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PLANES WITH ANALOGUES TO EUCLIDEAN ANGULAR BISECTORS

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To Werner Fenchel on his 70th birthday.

1.

In the euclidean plane the bisector of an angle in a triangle divides the opposite side in the ratio of the adjacent sides. Avoiding angular measure this may be expressed as the following property:

(P) *Inside a (nonstraight) convex angle with legs N_1, N_2 and vertex v there is a ray M with origin v such that any segment $T(a_1,a_2)$ with $a_i \in N_i$, $a_i \neq v$, intersects M in a point $b=b(a_1,a_2)$ for which the distances satisfy*

$$va_1 : va_2 = ba_1 : ba_2 .$$

P is so strong that one might expect it to distinguish the euclidean plane at least among all Desarguesian planes, i.e. the planes whose geodesics fall on the ordinary affine lines. Actually one verifies easily that *every Minkowski plane has property* P, see Note 1.

It is the purpose of this paper to show that P *characterizes the Minkowski planes among all Desarguesian planes and, if the* (by P convex) *circles are differentiable, even among all two-dimensional straight G-spaces,* which we will briefly call straight planes[1].

Whether the differentiability hypothesis is necessary is an open question closely related to a problem which has remained unsolved for more than forty years, namely, whether the Minkowski planes are the only straight planes with convex circles satisfying the parallel axiom.

2.

A straight plane R is given by a metrization xy of the ordinary plane such that any two distinct points a,b lie on exactly one curve, denoted as the *line* $L(a,b)$, which is isometric to the real axis, i.e. representable, given an arbitrary t_0, as $z(t)$ with

(1) $$z(t_0) = a, \quad z(t_0+ab) = b, \quad z(t_1)z(t_2) = |t_1-t_2|$$

[1] All concepts occurring here can be found in [1], but for the convenience of the reader we will briefly recapitulate in Section 2 the properties of straight planes used later.

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and $-\infty < t < \infty$. The arc from a to b on $L(a,b)$ is the *segment* $T(a,b)$ and is represented by (1) restricted to $[t_0, t_0 + ab]$.

Orient the line L and let $p \notin L$. If x traverses L in the positive direction then the line $L(p,x)$ tends to the *asymptote* A^+ to the positive orientation of L and is *independent of* p in the sense that any $q \in A^+$ also leads to A^+. The asymptote A^- to the negative orientation of L is defined analogously. If A^+ and A^- lie on the same line M, then M is called *parallel* to L. But notice that this will, in general, not imply that L is parallel to M, see [1, p. 141].

Convexity of curves or sets in R is defined in the usual way, but, of course, with respect to the $T(x,y)$ as segments. The *circle* $K(p,\varrho)$, $\varrho > 0$, is the locus $px = \varrho$. *The convexity of all circles is equivalent to the existence of exactly one foot f of a given point p on a given line L* (i.e. $pf = \min_{x \in L} px$, see [1, p. 121]).

If L^+ is an oriented line, $p \in L^+$ and x traverses L^+ in the positive sense then $K(x,xp)$ converges for $px \to \infty$ to the so-called *limit circle* $K_\infty(L^+,p)$ through p with central line L^+. Limit-circles $K_\infty(L^+,p_1)$ and $K_\infty(L^+,p_2)$, $p_i \in L^+$, are *equidistant*; they intersect any asymptote A to L^+ in points q_1, q_2 with $p_1p_2 = q_1q_2$ and q_i is the foot of q_j $(i \neq j)$ on $K_\infty(L^+,p_i)$, see [1, p. 138].

An *angular domain* D in R is defined as the convex set bounded by two distinct rays N_1, N_2 (a ray is a halfline) with the same origin v which do not form a line. A ray M in D with origin v which has the property P is called the *bisector of* D or also of N_1 *and* N_2. We say that R has the property P if every angular domain in R possesses a bisector.

3.

We begin by showing that P implies another property P′ from which our assertions will follow.

(2) *If the straight plane R has property* P *then it also satisfies*:

(P′) *The parallel axiom holds. For two distinct parallel lines L_1, L_2 there is a line L (parallel to the L_i) which contains the centers of all segments $T(p_1,p_2)$ with $p_i \in L_i$.*

Let L_1 be given and $p_2 \notin L_1$, $p_1 \in L_1$ and orient L_1. If x traverses L_1 in the positive direction, $L(p_2,x)$ tends to the asymptote $A_2{}^+$ to the positive orientation $L_1{}^+$ of L_1. The bisector of the domain bounded by the rays from x and through p_1, p_2 intersects $T(p_1,p_2)$ in a point y with

$$xp_1 : xp_2 = yp_1 : yp_2 \,.$$

The inequality $|xp_1 - xp_2| \leqq p_1p_2$ implies

$$xp_1 : xp_2 \to 1 \quad \text{as } p_1x \to \infty ,$$

therefore y tends to the center p of $T(p_1,p_2)$ and $L(y,x)$ to the asymptote A^+ to L_1^+ through p, see [1, p. 138].

If $q_1 \in L_1$, $q_2 \in A_2^+$ then by the same argument A^+ passes through the center q of $T(q_1,q_2)$. For, A^+ does not depend on p_1 and the asymptote to L_1^+ through q_2 is also A_2^+.

Also by the same argument, if x traverses L_1 in the negative direction then $L(p_2,x)$ tends to the asymptote A_2^- to the opposite orientation L_1^- of L_1 and $L(y,x)$ to the asymptote A^- to L_1^- through p.

Choose q_1 on L_1 so close to (but different from) p_1 that $L(q_1,p)$ intersects A_2^+ and A_2^- in points q_2^+, q_2^-. Then $T(q_1,q_2^+)$ and $T(q_1,q_2^-)$ have p as common center, so that $q_2^+ = q_2^-$ and A_2^+ and A_2^- fall on the same line L_2 parallel to L_1. Obviously L_2 is the only line through p_2 not intersecting L_1, and since L_1 and p_2 were arbitrary this implies the parallel axiom, in particular L_1 is also parallel to L_2.

The asymptotes A^+, A^- to L_1 through p then also fall on the same line L parallel to the L_i and L contains by the preceding discussion the centers of all $T(q_1,q_2)$ with $q_i \in L_i$.

(3) *If a straight plane has one of the properties* P, P′ *then its circles are convex.*

Because of (2) it suffices to prove that P′ implies the convexity of the circles, or according to Section 2, that a given point $p \notin L$ has exactly one foot f on L.

Let f be a foot of p on L and L_1 the parallel to L through p. The parallel L_0 to L through the center c of $T(p,f)$ bisects by (2) every segment leading from L to L_1, therefore p is a foot of c on L_1. Choose f_2 such that p is the center of $T(f,f_2)$. Since f is a foot of p on L, the point f_2 is a foot on the parallel L_2 to L through f_2. If c_2 is the center of $T(p,f_2)$ then, as before, p is a foot of c_2 on L_1.

If now p had another foot g on L then $T(p,g)$ would intersect L_0 in the center c_0 of $T(p,g)$ and p would be a foot of c_0 on L_1. This leads to a contradiction because c_2, p, c_0 are not collinear so that $T(c_2,c_0)$ intersects L_1 in $q \neq p$ and

$$c_2c_0 = c_2q + qc_0 < c_2p + pc_0 ,$$

but $c_2p \leqq c_2q$ and $pc_0 \leqq qc_0$.

Theorem 1. *A Desarguesian straight plane is Minkowskian if, and only if, it has one of the properties* P, P′.

This follows from the fact, see [1, p. 144], that Desarguesian spaces with convex spheres satisfying the parallel axiom are Minkowskian.

Without the assumption that the space be Desarguesian we derive from the preceding arguments the following additional fact:

(4) *If a straight plane has one of the properties* P,P′, *then a family of parallel lines has a family of parallels lines as common perpendiculars.*

A *perpendicular* to L is a line H intersecting L at a point f such that every point of H has f as foot on L. The existence of a unique perpendicular to a given line through a given point (in two dimensions) follows from the convexity of the circles, see [1, pp. 121, 122].

With the same notation as in the proof of (3) the center of $T(c,f)$ has c as unique foot on L_0. By iterated bisecting and doubling of distances on $L(p,f)$ we obtain a set S dense on $L(p,f)$ such that $L(p,f)$ is perpendicular to every parallel to L through a point of S, so that by continuity $L(p,f)$ is perpendicular to every parallel to L.

Since perpendiculars to the same line do not intersect and the parallel axiom holds, the common perpendiculars must be parallel.

4.

This is as far as we have been able to proceed without differentiability assumptions or the Desargues property. The circles being convex they are differentiable (i.e. have a unique supporting line) except at an at most countable number of points.

(5) *In a straight plane with the parallel axiom and differentiable convex circles parallel lines are equidistant.*

If L is perpendicular to H we call H transversal to L. If $H \cap L = p$ and $x \in L$, $x \neq p$ then H is the supporting line of $K(x,xp)$ at p and hence the only transversal to L at P. This implies that transversals to a given line L are parallel. For, if two distinct transversals H_1, H_2 to L intersected at a point q then the parallel L_0 to L through q would be perpendicular to both H_1 and H_2 at q, and L_0 would have two transversals at q.

Let H be given and L perpendicular to H at p. Orient L obtaining L^+ and let x follow p on L^+. For $px \to \infty$ the circle $K(x,xp)$ tends to the limitcircle $K_\infty(L^+,p)$. On the other hand it follows from the general theory (see [1, p. 147]) that $K_\infty(L^+,p) = H$.

The limitspheres with L^+ as central line are therefore transversals of L. Consequently, see Section 2, they are equidistant and have the perpendiculars to H as common perpendiculars.

THEOREM 2. *If the straight plane R has one of the properties* P, P′ *and differentiable circles then it is Minkowskian.*

The proof which follows can be simplified if it is known beforehand that a pair of mutually perpendicular lines exists, as it does in every Minkowskian geometry.

Let $u(x)$, $-\infty < x < \infty$, represent an arbitrary line M in the form (1). Denote the two sides of M by π^+ and π^-. To any point p we assign coordinates x, y as follows: x is determined by the intersection $u(x)$ of the transversal to M through p. For $p \in M$ put $y = 0$. For $p \notin M$ let f be the foot of p on M and put $y = pf$ if $p \in \pi^+$, $y = -pf$ if $p \in \pi^-$. There is exactly one point with given coordinates x, y, because for a fixed x_0 the ordinate y increases monotonically from $-\infty$ to ∞ on the transversal $x = x_0$ to M, since the parallels to M are by (5) the lines $y = \text{const.}$

Let L be any line which is neither transversal nor parallel to M and $a_i \in L$ $(i = 1, 2, 3)$ with $a_1a_2 = a_2a_3$. If $a_i = (x_i, y_i)$ it is no restriction to assume that $x_1 < x_2 < x_3$ and $y_1 < y_2 < y_3$.

For $i = 1, 3$ let the parallels to M through a_i intersect the perpendicular to M through a_2 at b_i and let the transversals to M through a_i intersect the parallel to M through a_2 at c_i. Then $b_i = (w_i, y_i)$, $c_i = (x_i, z_i)$ with suitable w_i, z_i.

Because of (4) and P′ we have

$$b_1a_2 = a_2b_3 \quad \text{and} \quad c_1a_2 = a_2c_3 .$$

Moreover by (5) $c_1a_2 = u(x_1)u(x_2) = x_2 - x_1$ and similarly $a_2c_3 = x_3 - x_2$.

By definition $b_1a_2 = y_2 - y_1$, $a_2b_3 = y_3 - y_2$, so that

$$(y_1 - y_2) : (x_1 - x_2) = (y_3 - y_2) : (x_3 - x_2) . \tag{6}$$

Bisecting and doubling segments on L and using (6) we obtain a dense set S on L such that for $(x, y) \in S$ the relation

$$\frac{y - y_2}{x - x_2} = \frac{y_1 - y_2}{x_1 - x_2} \tag{7}$$

holds. Therefore (7) is the equation of L. The fact that all lines have linear equations is obviously equivalent to the Desarguesian character of R and the assertion follows from Theorem 1.

NOTE 1. To show that *a Minkowski plane has property* P denote (using the notations of P) by u_i the point $u_i \in N_i$ with $vu_i = 1$ and select an ellipse with center v passing through u_1 and u_2. The ellipse is the unitcircle of a euclidean metric $\varepsilon(x,y)$ (invariant under translations). Let M be the angular bisector of N_1 and N_2 with respect to $\varepsilon(x,y)$. Then

$$\varepsilon(v,a_1):\varepsilon(v,a_2) = \varepsilon(b,a_1):\varepsilon(b,a_2) .$$

Now $\varepsilon(v,u_i)=vu_i=1$ implies $\varepsilon(v,a_i)=va_i$ and $\varepsilon(b,a_1):\varepsilon(b,a_2)=ba_1:ba_2$ holds because a_1,b,a_2 are collinear.

NOTE 2. Since the question probably occurred to the reader we observe:

(8) *If in a straight plane the angles of a triangle possess bisectors then these are concurrent.*

The proof is obvious: If abc is the triangle and the bisectors of the angles at a and b intersect at p, put $L(c,p)\cap T(a,b)=q$. Then

$$cp:qp = ac:aq=bc:bq ,$$

hence

$$ac:bc = aq:bq ,$$

so that q and hence p lies on the bisector of the angle at c.

NOTE 3. The condition P′ can be replaced by a seemingly much weaker one.

(9) *In a straight plane* P′ *is equivalent to:*
(P″) *For any line L and any point $p \notin L$ the centers of the segments $T(p,x)$, $x \in L$, lie on a line L_1.*

That P′ implies P″ is obvious. Let P″ hold and orient L obtaining L^+. When x traverses L^+ in the positive direction L_1 intersects $T(p,x)$ in its center x_1 and the line $L(p,x)$ tends to the asymptote M to L^+ through p. Either M intersects L_1 or M is asymptote to the induced orientation L_1^+ of L_1. The first case is impossible, because px_1 would tend to a finite limit whereas $px \to \infty$.

Repeating the argument yields a sequence of lines L_i, $i=1,2,\ldots$, with induced orientations L_i^+ such that L_{i+1} contains the centers x_{i+1} of all $T(p,x_i)$, $x_i \in L_i$, and M is asymptote to all L_i^+. Consequently L_i converges to M. Since L_i is independent of the orientation L^+ of L, the line M is also an asymptote to the opposite orientation of L, hence parallel to L. Because L and p were arbitrary, the parallel axiom holds, in particular L_1 is parallel to L and M.

Let $x \in L$, then the centers of all $T(x,y)$, $y \in M$, lie on a line L_1', so L_1'

passes through the center x_1 of $T(p,x)$, whence $L_1' = L_1$. Therefore L_1 contains the centers of all $T(x,y)$, $x \in L$, $y \in M$, and P′ follows.

Thus we have as a corollary of Theorem 2 and (9)

THEOREM 3. *If, in a straight plane, for a given line* L *and a given point* $p \notin L$ *the centers of the segments* $T(p,x)$, $x \in L$, *lie on a line and the* (by (3) and (9) convex) *circles are differentiable, then the metric is Minkowskian.*

REFERENCE

1. H. Busemann, *The Geometry of Geodesics*, Academic Press, New York, 1955.

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