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DEFORMATIONS AND MODULI OF RIEMANN SURFACES WITH NODES AND SIGNATURES*

LIPMAN BERS

To Werner Fenchel on his 70th birthday.

In two recent notes [2, 3] I outlined a function theoretical approach to deformation spaces and moduli spaces of compact Riemann surfaces with nodes. The method makes essential use of the Fenchel-Nielsen parametrization of Fuchsian groups. Here I describe a simple device which permits one to extend the results (and also certain results obtained by Mumford and others by methods of algebraic geometry) to the case of Riemann surfaces of finite type with nodes, punctures, and signatures.

A *Riemann surface with nodes*, S, is a connected complex space such that every $P \in S$ has either a fundamental system of neighborhoods isomorphic to the unit disc $|z|<1$, or a fundamental system of neighborhoods isomorphic to the set $z_1 z_2 = 0$ in the unit bicylinder $|z_1|<1$, $|z_2|<1$. In the latter case P is called a *node*. Every component $\sum$ of the complement of the set of nodes is called a *part* of S, and S is called *stable* if every part has the upper half-plane as its universal covering surface, and therefore carries a canonical Poincaré metric.

By a Riemann surface S of *finite type* we mean a stable Riemann surface with or without nodes, such that either $n=0$ and S is compact, or $n>0$ and S is compact except for n punctures. (A puncture can never be at a node.) Such an S has finitely many parts $\sum_1,\ldots,\sum_r$, each part $\sum_j$ is compact of some genus p_j, except for n_j punctures, $3p_j-3-n_j \geqq 0$ (this is the stability condition), and

$$\textstyle\sum_{j=1}^r n_j \;=\; 2k+n$$

where k is the number of nodes. Also, the total Poincaré area of S equals

$$A \;=\; 2\pi \textstyle\sum_{j=1}^r (2p_j-2+n_j)\;.$$

The (arithmetic) genus p of S is defined by the relation

$$A \;=\; 2\pi(2p-2+n)\;.$$

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If one "thickens" each node so as to obtain a smooth surface $\check{S}$, $\check{S}$ is homeomorphic to a compact surface of genus p with n punctures. The pair

$$(p,n)$$

is called the type of S.

By a Riemann surface *with signature* we mean a Riemann surface S of finite type (p,n) together with an assignment of a "ramification number" ν to each puncture; this ν is either an integer >1 or the symbol ∞. The signature of S is the sequence

$$\sigma = (p,n;\nu_1,\nu_2,\ldots,\nu_n)$$

where the ν_j are the ramification numbers, and $\nu_1 \leqq \nu_2 \leqq \ldots \leqq \nu_n$.

(The terms 'ramification number' and 'signature' come from the theory of Fuchsian groups. For the theory of deformations the actual values of the ramification numbers do not matter; only the equivalence classes $L_1,\ldots,L_s$ of punctures with the same ramification numbers are of significance. It would be better to talk not about the signature but about the *signature type* (cf. Engber [4]) of S, that is, about the sequence $[p,n;l_1,\ldots,l_s]$ where l_i is the cardinality of L_i, and $l_1 \leqq l_2 \leqq \ldots \leqq l_s$, $l_1+\ldots+l_s=n$. We retain, however, the traditional terminology, and note explicitely that the Poincaré metric we use has no relation to the signature.)

Let S and S' be two Riemann surfaces of the same signature. A continuous surjection $f\colon S' \to S$ is called a *deformation* if the inverse image of every node of S is either a node of S' or a Jordan curve on a part of S', if, for every part $\sum$ of S, the restriction $f^{-1}|\sum$ is an orientation preserving homeomorphism onto $f^{-1}(\sum)$, and if every puncture of S' corresponds, under f, to a puncture of S with the same ramification number. The *index* of f is the difference $k-k'$ where k and k' are the numbers of nodes of S and S', respectively. A holomorphic deformation is called an *isomorphism*. Its index is, of course, 0.

The *moduli space* M_σ of a signature σ is the set of all isomorphism classes $[S]$ of Riemann surfaces of signature σ.

The equivalence class $[f]$ of a deformation $f\colon S' \to S$ consists of all deformations $S'' \to S$ of the form $\psi \circ f \circ \varphi^{-1}$ where $\varphi\colon S' \to S''$ is a deformation isotopic to an isomorphism and $\psi\colon S \to S$ is a deformation isotopic to the identity. The *deformation space* $D(S)$ consists of all equivalence classes $[f]$ of deformations onto S. To every node $P \in S$ there belongs a *distinguished subset* $\langle P \rangle \subset D(S)$ consisting of all $[f] \in D(S)$ such that $f^{-1}(P)$ is a node of $f^{-1}(S)$.

Every deformation $g\colon S \to S_0$ induces an *allowable mapping* $g_*\colon D(S) \to$

$D(S_0)$ which sends $[f] \in D(S)$ into $[g \circ f] \in D(S_0)$. Also, there is a *canonical mapping* $D(S) \to M_\sigma$, $\sigma =$ signature of S; it takes $[f] \in D(S)$ into $[f^{-1}(S)] \in M_\sigma$.

Following the method of [3] one can define in the deformation spaces and moduli spaces canonical structures of ringed spaces. Then all allowable mappings between deformation spaces and all canonical mappings into moduli spaces become morphisms of ringed spaces.

THEOREM A. *The deformation space $D(S)$ of a Riemann surface S of signature $\sigma = (p, n; \nu_1, \ldots, \nu_n)$ is a complex manifold homeomorphic to $\mathbf{C}^{3p-3+n}$, under a homeomorphism which takes every distinguished subset $\langle P \rangle$ into a coordinate hyperplane. The deformation space is isomorphic to a bounded domain in $\mathbf{C}^{3p-3+n}$. Each distinguished subset $\langle P \rangle$ is a nonsingular hypersurface in $D(S)$.*

THEOREM B. *An allowable mapping $g_*: D(S) \to D(S_0)$ is a (holomorphic) universal covering of the complement of l distinguished subsets, where l is the index of the deformation $g: S \to S_0$. If $l = 0$, g_* is an isomorphism.*

THEOREM C. *The moduli space M_σ is a compact normal complex space (and a V-manifold).*

For the case $\sigma = (p, 0)$, $p > 1$, that is, for compact S, function-theoretical proofs of Theorems A, B, C are sketched in [2, 3]; a detailed presentation will appear elsewhere. Theorem C was proved originally, for $\sigma = (p, 0)$, by Mayer and Mumford (cf. [7]).

THEOREM D. *The moduli space M_σ is a projective variety.*

For $\sigma = (p, 0)$, $p > 1$, the only known (algebraic-geometric) proof of this will appear in the forthcoming paper by F. Knudson and Mumford [7].

In order to extend Theorems A through D to the case $n > 0$, we shall associate to every Riemann surface S of signature $\sigma = (p, n; \nu_1, \ldots, \nu_n)$ a Riemann surface $\alpha(S)$ of signature $(\tau, 0)$, that is, a compact Riemann surface of genus $\tau = \tau(\sigma)$. This is accomplished by attaching to each of the punctures on S a "tagging" Riemann surface determined by the ramification number.

A Riemann surface is called *terminal* if every part is of type $(0, 3)$. Every deformation of such a surface is equivalent to an isomorphism. The topology, and hence also the complex structure, of a terminal Riemann surface can be described by a one-dimensional connected graph with the following properties. An edge of the graph has either two distinct endpoints (and is called open) or its two endpoints are identified (and it is called closed). Every vertex of the graph is either incident with exactly one open edge (such a vertex is called exceptional) or with exactly one closed edge, or with exactly three distinct open edges. The non-exceptional vertices are in a one-to-one correspondence with the parts of S. An open edge is either incident with an exceptional vertex v and with a non-exceptional vertex v', and corresponds to a puncture on the part corresponding to v', or is incident with two non-exceptional vertices, v_1 and v_2, and corresponds to a node joining the parts corresponding to v_1 and v_2. A closed edge, finally, incident with a vertex v, corresponds to a node joining the part corresponding to v to itself.

Let $t \geqq 3$ be an integer. We denote by V_t the terminal surface of type $(2t+2, 1)$, with $6t+4$ nodes, defined by a graph with $4t+4$ vertices which we denote by $a_0, a_1, \ldots, a_{t+2}$; $b_1, b_2, \ldots, b_t$; $c_1, c_2, \ldots, c_{t+1}$, $d_1, d_2, \ldots, d_{t-1}$ and e, and with the following edges: $4t+6$ open edges joining a_i to a_{i-1} and to a_{i+1} $(1 \leqq i \leqq t+1)$, a_{t+2} to a_0, a_0 to e, a_i to b_i $(1 \leqq i \leqq t)$, a_{t+1} to c_t, a_{t+2} to c_{t+1}, c_i to c_{i+1} and to c_{i-1} $(2 \leqq i \leqq t)$, c_{t+1} to c_1, c_i to d_i $(1 \leqq i \leqq t-1)$, and $2t-1$ closed edges incident with b_i $(1 \leqq i \leqq t)$ and with d_i $(1 \leqq i \leqq t-1)$.

If S is a Riemann surface with signature σ, let $Q_1, \ldots, Q_n$ be its punctures, ordered so that Q_j has ramification number ν_j. Let $V^1, \ldots, V^n$ be disjoint Riemann surfaces, V^j isomorphic to V_{2p+3} if $\nu_j = \infty$, to $V_{2p+2+\nu_j}$ if $\nu_j < \infty$. We denote by $\alpha(S)$ the Riemann surface obtained by joining each puncture Q_j with the unique puncture of V^j into a node. Then $\alpha(S)$ is a compact Riemann surface with nodes, of genus $\tau = p + p_0$ where p_0 is the sum of the genera of $V^1, \ldots, V^n$. If S has k nodes, $\alpha(S)$ has $k+n+k_0$ nodes where k_0 is the number of nodes on $V^1, \ldots, V^n$. If $n = 0$, then $\alpha(S) = S$.

If S' is another Riemann surface with signature σ, and $f: S' \to S$ is a deformation, we denote by $\alpha(f)$ the unique deformation of $\alpha(S')$ onto $\alpha(S)$ such that

$$\alpha(f) \mid S = f, \ \alpha(f) \mid V^j \text{ is an isomorphism}, \quad 1 \leqq j \leqq n .$$

If S and S' are compact, $\alpha(f) = f$. In all cases, f and $\alpha(f)$ have the same index.

We observe that α is a functor from the category of Riemann surfaces (with signature) and deformations to the subcategory of compact surfaces; this means that $\alpha(\mathrm{id}) = \mathrm{id}$, $\alpha(f_1 \circ f_2) = \alpha(f_1) \circ \alpha(f_2)$.

Also, the following statements are true for a *fixed* signature σ. Two Riemann surfaces, $\alpha(S)$ and $\alpha(S')$ are isomorphic if and *only* if S and S'

are, and two deformations, $\alpha(f)$ and $\alpha(f')$ are equivalent if and *only* if f and f' are. (Here isomorphism means holomorphic deformation, not isomorphism in the category mentioned above.) Every deformation $\alpha(S') \to \alpha(S)$ is equivalent to one of the form $\alpha(f)$.

It follows that the deformation space $D(S)$ of a Riemann surface S of signature σ (with $n > 0$) may be identified with the intersection of k distinguished subsets of $D(\alpha(S))$, and that every allowable mapping $g_*: D(S) \to D(S_0)$ is the restriction of the allowable mapping $\alpha(g)_*$: $D(\alpha(S)) \to D(\alpha(S_0))$.

It is now not difficult to extend the validity of Theorems A, B, C to the case of signatures σ with $n > 0$. (For Theorem C, it is necessary to go back to the proof of the theorem for $n = 0$). To obtain Theorem D one shows (by going back to the proof of Theorem C for $n = 0$) that M_σ is an analytic subvariety of $M_{(\tau, 0)}$, and appeals to Theorem D for $n = 0$ (which asserts that $M_{(\tau, 0)}$ is projective) and to Chow's theorem.

A similar argument gives the following

COROLLARY. *The set R_σ of conjugacy classes of Fuchsian groups of signature σ has the structure of a quasi-projective algebraic variety.*

The proof hinges on the observation that R_σ can be identified with the subset of M_σ corresponding to Riemann surfaces without nodes.

For $\sigma = (p, 0)$, the Corollary is the well-known theorem of Baily [1]. For $\sigma = (p, n; \infty, \ldots, \infty)$ the Corollary has been proved, by a different method, by J. Gilman [5].

We note in conclusion that the functor α gives, at once, an extension to the case of arbitrary signatures of the results in [3] concerning holomorphic families of q-canonical embeddings.

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COLUMBIA UNIVERSITY, NEW YORK, U.S.A.