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FUNDAMENTAL DOMAINS FOR FINITELY GENERATED KLEINIAN GROUPS

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To Werner Fenchel on his 70th birthday

1. Introduction.

Let G be any non-elementary Kleinian group acting on the extended complex plane $\mathbf{C}\cup\{\infty\}$. Thus $\mathbf{C}\cup\{\infty\}$ is the disjoint union of the non-empty sets L (the limit points of G) and Ω (the ordinary points of G).

For any set $E\subseteq\mathbf{C}\cup\{\infty\}$ we denote by $\bar{E}$ and $\tilde{E}$ the closure of E in $\mathbf{C}\cup\{\infty\}$ and Ω respectively. The boundary of E in $\mathbf{C}\cup\{\infty\}$ is denoted by ∂E.

An open (not necessarily connected) subset D of Ω is a fundamental domain for G if and only if each point in Ω is G-equivalent to at least one point in $\tilde{D}$ and at most one point in D. When reasonably defined D has "sides"; it is of interest to know whether D has only finitely many sides. When G is Fuchsian it is well known that this property is equivalent to G being finitely generated.

In this paper we apply *Ahlfors' Finiteness Theorem* [1] to discuss the general situation for finitely generated groups. Kra [3, p. 74] has shown that if G is finitely generated then D may be chosen to be finite sided in each component of Ω and MacMillan indicated the more explicit result that the Ford fundamental region for G (when defined) is finite sided (seminar talk at the Mittag-Leffler Institute, the autumn 1971). We shall prove MacMillan's Theorem.

2. Preliminaries.

We introduce the ideas required for our results.

We shall say that D is *locally finite in* Ω if and only if each point of Ω has a neighbourhood which meets only finitely many images $g(D)$ of D for g belonging to G (see [2] for a discussion of locally finite domains). If D is locally finite in Ω then each compact subset of Ω meets only finitely many images of D.

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Next, we consider the boundary of D. There is no reason to assume that D has sides. We shall call $g(\tilde{D})$ a *neighbour* of $\tilde{D}$ if and only if $\tilde{D} \cap g(\tilde{D}) \neq \emptyset$. We seek conditions under which $\tilde{D}$ has only finitely many neighbours. If this is so and if $\tilde{D}$ meets its neighbours in connected sides, then D has only a finite number of sides. If D is locally finite in Ω, then D meets only finitely many neighbours in any compact subset of Ω.

Both Kra and MacMillan rely on Ahlfors' Finiteness Theorem and we too shall use it. We assume that G is finitely generated and adjoin to Ω a finite number of orbits of parabolic fixed points called *cusps*; the resulting space is denoted by Ω^* and Ω^*/G is a finite union of compact Riemann surfaces. If p is a cusp there is a neighbourhood N of $\pi(p)$ on Ω^*/G (π is the natural projection of Ω^* onto Ω^*/G) such that

$$\pi^{-1}(N) = \bigcup_{g \in G} g(H \cup \{p\})$$

where H is an open disc (called a *horocycle*) in Ω containing p on its boundary and where for any two elements g_1 and g_2 of G, $g_1(H)$ meets $g_2(H)$ if and only if ${g_1}^{-1}g_2$ is a parabolic element of G fixing b in which case $g_1(H) = g_2(H)$. In particular, H contains no fixed points of elements in $G \setminus \{\mathrm{id}\}$. We may assume, of course, that $\bar{H} \setminus \{p\} \subset \Omega$.

It is natural to try to extend the definition of locally finite to Ω^* rather than Ω. If H is a horocycle at a cusp p and if H meets $\tilde{D}$, then H meets all images of $\tilde{D}$ under the parabolic elements fixing p. It is natural, therefore, to say that D is *locally finite in* Ω^* if and only if

(i) D is locally finite in Ω and

(ii) each cusp has an associated horocycle H which meets $g(\tilde{D})$ for those g lying only in a finite set of cosets, say $G_p h_1, \ldots, G_p h_r$, where $G_p = \{P^n\}$ is the cyclic subgroup of parabolic elements fixing p.

Observe that $\tilde{D}$ meets $g(H)$ if and only if

$$(1) \qquad g^{-1} \in G_p h_1 \cup \ldots \cup G_p h_r$$

that is if and only if $g(H) = h_j^{-1}(H)$ for some j. Thus D is locally finite in Ω^* if and only if, firstly, it is locally finite in Ω and, secondly, $\tilde{D}$ meets only a finite number of horocycles from each equivalence class of horocycles.

We can now establish our first result.

THEOREM 1. *Let G be a finitely generated Kleinian group and let D be a fundamental domain which is locally finite in Ω^*. Then D is contained in the union of a compact subset of Ω and a finite number of horocycles.*

Proof. As G is finitely generated, there are only a finite number of equivalence classes of horocycles. As D is locally finite in Ω^*, D meets only a finite number of horocycles, say $H_1,\ldots,H_S$.

Now select any sequence of points z_n in $\tilde{D}\setminus(H_1\cup\ldots\cup H_S)$; after passing to a subsequence and relabeling we may assume that $\pi(z_n)$ converges to some point ω in Ω^*/G. If ω were a point on Ω^*/G corresponding to a cusp, then z_n would eventually lie in the union of the horocycles and this is not so. Thus ω is $\pi(z)$ for some z in $\tilde{D}\setminus(H_1\cup\ldots\cup H_S)$. We select a small neighbourhood N of z in Ω and conclude that for all but a finite set of indices n, there are elements g_n in G with $z_n\in g_n(N)$. This means that $g_n^{-1}(\tilde{D})$ meets N (in $g_n^{-1}(z_n)$) and so g_n belongs to a finite set of elements of G (only finitely many images of D meet N). We conclude that for infinitely many n, $g_n=g$, say and so $z_n\in g(N)$. This shows that the original sequence has a convergent subsequence and so $\tilde{D}\setminus(H_1\cup\ldots\cup H_S)$ is a compact subset of Ω.

If D meets infinitely many components of Ω, say in points z_n, then a subsequence of the z_n converges to a point of L. We thus have the following

Corollary. *Let G and D be as in Theorem* 1. *Then D meets only a finite number of components of Ω.*

3. The main result.

We continue to explore the situation described in Theorem 1. If $\tilde{D}$ has infinitely many neighbours, then it must meet infinitely many of these in *one* horocycle H. Using (1), this means that $\tilde{D}$ must meet infinitely many neighbours of the form $P^n h_j(\tilde{D})$ in H (where j is a given integer).

This latter situation may occur unless we impose further restrictions on D. For example this is so if we consider the cyclic group generated by $z\to z+1$ and put $Q_n=\{|z-in|<\frac{1}{2}\}$ and

$$D = (\{0<x<1\}\setminus \textstyle\bigcup_{n=1}^{\infty} Q_n)\cup(\bigcup_{n=1}^{\infty} P^n Q_n)\,.$$

We shall say that D is *cusped* if and only if there exists a system of horocycles H as described earlier such that if G is transformed so that H becomes the upper half-plane $\{y>0\}$ and p the point at infinity, then $D\cap H$ is contained in some half-strip $[a,b]\times]0,\infty[$ and further, if $D\cap H\neq\emptyset$, then $D\cap\partial H\neq\emptyset$.

It seems difficult to obtain a suitable elegant condition for our purposes; however, we can now prove the following result.

THEOREM 2. *Let G be a finitely generated Kleinian group and D a fundamental domain which is cusped and locally finite in Ω. Then D has only a finite number of neighbours.*

PROOF. We first prove that D is locally finite in Ω^*. If not, then there is a horocycle H which meets $g_i(\tilde{D})$, $i=1,2,\ldots$, where the cosets $G_p g_i$ are distinct. It follows that $\tilde{D}$ meets $g_i^{-1}(H)$ and so meets $\partial g_i^{-1}(H)$ in Ω (this is because D is cusped). Thus $g_i(\tilde{D})$ meets ∂H and so for some n_i, $P^{n_i} g_i(\tilde{D})$ meets a fixed compact subset of $\Omega \cap \partial H$, where P generates G_p. If the cosets $G_p g_i$ are distinct, then the elements $P^{n_i} g_i$ are distinct and the above result contradicts the fact that D is locally finite in Ω. Thus D is locally finite in Ω^*.

We have seen above that if this is so and if D has infinitely many neighbours, then $\tilde{D}$ meets infinitely many neighbours of the form $P^n g(\tilde{D})$ in H (where g is given). Thus $g^{-1}(H)$ meets $\tilde{D}$ and so $\tilde{D} \cap g^{-1}(H)$ lies in some half-strip (determined by two tangent circular axes) in $g^{-1}(H)$. Thus $g(\tilde{D}) \cap H$ lies in some half-strip in H as does $\tilde{D} \cap H$. We now see that $P^n g(\tilde{D})$ can only meet $\tilde{D}$ for a finite set of values of n and the proof is complete.

4. MacMillan's Theorem.

If ∞ is an ordinary point of G fixed only by the identity in G (as we shall assume in the following), then G has a Ford fundamental region F. It is defined as the set of points lying exterior to the isometric circle of each element in $G \setminus \{\mathrm{id}\}$ (for instance, see [4]).

Certainly F is locally finite in Ω for if K is a compact subset of Ω, then K lies outside all but a finite number of isometric circles, and if g is not the identity, then $g(F)$ lies inside the isometric circle of g^{-1}. Thus we see that K meets only finitely many images of F.

We shall prove

THEOREM 3. *If G is finitely generated, then F is bounded by a finite number of sides (circular arcs).*

PROOF. Let R be the function which to a transformation (not fixing ∞) assigns the radius of its isometric circle.

If P is parabolic with fixed point p and g does not fix ∞, then an easy computation yields

$$R(P)R(g)^2 = R(gPg^{-1})|g^{-1}(\infty)-p|^2 .$$

Thus, since the pole of g is the center of the isometric circle of g, we see that a necessary and sufficient condition for p not to lie interior to some isometric circle is that

$$R(P) = \max\{R(gPg^{-1}) \mid g \in G\} .$$

Since almost all the isometric circles are extremely small it also follows that only finitely many points can be equivalent to p and not lie inside some isometric circle; in other words, each parabolic cycle is finite. Hence, using Ahlfors' Finiteness Theorem, we see that only finitely many parabolic fixed points can appear on the boundary of F.

Next, we suppose that F meets a horocycle H and that the parabolic fixed point p belongs to the boundary of $F \cap H$. We may assume that H does not contain any pole.

Define an equivalence relation on G by

$$g \sim h \Leftrightarrow g^{-1}h \in G_p .$$

Thereby, G splits into cosets:

$$G = G_p + g_1G_p + g_2G_p + \ldots$$

We pick from each coset a representative whose pole does not lie inside the isometric circles of P or P^{-1} and consider those, say $h_1, h_2, \ldots$, whose poles lie in the same open half-plane, invariant under G_p, as H does. As before, P denotes a generator of G_p.

Since $p \in \partial(F \cap H)$, we see that there exists a natural number n_0 such that the isometric circles of P^{n_0} and P^{-n_0} lie exterior to the isometric circle of h_j for each $j=1,2,\ldots$. It implies that the isometric circle of h_jP^n lies inside the isometric circle of P^{n_0} or P^{-n_0} as soon as $|n| \geqq n_0$. For $|n| < n_0$, the isometric circles of the elements h_jP^n have centres lying in the open G_p-invariant half-plane containing H and exterior to H as well as to the isometric circles of P^{2n_0} and P^{-2n_0}. Thus we see that none of the isometric circles with centres in the open G_p-invariant half-plane containing H can contribute to the boundary of $F \cap H$ sufficiently near p and, of course, the same is true for those circles with centres lying in the opposite open half-plane. Consequently, the boundary of F consists of a smaller horocycle of two circular arcs which are tangent at p, since only finitely many isometric circles with centres on the G_p-invariant straight line can come close to p without lying inside the isometric circles of P or P^{-1}.

We conclude from what has been shown that F is cusped and so, being locally finite, F has only finitely many neighbours and, obviously, is bounded by finitely many sides.

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