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A REMARK ON DEGENERATE GROUPS

BERNARD MASKIT*

To Werner Fenchel on his 70th birthday.

Aside from existence and non-uniqueness (both due to Bers [3]), very little is known about degenerate Kleinian groups. It was shown by Abikoff [1] that the limit set is not locally connected. In this note we show that the limit set remains connected if we remove from it the limit set of a Schottky subgroup. The main ingredient in the proof is an observation concerning simply-connected regions which are invariant under Schottky groups.

1. Definitions.

Let G be a Kleinian group (i.e., a discrete subgroup of $\mathrm{PSL}(2;\mathbf{C})$ which operates discontinuously at some point of $\hat{\mathbf{C}}=\mathbf{C}\cup\{\infty\}$).

A point z is a *limit point* of G if there is a sequence $\{g_n\}$ of distinct elements of G and there is a point $x\in\hat{\mathbf{C}}$ with $g_n(x)\to z$. The limit set is denoted by $\Lambda(G)$ and its complement, the *set of discontinuity* is denoted by $\Omega(G)$.

A finitely generated Kleinian group G is a *degenerate group* if $\Omega(G)$ is both connected and simply-connected, and $\Lambda(G)$ contains at least 3 points.

2. Schottky groups.

Let D be a region of $\hat{\mathbf{C}}$ bounded by $2n$ disjoint Jordan curves $C_1,C_1',\ldots,C_n,C_n'$. Suppose there are fractional linear transformations $A_1,\ldots,A_n$ so that

$$A_i(C_i) = C_i', \quad \text{and} \quad A_i(D)\cap D = \emptyset, \qquad i=1,\ldots,n\,.$$

Then H, the group generated by $A_1,\ldots,A_n$ is called a *Schottky group.*

We will need the following well-known facts about Schottky groups. H is Kleinian; D is a fundamental domain for H (i.e., $h(D)\cap D=\emptyset$, for all $h\in H$, $h\neq 1$, and $\bigcup_{h\in H} h(\bar{D})\supset\Omega(H)$); D is relatively compact in $\Omega(H)$.

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It was shown by Chuckrow [4] that if H' is another Schottky group and if $\psi: H \to H'$ is an isomorphism, then there is a quasiconformal homeomorphism $w: \mathbf{C} \to \mathbf{C}$ so that $w \circ h \circ w^{-1} = \psi(h)$ for all $h \in H$.

It was shown in [5] that a finitely generated Kleinian group is a Schottky group if and only if it is free and purely loxodromic.

3. Invariant discs.

THEOREM 1. *Let H be a Schottky group and let T be a connected and simply connected open set which is invariant under H, where $\partial T \cap \Omega(H)$ is a collection of disjoint Jordan arcs, each of which appears only once on the boundary of T. Then ∂T is a Jordan curve.*

PROOF. Let ζ be the Riemann map from the unit disc U onto T. Then ζ^{-1} conjugates H onto a Fuchsian group Γ. We denote the isomorphism from Γ to H defined by $\gamma \to \zeta \circ \gamma \circ \zeta^{-1}$ by ψ.

Since H is purely loxodromic, by a lemma of Ahlfors [2], Γ must be purely hyperbolic; hence [5] Γ is a Schottky group. By Chuckrow's theorem [4], there is a quasiconformal homeomorphism $w: \hat{\mathbf{C}} \to \hat{\mathbf{C}}$ with $\psi(\gamma) = w \circ \gamma \circ w^{-1}$ for all $\gamma \in \Gamma$.

We next extend ζ to act on ∂U. If $z \in \partial U \cap \Omega(\Gamma)$, then using the strong form of the Riemann Mapping Theorem we can define ζ to be continuous at z, and $\zeta(z) \in \partial T \cap \Omega(H)$. If $z \in \Lambda(\Gamma)$, we set $\zeta(z) = w(z)$.

In order to complete the proof of this theorem, we need to show that ζ is continuous at every point of $\Lambda(\Gamma)$, for we already know that ζ is injective on both $\Lambda(\Gamma)$ and $\Omega(\Gamma)$, and that $\zeta(\Omega(\Gamma)) \cap \zeta(\Lambda(\Gamma)) = \emptyset$.

Let D be a fundamental domain for H, and let $D' = \zeta^{-1}(D \cap \zeta(U))$; then D' is a fundamental domain for the action of Γ on U, and D' is relatively compact in $\Omega(\Gamma)$.

Let z be some point of $\Lambda(\Gamma)$ and let $\{z_n\}$ be a sequence of points of U with $z_n \to z$. Choose $\gamma_n \in \Gamma$ and $s_n \in \bar{D}'$ so that $\gamma_n(s_n) = z_n$. Since D' is relatively compact in $\Omega(\Gamma)$, $\gamma_n(s) \to z$ for all $s \in \Omega(\Gamma)$ (see Lemmas 1 and 2 in [6]). Now

$$\zeta(z_n) = \zeta \circ \gamma_n(s_n) = \psi(\gamma_n) \circ \zeta(s_n) = w \circ \gamma_n \circ w^{-1} \circ \zeta(s_n) .$$

The points $\zeta(s_n)$ lie in D and so $w^{-1} \circ \zeta(s_n)$ all lie in a relatively compact subset of $\Omega(\Gamma)$; hence

$$\gamma_n \circ w^{-1} \circ \zeta(s_n) \to z .$$

Hence $\zeta(z_n) = w \circ \gamma_n \circ w^{-1} \circ \zeta(s_n) \to w(z)$.

COROLLARY. *Let T be a disc as in Theorem 1 where each of the Jordan arcs of $\partial T \cap \Omega(H)$ is smooth. Then T is a quasi-disc.*

PROOF. T and its exterior are both connected invariant open sets, and so the result in [7] is applicable.

4. Degenerate groups.

THEOREM 2. *Let G be a degenerate group and let H be a Schottky subgroup of G. Then $\Lambda^* = \Lambda(G) - \Lambda(H)$ is connected.*

We set $X = \Omega(H)/H$; X is then a closed Riemann surface and we denote the natural projection $\Omega(H) \to X$ by p.

Since $H \subset G$, $\Omega(G) \subset \Omega(H)$. By assumption, $\Omega(G)$ is connected, hence $p(\Omega(G))$, the complement of $p(\Lambda^*)$ is also connected.

LEMMA 1. *Every neighborhood of $p(\Lambda^*)$ contains a neighborhood N of $p(\Lambda^*)$ with the following properties:*

i) *The complement of N is connected.*

ii) *N is bounded by a finite number of smooth arcs.*

iii) *N carries no unnecessary homotopy; i.e., if $N' \subset N$ is another neighborhood of $p(\Lambda^*)$ then the inclusion induces an isomorphism of fundamental groups.*

PROOF. We use the Poincaré metric on X, and observe that since $p(\Lambda^*)$ is compact, for ε sufficiently small, the ε-neighborhood of $p(\Lambda^*)$ satisfies iii); of course any smaller neighborhood also satisfies iii). Next, since the complement of $p(\Lambda^*)$ is connected, any neighborhood contains one satisfying i). Finally, we approximate the boundary so as to satisfy ii).

We choose a neighborhood N as in Lemma 1, and let Y be the complement in X of $p(N)$. Let T be a component of $p^{-1}(Y)$.

LEMMA 2. *T is simply-connected.*

PROOF. Let v be some loop in T. Since T is contained in $\Omega(G)$ which is simply connected, v is contractible in $\Omega(G)$; hence v is contractible in $\Omega(H) - \Lambda^*$, and so v is contractible in T.

LEMMA 3. *For N sufficiently small, T is invariant under H.*

PROOF. Let z_0 be some point of T, and let $A_1, \ldots, A_n$ be generators of H. Since $\Omega(G)$ is connected, we can find paths connecting z_0 to $A_i(z_0)$ which lie in $\Omega(G)$. These paths are at some finite distance, in the Poincaré metric, from Λ^*, and so for N sufficiently small, these paths all miss $p^{-1}(N)$. Then these paths all lie in T, and so $A_i(T) = T$, for each of the generators $A_1, \ldots, A_n$.

We now know that for N sufficiently small, T satisfies the hypotheses of Theorem 1; hence $p^{-1}(Y)$ is connected, simply-connected, and invariant under H. We conclude that $p^{-1}(N)$ is connected, simply-connected, and invariant under H. Therefore $p(\Lambda^*)$ is connected and since N contains no unnecessary homotopy, $p^{-1}(p(\Lambda^*))$ is also connected; i.e., Λ^* is connected. This completes the proof of Theorem 2.

We remark in conclusion that in this proof we did not need G to be finitely generated. We needed only that $\Omega(G)$ is connected, simply-connected and hyperbolic.

REFERENCES

1. W. Abikoff, *Two theorems on totally degenerate groups*, Amer. J. Math., to appear.
2. L. V. Ahlfors, *Finitely generated Kleinian groups*, Amer. J. Math. 86 (1964), 413–429.
3. L. Bers, *On boundaries of Teichmüller spaces and on Kleinian groups*: I, Ann. of Math. 91 (1970), 570–600.
4. V. Chuckrow, *On Schottky groups with applications to Kleinian groups*, Ann. of Math. 88 (1968), 47–61.
5. B. Maskit, *A characterization of Schottky groups*, J. Analyse Math. 19 (1967), 227–230.
6. B. Maskit, *On Klein's combination theorem*, Trans. Amer. Math. Soc. 120 (1965), 499–509.
7. B. Maskit, *On boundaries of Teichmüller spaces and on Kleinian groups*: II, Ann. of Math. 91 (1970), 607–639.

STATE UNIVERSITY OF NEW YORK
STONY BROOK, NEW YORK, U.S.A.