqq 1 + \lambda(K)$.

PROOF. If K is orderable or if K has characteristic 2, the assertion is true. Now we assume that $\lambda(K) = 2^n$ and K has characteristic unequal to 2. Since $-1 \in [2^n]_K$, for every $x \in K$ we have

$$x = (\tfrac{1}{2}(x+1))^2 + (-1)(\tfrac{1}{2}(x-1))^2 \in [2^n + 1]_K .$$

In particular $[\infty]_K = [2^n + 1]_K$ and so $\mathrm{Pf}(K) \leqq 1 + \lambda(K)$.

(d) *If* $\mathrm{Pf}(K)-1$ *is not a power of* 2 *then* $\mathrm{Pf}(K) \leqq \lambda(K)$.

PROOF. This is true when K is orderable, and also when K is not orderable, as follows from (c) and the hypothesis.

(e) $\mathrm{Pf}(K(X_1,\ldots,X_n)) \leqq 1+\lambda(K)$.

PROOF. This follows from (c) and (b).

(f) *If K has characteristic* 2, $\mathrm{Pf}(K)=\mathrm{Pf}(K(X))=1$. *If K is an orderable field, then* $\mathrm{Pf}(K) \leqq \mathrm{Pf}(K(X))$.

PROOF. We may assume that $\mathrm{Pf}(K(X))$ is finite, say equal to m. Let $a \in [\infty]_K \subseteqq [\infty]_{K(X)} = [m]_{K(X)}$. After eliminating denominators, there exist non-zero polynomials

$$g, f_i \in K[X], \quad i-1,\ldots,m'$$

(where $m' \leqq m$) such that $g^2 a = \sum_{i=1}^{m'} f_i^2$. Let $g = X^r g'$ with $g'(0) \neq 0$, $r \geqq 0$; similarly, let $f_i = X^{r_i} f_i'$ with $f_i'(0) \neq 0$, $r_i \geqq 0$. If $r' = \min\{r_i\} < r$, comparing the terms of degree $2r'$ we would have a non-trivial sum of squares in K equal to 0, which is not possible since K is orderable. Then we may write

$$f_i = X^r(b_{i0}+b_{i1}X+\ldots+b_{is_i}X^{s_i})$$

and comparing the terms of degree $2r$, we have the relation

$$g'(0)^2 a = \textstyle\sum_{i=1}^{m'} b_{i0}^2 ,$$

hence

$$a = \textstyle\sum_{i=1}^{m'} (b_{i0}/g'(0))^2 \in [m]_K .$$

This proves that $\mathrm{Pf}(K) \leqq m$.

It may happen that $\mathrm{Pf}(K) \neq \mathrm{Pf}(K(X))$; for example, $\mathrm{Pf}(\mathsf{Q})=4$, $\mathrm{Pf}(\mathsf{Q}(X))=5$ or also $\mathrm{Pf}(\mathsf{R})=1$, $\mathrm{Pf}(\mathsf{R}(X))=2$.

Similarly, $\mathrm{Pf}(\mathsf{R}(X_1))=2$ and $\mathrm{Pf}(\mathsf{R}(X_1,X_2))=4$ (see [5, page 211]; this is a result of Cassels, Ellison and Pfister). Hence $\mathrm{Pf}(K(X))$ may be larger than $\mathrm{Pf}(K)+1$.

We now discuss what happens for non-orderable fields.

(g) *If K is not an orderable field and has characteristic different from* 2 *then*

$$\mathrm{Pf}(K(X_1,\ldots,X_n)) = 1+\lambda(K) .$$

PROOF. It is enough to consider $n=1$, in view of (b). If $\lambda(K)=1$, then $\operatorname{Pf}(K(X)) \geqq 2$ since X^2+1 is not a square in $K(X)$, because K has characteristic different from 2.

If $\lambda(K)=2^m$, with $m \geqq 1$, we may write

$$-1 = \sum_{i=1}^{2m} a_i^2 \quad \text{with } a_i \in K .$$

Then

$$X^2-1 = X^2+\sum_{i=1}^{2m} a_i^2 \in [2^m+1]_{K(X)} .$$

However $X^2-1 \notin [2^m]_{K(X)}$, otherwise by a lemma of Cassels (see [5, page 194]), we would have $-1 \in [2^m-1]_K$, against the hypothesis on the level of K. This shows that $\operatorname{Pf}(K(X)) \geqq \lambda(K)+1$, hence by (e) $\operatorname{Pf}(K(X)) = 1+\lambda(K)$.

Hence for an arbitrary field K we have $\operatorname{Pf}(K) \leqq \operatorname{Pf}(K(X))$.

If K is a finite field then $\lambda(K)$ is 1 or 2, and correspondingly $\operatorname{Pf}(K(X_1,\ldots,X_n))$ is 2 or 3. If Q_p denotes the field of p-adic numbers then $\lambda(\mathsf{Q}_p) \leqq 4$ (as follows from Hasse's theorem: the polynomial

$$X_1^2+X_2^2+X_3^2+X_4^2+X_5^2$$

has a non-trivial zero in Q_p); hence $\operatorname{Pf}(\mathsf{Q}_p(X_1,\ldots,X_n)) \leqq 5$. Similarly, if K is a totally imaginary algebraic number field, then

$$\operatorname{Pf}(K(X_1,\ldots,X_n)) \leqq 5 .$$

In [5, page 208], we have given the proof of the following result of Pfister:

(h) *Let K be an orderable field, $d \geqq 0$ an integer and assume that for every non-orderable algebraic extension L of K, $-1 \in [2^d]_L$. Then* $\operatorname{Pf}(K(X)) \leqq 2^{d+1}$.

With this result, we have a quick proof of Landau's theorem:

(i) $\operatorname{Pf}(\mathsf{Q}(X)) \leqq 8$.

PROOF. As we quoted, if L is a non-orderable, i.e., totally imaginary algebraic number field, then $\lambda(L) \leqq 4$. It follows from (h) that $\operatorname{Pf}(\mathsf{Q}(X)) \leqq 8$.

3.

Let A be a commutative ring (with unit) and $A[[X_1,\ldots,X_n]]$ the ring of formal power series in n indeterminates. This ring is canonically isomorphic to $(A[[X_1,\ldots,X_{n-1}]])[[X_n]]$. If A is a domain, then so is $A[[X_1,\ldots,X_n]]$. If K is a field, let $K((X_1,\ldots,X_n))$ denote the field of quotients of the domain $K[[X_1,\ldots,X_n]]$.

We shall also consider the following fields, which are defined inductively: $S_0=K$, $S_1=K((X_1))$, $S_n=S_{n-1}((X_n))$ and we denote this field also by $K((X_1))((X_2))\ldots((X_n))$. Up to isomorphism, this field is independent of the order of adjunction of the indeterminates.

Since S_n is the field of quotients of $S_{n-1}[[X_n]]$, it contains $K[[X_1,\ldots,X_n]]$, hence it contains its field of quotients $K((X_1,\ldots,X_n))$. However $S_n \neq K((X_1,\ldots,X_n))$ (when $n \geqq 2$); indeed, $\sum_{i=0}^{\infty} X_1^{-i!}X_2^{i}$ belongs to S_n but not to $K((X_1,X_2))$.

If K is an orderable field then $K((X))$ is also orderable. Indeed, let P be the set consisting of 0 and of all series $F=X^r(a_0+a_1X+\ldots)$ with $r \in \mathsf{Z}$, $a_i \in K$, $a_0>0$ (in a given order of K). Then P is the set of positive elements of a total order on $K((X))$ compatible with the operations and extending the given order of K.

Therefore if K is orderable so are the field $K((X_1))((X_2))\ldots((X_n))$ and the subfield $K((X_1,\ldots,X_n))$.

(j) $\lambda\big(K((X_1))((X_2))\ldots((X_n))\big)=\lambda\big(K((X_1,\ldots,X_n))\big)=\lambda(K)$.

Proof. We have

$$\lambda\big(K((X_1))((X_2))\ldots((X_n))\big) \leqq \lambda\big(K((X_1,\ldots,X_n))\big) \leqq \lambda(K)\,.$$

It is enough to prove the other inequality and we may assume $n=1$, and that K is not orderable. Let $\lambda(K)=2^m$, $\lambda\big(K((X))\big)=2^s$, so $2^m \geqq 2^s$. Since $-1 \in [2^s]_{K((X))}$, by eliminating denominators, we have

$$-G^2 = \sum_{i=1}^{2^s} H_i^2 \quad \text{with } G,H_i \in K[[X]]\,.$$

Let

$$G = X^r(a_0+a_1X+\ldots), \quad a_i \in K,\ a_0 \neq 0\,,$$

let

$$H^i = X^t(b_{i0}+b_{i1}X+\ldots), \quad b_{ij} \in K\,,$$

and for some index i_0, $1 \leqq i_0 \leqq 2^s$, $b_{i_0 0} \neq 0$.

If $t<r$ then $0=\sum_{i=1}^{2^s} b_{i0}^2$ and dividing by $b_{i_0 0}^2$ we have

$$-1 = \sum_{i \neq i_0} (b_{i0}/b_{i_0 0})^2 \in [2^s-1]_K\,,$$

so $2^m \leqq 2^s - 1 < 2^s$, which is a contradiction. Similarly, if $r < t$ then $a_0{}^2 = 0$, which is impossible. So $r = t$ and

$$-a_0{}^2 = \sum_{i=1}^{2^s} b_{i0}{}^2$$

hence

$$-1 = \sum_{i=1}^{2^s} (b_{i0}/a_0)^2 \in [2^s]_K \,.$$

So $2^m \leqq 2^s$, and this proves the equality.

(k) *Let K be a field of characteristic not equal to 2, let*

$$F = X^r(a_0 + a_1X + \ldots) \in K((X)), \quad \textit{where } r \in \mathbf{Z}\,,$$

$a_0 \neq 0$. *Then: F is a square in $K((X))$ if and only if r is even and a_0 is a square in K.*

Proof. If F is a square in $K((X))$, then r is clearly even and a_0 is a square in K. Conversely, let $r = 2s$, let $a_0 = b_0{}^2$, where $b_0 \in K$. We define $b_1, b_2, \ldots, b_m, \ldots$ inductively by the relation

$$\sum_{i+j=m} b_i b_j = a_m \,,$$

so

$$b_m = (2b_0)^{-1}(a_m - \sum_{\substack{i+j=m \\ i,j \neq 0}} b_i b_j) \,.$$

Let $G = X^s(b_0 + b_1X + b_2X^2 + \ldots)$. Then it is immediate that $G^2 = F$.

As a corollary, we have:

(l) *If K has characteristic different from 2, if $F, G \in K((X))$ have orders satisfying $2o(F) < o(G)$, then $F^2 + G$ is a square in $K((X))$.*

Proof. The order of $F^2 + G$ is $2o(F)$ and the coefficient of the term of lowest degree is a square. By (k) $F^2 + G$ is a square in $K((X))$.

We shall compare the Pfister dimensions of K and $K((X))$.

(m) *If K is an orderable field, $F = X^r(a_0 + a_1X + \ldots) \in K((X))$ with $r \in \mathbf{Z}$, $a_0 \neq 0$, then $F \in [m]_{K((X))}$ if and only if r is even and $a_0 \in [m]_K$.*

Proof. We assume $F = \sum_{i=1}^m G_i{}^2$ where $G_i \in K((K))$; let s be the minimum of the orders of the series G_i $(i = 1, \ldots, m)$. By comparing the coefficients of X^{2s} in F and $\sum_{i=1}^m G_i{}^2$, we see that $2s = r$ (since K is orderable and $a_0 \neq 0$) and $a_0 \in [m]_K$.

Let us assume that $r=2s$ and $a_0=\sum_{i=1}^m b_i^2$. Then

$$F = X^{2s}(\textstyle\sum_{i=1}^m b_i^2+a_1X+\ldots)$$
$$= X^{2s}(b_1^2+a_1X+\ldots)+X^{2s}b_2^2+\ldots+X^{2s}b_m^2 .$$

It follows from (k) that $F\in[m]_{K((X))}$.

(n) *If* K *has characteristic* 2, *or if* K *is orderable, then* $\mathrm{Pf}(K)=\mathrm{Pf}(K((X)))$.

Proof. This is trivial when K has characteristic 2, so we assume that K is orderable.

If $\mathrm{Pf}(K)=m$ then $\mathrm{Pf}(K((X)))\leqq m$. In fact, if $F\in[s]_{K((X))}$ then by (m), F has even order and the coefficient a_0 of the term of lowest degree of F is such that $a_0\in[s]_K\subseteqq[m]_K$. Hence by (m), $F\in[m]_{K((X))}$.

Now we show that if $\mathrm{Pf}(K((X)))=m$ then $\mathrm{Pf}(K)\leqq m$. Let

$$a\in[s]_K \subseteqq [s]_{K((X))} \subseteqq [m]_{K((X))} .$$

We apply (m) to the series $F=a$ and conclude that $a\in[m]_K$.

(o) *If* K *is not orderable and has characteristic different from* 2, *then*

$$\mathrm{Pf}(K((X))) = 1+\lambda(K) = \mathrm{Pf}(K(X)) .$$

Proof. Under these hypotheses, we have

$$\mathrm{Pf}(K((X))) \leqq 1+\lambda(K((X))) = 1+\lambda(K) = \mathrm{Pf}(K(X)) ,$$

as follows from (c), (k), and (h).

Conversely, if $\lambda(K)=2^m$ then $-1\in[2^m]_K\subseteqq[2^m]_{K((X))}$. Hence $X\in[2^m+1]_{K((X))}$, since

$$X = (\tfrac{1}{2}(X+1))^2+(-1)(\tfrac{1}{2}(X-1))^2 .$$

We shall prove that $X\notin[2^m]_{K((X))}$. Indeed, if $X=\sum_{i=1}^{2^m}F_i^2$, where $s=o(F_1)$ is the minimum of the orders of the series F_i, by (l),

$$F_1^2+F_{i_1}^2+\ldots+F_{i_k}^2 \quad (\text{with } o(F_{i_1})>s,\ldots o(F_{i_k})>s)$$

is a square in $K((X))$. Hence we may write $X=\sum_{i=1}^{m'}G_i^2$ with $m'\leqq 2^m$ and $o(G_1)=\ldots=o(G_{m'})=s$. Letting

$$G_i = X^s(b_{i0}+b_{i1}X+\ldots) \quad (\text{with } b_{i0}\neq 0)$$

it follows that

$$X = X^{2s}(\textstyle\sum_{i=0}^{m'} b_{i0}^2+c_1X+\ldots) .$$

Hence $\sum_{i=0}^{m'} b_{i0}^2 = 0$ and so

$$-1 \in [m'-1]_K \subseteq [2^m - 1]_K ,$$

against the hypothesis. This proves the statement.

In particular, if K is a Pythagorean field, i.e. $\mathrm{Pf}(K)=1$, then $K((X))$ is Pythagorean if and only if K is orderable. This result was proved already by Griffin (see [2]).

We conclude with a remark and a problem.

The polynomial $f = 1 + X^2 + Y^2 \in \mathsf{R}[X, Y]$ is such that for every $x \in \mathsf{R}$, $y \in \mathsf{R}$,

$$f(x, Y) \in [2]_{\mathsf{R}[Y]}, \quad f(X, y) \in [2]_{\mathsf{R}[X]} ,$$

however by Cassel's result $1 + X^2 + Y^2 \notin [2]_{\mathsf{R}(X, Y)}$.

Is it true that if $f \in \mathsf{Q}[X, Y]$ is such that

$$f(x, Y) \in [2]_{\mathsf{Q}[Y]}, \; f(X, y) \in [2]_{\mathsf{Q}[X]}$$

(for every $x, y \in \mathsf{Q}$) then $f \in [2]_{\mathsf{Q}(X, Y)}$?

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