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INVARIANT OPERATORS AND INTEGRAL REPRESENTATIONS IN HYPERBOLIC SPACE

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To Werner Fenchel on his 70th birthday.

1. Introduction.

1.1. The complex derivative $f_{\bar z} = \frac{1}{2}(f_x + i f_y)$ measures the lack of conformality of an infinitesimal deformation $f\colon \mathsf{C} \to \mathsf{C}$. Indeed,

$$F(z,\varepsilon) = z + \varepsilon f(z) + o(\varepsilon)$$

has the complex dilatation $F_{\bar z}/F_z = \varepsilon f_{\bar z} + o(\varepsilon)$.

There is a natural generalization of this notion, not to several complex variables, but to n real variables. Suppose that $F\colon \mathsf{R}^n \to \mathsf{R}^n$ is a one parameter family of mappings with the development

$$F(x,\varepsilon) = x + \varepsilon f(x) + o(\varepsilon)\,.$$

With F we associate its Jacobian matrix F' and the positive definite symmetric matrix ${}^tF' \cdot F'$. Because conformality disregards size we normalize by passing to

$$(\det F')^{-2/n}\, {}^tF' \cdot F' = 1_n + \varepsilon[f' + {}^tf' - 2n^{-1}(\operatorname{tr} f')1_n] + o(\varepsilon)\,.$$

This is an indication that the matrix

$$Sf(x)_{ij} = \tfrac{1}{2}(\partial f_i/\partial x_j + \partial f_j/\partial x_i) - n^{-1}\delta_{ij} \textstyle\sum_1^n (\partial f_k/\partial x_k) \tag{1.1}$$

is a suitable measure for the deviation from conformality of the infinitesimal deformation f.

For $n=2$ it is customary to use the complex notation $f = f_1 + i f_2$, $z = x_1 + i x_2$. In this notation

$$Sf = \begin{pmatrix} \operatorname{Re} f_{\bar z} & \operatorname{Im} f_{\bar z} \\ \operatorname{Im} f_{\bar z} & -\operatorname{Re} f_{\bar z} \end{pmatrix}.$$

This confirms our contention that Sf is a natural generalization of $f_{\bar z}$.

1.2. The operator S maps vector fields on matrix valued functions. We shall use the customary notations of euclidean geometry:

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$$x = (x_1,\dots,x_n), \quad f = (f_1,\dots,f_n)\,,$$
$$xy = x_1y_1+\dots+x_ny_n, \quad |x| = (xx)^{\frac{1}{2}}\,.$$

The dimension n shall be fixed and $\geqq 2$. For the purpose of this paper we may assume that all functions are C^∞. In a preliminary way $\mathscr{V}$ will denote the vector space of mappings $f\colon \mathsf{R}^n \to \mathsf{R}^n$, and $\mathscr{W}$ will be the space of $n\times n$ matrixvalued functions φ with $\varphi = {}^t\varphi$ and $\operatorname{tr}\varphi = 0$. We regard S, defined by (1.1), as a mapping from $\mathscr{V}$ to $\mathscr{W}$.

From now on the summation convention will be in force. Stokes' theorem yields

$$\int (Sf)_{ij}\varphi_{ij}\,dx = -\int f_i(\partial\varphi_{ij}/\partial x_j)\,dx \tag{1.2}$$

if either $f\in\mathscr{V}$ or $\varphi\in\mathscr{W}$ has compact support (dx is the volume element and the integrals are over R^n). Consequently, $S^*\colon\mathscr{W}\to\mathscr{V}$ with components

$$(S^*\varphi)_i = \partial\varphi_{ij}/\partial x_j \tag{1.3}$$

is the adjoint of $-S$. For $n=2$ there is again a connection with the complex derivatives. Namely, if

$$\varphi = \begin{pmatrix}\varphi_1 & \varphi_2\\ \varphi_2 & -\varphi_1\end{pmatrix}$$

is identified with $\varphi_1 + i\varphi_2$, then S^* can be identified with $2\varphi_z$.

The operators combine to

$$S^*Sf = \tfrac{1}{2}\Delta f + (\tfrac{1}{2} - 1/n)\operatorname{grad}\operatorname{div} f\,.$$

This is a special case of the elasticity operator $(a-b)\operatorname{grad}\operatorname{div} + b\Delta$ (see [4]). It can be expected that S^*S plays a role similar to that of the Laplace operator in potential theory.

1.3. The operators S and S^* are invariant only with respect to euclidean motions. It would not be difficult to modify the definitions so as to apply to deformations of an arbitrary Riemannian space. However, there are special reasons to study infinitesimal deformations in their relation to conformal mappings, and this turns out to be a good deal more tractable than the general situation.

We shall regard the unit ball $B(1) = \{|x| < 1\}$ as hyperbolic space with the metric

$$ds = \varrho|dx|, \quad \varrho = (1-|x|^2)^{-1}\,.$$

The conformal self-mappings of $B(1)$ are Möbius transformations generated by reflections in spheres or planes orthogonal to the unit sphere

$S(1)$. A somewhat detailed study of these transformations will be given in Section 2.

For simplicity we shall consider only the subgroup G of orientation preserving conformal mappings. The image of x under $A \in G$ is mostly written as Ax, but we use the notation $A'(x)$ for the Jacobian matrix. The conformality is expressed by

$$(1.4) \qquad {}^tA'(x)\cdot A'(x) = |A'(x)|^2$$

where $|A'(x)| > 0$ denotes the ratio of linear stretching. In (1.4) we have followed the practice of suppressing the unit matrix 1_n when multiplied by a scalar.

1.4. The mapping $x \to Ax$ may be viewed as a change of coordinates. If $f(x)$ represents a contravariant vector field in the original variable, then the components of the same vector field in the new coordinates are given by the column vector

$$(1.5) \qquad f_A(x) = A'(x)^{-1}f(Ax)\,.$$

Note that $(f_A)_B = f_{AB}$.

With the customary use of upper and lower indices $f^i = \varrho^{-2}f_i$, and the operator S can be invariantly defined in terms of covariant derivatives by

$$(1.6) \qquad (Sf)^i{}_j = \tfrac{1}{2}(f^i{}_{,j} + f^j{}_{,i}) - n^{-1}\delta^i{}_j f^k{}_{,k}\,.$$

It is easily verified that this expression does not change if the covariant derivatives are replaced by ordinary derivatives. This means that Sf, as originally defined by (1.1), behaves like a mixed tensor, and hence that

$$(1.7) \qquad S(f_A) = A'(x)^{-1}(Sf\circ A)A'(x)\,.$$

The notation (1.6) can now be abandoned in favor of a matrix notation that uses only lower indices.

1.5. Consider the formula (1.2) which we now write as

$$\int_{B(1)} (Sf)_{ij}\varphi_{ij}\,dx = -\int_{B(1)} f_i(S^*\varphi)_i\,dx$$

under the tacit assumption that the integral over the boundary $S(1)$ vanishes. The left hand side is invariant under conformal mappings if φ transforms as a mixed tensor density according to the rule

$$(1.8) \qquad \varphi_A(x) = |A'(x)|^n A'(x)^{-1}(\varphi\circ A)A'(x)\,.$$

When this is so the invariance of the right hand side shows that $S^*\varphi$ behaves like a covariant vector density

We now redefine the vector spaces $\mathscr{V}$ and $\mathscr{W}$ as follows:

$\mathscr{V}$ is the space of contravariant vector fields on $B(1)$, and $\mathscr{W}$ is the space of symmetric mixed density fields on $B(1)$ with zero trace.

In other words, $f \in \mathscr{V}$ and $\varphi \in \mathscr{W}$ shall transform according to the rules (1.5) and (1.8). Observe that the symmetry of φ is invariant only because we are restricting ourselves to orthogonal coordinate changes. The operators S and S^* shall retain their original meaning as explicit differential operators defined by (1.1) and (1.3). Based on (1.7) and $(\varrho \circ A)|A'(x)| = \varrho$ we introduce in addition the invariant operators $P: \mathscr{V} \to \mathscr{W}$ and $P^*: \mathscr{W} \to \mathscr{V}$ defined by

$$Pf = \varrho^n Sf, \quad P^*\varphi = \varrho^{-n-2} S^*\varphi .$$

The invariant inner products are

$$\langle f, g \rangle_{\mathscr{V}} = \int_{B(1)} f_i g_i \varrho^{n+2}\, dx$$
$$\langle \varphi, \psi \rangle_{\mathscr{W}} = \int_{B(1)} \varphi_{ij} \psi_{ij} \varrho^{-n}\, dx ,$$

and Stokes' theorem implies $\langle Pf, \varphi \rangle_{\mathscr{W}} = -\langle f, P^*\varphi \rangle_{\mathscr{V}}$ under suitable boundary condtions.

2. Explicit formulas in hyperbolic geometry.

2.1. In this section we introduce some special notations and derive formulas that will be used in the rest of the paper. Most of the formulas can be found in [1].

We shall use $x^* = x/|x|^2$ to denote inversion in the unit sphere $S(1)$. The Jacobian matrix of x^* is

$$(x^*)' = |x|^{-2}(\delta_{ij} - 2x_i x_j/|x|^2) . \tag{2.1}$$

Because it occurs so frequently we introduce a special notation for the matrix

$$Q(x)_{ij} = x_i x_j/|x|^2$$

and write (2.1) as

$$(x^*)' = |x|^{-2}(1 - 2Q(x)) . \tag{2.2}$$

Note the relations $Q^2 = Q$ and $(1-2Q)^2 = 1$. The latter shows that $1-2Q$ is an orthogonal matrix.

2.2. We shall now give a formula for the most general conformal self-mapping A of $B(1)$. Set $y = A^{-1}0$ and assume that $y \neq 0$. Let S_y be the sphere with center y^* and radius $(|y^*|^2 - 1)^{\frac{1}{2}}$. Because S_y is orthogonal to $S(1)$ reflection in S_y maps $B(1)$ on itself, y going to 0. Explicitly, x is carried to

$$y^* + (|y^*|^2 - 1)(x - y^*)^* .$$

To make this transformation sense preserving we let it be followed by reflection in the hyperplane through 0 perpendicular to y. This is accomplished by multiplication with $1 - 2Q(y)$. The resulting mapping is denoted by T_y, and it is given by the formula

$$T_y x = (1 - 2Q(y))[y^* + (|y^*|^2 - 1)(x - y^*)^*] . \tag{2.3}$$

The definition is completed by setting $T_0 x = x$. Note that $T_y y = 0$ and $T_y 0 = -y$. Furthermore, $T_y^{-1} = T_{-y}$.

A more explicit version of (2.3) is

$$T_y x = \frac{(1 - |y|^2)x - (1 - 2xy + |x|^2)y}{1 - 2xy + |x|^2|y|^2} . \tag{2.4}$$

The denominator in (2.4) occurs in many formulas and will be denoted by $[x,y]^2$. This means that we write

$$[x,y] = |y||x - y^*| = |x||y - x^*| . \tag{2.5}$$

Notice the symmetry in x and y.

From $T_y A^{-1} 0 = T_y y = 0$ we conclude that $T_y A^{-1}$ is a rotation. Hence

$$A = UT_y \tag{2.6}$$

with a constant orthogonal matrix U. We have shown that (2.6) represents the most general conformal self-mapping.

2.3. Together with (2.2) and (2.5) differentiation of (2.3) yields

$$T_y'(x) = \frac{1 - |y|^2}{[x,y]^2}(1 - 2Q(y))(1 - 2Q(x - y^*)) . \tag{2.7}$$

The matrix part on the right will play an important part and deserves a notation of its own:

$$\Delta(x,y) = (1 - 2Q(y))(1 - 2Q(x - y^*)) . \tag{2.8}$$

This is a proper orthogonal matrix.

It is often convenient to write (2.7) in two parts:

$$|T_y'(x)| = (1 - |y|^2)/[x,y]^2 \tag{2.9}$$

$$T_y'(x) = |T_y'(x)| \Delta(x,y) . \tag{2.10}$$

We leave it to the reader to derive formulas (2.11)–(2.15), the last three for arbitrary $A \in G$:

$$|T_y x| = |x-y|/[x,y] \tag{2.11}$$

$$1-|T_y x|^2 = (1-|x|^2)(1-|y|^2)/[x,y]^2 \tag{2.12}$$

$$\frac{|A'(x)|}{1-|Ax|^2} = \frac{1}{1-|x|^2} \tag{2.13}$$

$$|Ax-Ay|^2 = |A'(x)|\,|A'(y)|\,|x-y|^2 \tag{2.14}$$

$$[Ax,Ay]^2 = |A'(x)|\,|A'(y)|[x,y]^2 . \tag{2.15}$$

Note that (2.13) expresses the invariance of the Poincaré metric $\varrho|dx|$.

2.4. There is a close relation between $T_y x$ and $T_x y$. We shall show that

$$T_y x = -\Delta(x,y)T_x y . \tag{2.16}$$

To obtain this we observe that for any $A \in G$ and any $y \in B(1)$

$$T_{Ay}(Ax) = \frac{A'(y)}{|A'(y)|}T_y x . \tag{2.17}$$

This is so because both sides map y on 0, and the Jacobian matrices at y are both multiples of $A'(y)$, and hence equal. We apply (2.17) with $A = T_x$. Because $T_x x = 0$ and $T_y 0 = -y$ one obtains

$$T_x y = -\frac{T_x'(y)}{|T_x'(y)|}T_y x$$

which is (2.16) with x and y reversed.

Repetition of (2.16) yields

$$T_y x = \Delta(x,y)\Delta(y,x)T_y x . \tag{2.18}$$

To show that this implies $\Delta(x,y)\Delta(y,x) = 1$ or, equivalently,

$$\Delta(x,y) = {}^t\Delta(y,x) \tag{2.19}$$

we replace x,y in (2.18) by Ux, Uy where U is a constant orthogonal matrix. It is evident from (2.8) that $\Delta(Ux,Uy) = \Delta(x,y)$, and (2.17) yields $T_{Uy}Ux = UT_y x$. Therefore (2.18) remains true when T_y is replaced by UT_y, and since $UT_y x$ ranges over all directions (2.19) follows.

Explicitly, (2.18) amounts to a not quite obvious identity

$$(1-2Q(y))(1-2Q(x-y^*)) = (1-2Q(y-x^*))(1-2Q(x)) . \tag{2.20}$$

The identity can be proved more geometrically by interpreting each side as a product of two reflections in hyperplanes through the origin.

A final remark: Although $T_y x$ is not defined when $y \in S(1)$ it is clear by continuity, for instance from (2.4), that we must set $T_y x = -y$ when $|y| = 1$. On interchanging x and y it then follows from (2.16) that

$$T_y x = \Delta(x,y)x \quad \text{for } |x| = 1 . \tag{2.21}$$

This will prove quite helpful.

3. Statement of the theorems.

3.1. We are interested in solutions of the equation $P^*Pf = 0$. The subclass of $\mathscr{V}$ consisting of these solutions will be denoted by $\mathscr{F}$. If $f \in \mathscr{F}$ has a continuous extension to $S(1)$ we shall say that f belongs to $\bar{\mathscr{F}}$. The classes $\bar{\mathscr{F}}$ and $\mathscr{F}$ are both invariant with respect to conformal mappings: If $f \in \mathscr{F}$ or $\bar{\mathscr{F}}$ and $A \in G$, then

$$f_A = A'(x)^{-1} f(Ax)$$

belongs to the same class. Similarly, if φ belongs to $P\mathscr{F}$ or $P\bar{\mathscr{F}}$, then

$$\varphi_A = |A'(x)|^n A'(x)^{-1} \varphi(Ax) A'(x)$$

has the same property.

It will be shown, first of all, that the continuous boundary value problem has a unique solution. This solution will be given explicitly in the form of a Poisson type integral. In the formulas below $d\omega$ refers to the $(n-1)$-dimensional area element on $S(1)$, ω_n denotes the area of $S(1)$, and $c_n = n/2(n-1)\omega_n$.

Theorem 1. *Every $f \in \bar{\mathscr{F}}$ can be represented as an integral*

$$f(y) = \int_{S(1)} K(x,y) f(x)\, d\omega(x) \tag{3.1}$$

with

$$K(x,y) = c_n \frac{(1-|y|^2)^{n+1}}{|y-x|^{2n}} \Delta(x,y)[1+(n-2)Q(x)] . \tag{3.2}$$

Conversely, for any given continuous f on $S(1)$ the right hand member of (3.1) is the unique element of $\bar{\mathscr{F}}$ which is equal to f on $S(1)$.

We recall that $\Delta(x,y)$ in (3.2) can be expressed in either of the forms (2.8). With the second form there is a slight simplification inasmuch as

$$(1-2Q(x))(1+(n-2)Q(x)) = 1 - nQ(x) .$$

3.2. For a larger class of harmonic functions the Poisson integral can be replaced by a Poisson-Stieltjes integral. This Herglotz representation generalizes to the present situation.

THEOREM 2.

a) *The subclass* $\mathscr{F}_1 \subset \mathscr{F}$ *of functions* f *satisfying*

$$\int_{S(1)} |f(rx)|\, d\omega(x) = O(1) \tag{3.3}$$

for $0 \leqq r < 1$ *is identical with the class of functions expressible in the form*

$$f(y) = \int_{S(1)} K(x,y)\, d\mu(x) \tag{3.4}$$

where the components μ_i *are finite Borel measures on* $S(1)$.

b) *The function* f *in* (3.4) *has non-tangential limit* $\mu'(x) = d\mu(x)/d\omega(x)$ *at all points* $x \in S(1)$ *where this derivative exists.*

The motivation for this more general result is that it may well be of interest to consider functions $f \in \mathscr{F}$ that vanish almost everywhere on $S(1)$ without being identically zero.

3.3. If f represents an infinitesimal mapping of the closed unit ball which maps $S(1)$ into itself, then $f(x)$ is orthogonal to x, that is, $xf = 0$, for $x \in S(1)$. The subclass of $\overline{\mathscr{F}}$ for which this very natural condition is fulfilled is denoted by $\overline{\mathscr{F}}_0$. The condition implies $Q(x)f(x) = 0$ on $S(1)$, and it becomes possible to omit the factor $1 - nQ(x)$. In other words, (3.1) can be replaced by the simpler formula

$$f(y) = c_n \int_{S(1)} \frac{(1-|y|^2)^{n+1}}{|y-x|^{2n}} (1 - 2Q(y-x)) f(x)\, d\omega(x)\,. \tag{3.1'}$$

More generally, we shall write $f \in \mathscr{F}_0$ as soon as $f \in \mathscr{F}$ and $xf \to 0$ when $|x| \to 1$. When $f \in \mathscr{F}_0 \cap \mathscr{F}_1$ it will be shown that the measure μ in (3.4) satisfies $xd\mu = 0$ on $S(1)$. The formula will then simplify to

$$f(y) = c_n \int_{S(1)} \frac{(1-|y|^2)^{n+1}}{|y-x|^{2n}} (1 - 2Q(y-x))\, d\mu(x)\,. \tag{3.4'}$$

3.4. Starting from (3.1) or (3.4) one can of course obtain boundary value representations of Sf or Pf by straight forward if tedious computation. We shall give the result only for the case that f is expressible in the form (3.4′).

THEOREM 3. *If* $f \in \mathscr{F}_0 \cap \mathscr{F}_1$, *then*

$$(3.5) \qquad Pf(y)_{ij} = (n+1)c_n \int_{S(1)} \frac{\Delta(x,y)_{ih}\Delta(x,y)_{jk}}{|x-y|^{2n}} (x_h d\mu_k + x_k d\mu_h)$$

with the same μ *as in* (3.4′).

If $f \in \overline{\mathscr{F}}_0$ it is easy to convert the integral in (3.5) to a volume integral. We shall show, however, that the resulting formula is valid under much weaker conditions. We shall use $\varphi = Pf$ as a standard notation.

THEOREM 4. *If* $\varphi \in P\mathscr{F}_0$ *is such that*

$$(3.6) \qquad \int_{B(1)} |\varphi(x)|(1-|x|^2)^n dx < \infty$$

then

$$(3.7) \qquad \varphi(y) = 2(n+1)c_n \int_{B(1)} \frac{\Delta(x,y)\varphi(x)\Delta(y,x)}{[x,y]^{2n}} (1-|x|^2)^n \, dx \, .$$

In (3.6) the norm of φ is defined either as $\max|\varphi_{ij}|$ or by $|\varphi|^2 = \operatorname{tr}(\varphi)^2$. The condition can also be written in the form

$$(3.6') \qquad \int_{B(1)} |Sf| \, dx < \infty \, .$$

Although the integral is not invariant it is easy to see that f and f_A satisfy the condition simultaneously.

It is instructive to compare (3.7) with the kernel formula

$$\varphi(y) = 3\pi^{-1} \iint_{|x|<1} \varphi(x)(1-|x|^2)^2(1-\bar{x}y)^{-4} \, dx_1 dx_2$$

for complex analytic functions in the unit disk. Note that the factors $\Delta(x,y)$ and $\Delta(y,x)$ in (3.7) correspond to the argument of the kernel.

4. Proofs.

4.1. The equation $S^*\varphi = 0$ has n linearly independent fundamental solutions with integrable singularity at the origin, namely the matrices $\gamma^k(x)$, $k = 1, \ldots, n$ with entries

$$(4.1) \qquad \gamma^k(x)_{ij} = (\delta_{ik}x_j + \delta_{jk}x_i - \delta_{ij}x_k)|x|^{-n} + (n-2)x_i x_j x_k |x|^{-(n+2)} \, .$$

The reader may verify that both terms on the right are annihilated by S^*. The reason for forming a linear combination is to obtain solutions with vanishing trace.

There are also Green's functions g^k with $Pg^k = \gamma^k$ and $g^k(x) = 0$ on $S(1)$. However, they will not be needed in this paper.

4.2. We shall make repeated applications of Stokes' theorem. By way of notation we shall write $d\sigma(x) = r^{n-1}d\omega(x)$ for the euclidean measure on $S(r)$. In other words, $d\omega$ refers to solid angle.

LEMMA 4.1. *If* $\varphi \in \mathscr{W}$ *and* $S^*\varphi = 0$, *then*

$$\int_{S(r)} \varphi_{ij} x_i d\omega = 0$$

$$\int_{S(r)} \varphi_{ij} x_i x_j d\omega = 0 \tag{4.2}$$

$$\int_{S(r)} \varphi_{ij} x_i x_j x_k d\omega = 0$$

for all $r < 1$.

PROOF. By Stokes' theorem

$$r^{n-2} \int_{S(r)} \varphi_{ij} x_i d\omega = \int_{S(r)} \varphi_{ij} x_i r^{-1} d\sigma = \int_{B(r)} (\partial\varphi_{ij}/\partial x_i) dx = 0$$

and similarly

$$r^{n-2} \int_{S(r)} \varphi_{ij} x_i x_j d\omega = \int_{B(r)} \delta_{ij} \varphi_{ij} dx = 0 ,$$

the latter because $\varphi_{ii} = 0$. In the same way

$$r^{n-2} \int_{S(r)} \varphi_{ij} x_i x_j x_k d\omega = \int_{B(r)} \varphi_{ij} (\delta_{ij} x_k + \delta_{ik} x_j) dx = \int_{B(r)} \varphi_{kj} x_j dx = 0$$

by virtue of the first equation (4.2).

4.3. For $f \in \mathscr{F}$ there is a simple relationship between the values of f on a sphere and at its center. We shall need it only for spheres $S(r)$.

LEMMA 2. *If* $f \in \mathscr{F}$, *then*

$$f(0) = c_n \int_{S(r)} [1 + (n-2)Q(x)] f(x) d\omega(x) . \tag{4.3}$$

PROOF. Another application of Stokes' theorem yields

$$\int_{B(r)-B(r_0)} (Sf)_{ij} \gamma_{ij}^k \, dx = \int_{B(r)-B(r_0)} (\partial f_i/\partial x_j) \gamma_{ij}^k \, dx$$

$$= \left(\int_{S(r)} - \int_{S(r_0)}\right) f_i \gamma_{ij}^k x_j |x|^{-1} \, d\sigma .$$

The integral on the left is zero by virtue of (4.2) and the fact that Sf differs from a φ with $S^*\varphi = 0$ only by a factor that depends only on $|x|$. In the resulting equality of surface integrals we let r_0 tend to zero and note that $|x|^{n-2} \gamma_{ij}^k x_i$ is homogeneous of degree zero and equal to $\delta_{ik} + (n-2) x_i x_k$ on $S(1)$. Equation (4.3) follows on account of

$$\int_{S(1)} x_i x_k d\omega = \delta_{ik} \omega_n / n .$$

4.4. If $f \in \bar{\mathscr{F}}$ it is clear that (4.3) remains true for $r=1$. We shall prove Theorem 1 by applying (4.3) with $r=1$ to f_A with the choice $A=T_y^{-1}$. By definition (see (1.5))

$$f_{(T_y^{-1})}(x) = [(T_y^{-1})'(x)]^{-1} f(T_y^{-1}x) .$$

Furthermore, $T_y^{-1}0=y$ and (2.7) gives

$$(T_y^{-1})'(0) = T_y'(y)^{-1} = 1-|y|^2,$$

and we obtain

$$f(y) = c_n(1-|y|^2)\int_{S(1)}[1+(n-2)Q(x)][(T_y^{-1})'(x)]^{-1}f(T_y^{-1}x)\,d\omega(x) .$$

Replace x by $T_y x$ in the integral, thereby transforming it into

$$\int_{S(1)}|T_y'(x)|^{n-1}[1+(n-2)Q(T_y x)]\,T_y'(x)f(x)\,d\omega(x) .$$

There is a simplification due to (2.21) together with (2.10) and (2.19). Indeed, it follows that

$$Q(T_y x)T_y'(x) = T_y'(x)Q(x)$$

and hence

$$[1+(n-2)Q(T_y x)]T_y'(x) = T_y'(x)[1+(n-2)Q(x)] .$$

In this way we obtain

$$f(y) = c_n(1-|y|^2)\int_{S(1)}|T_y'(x)|^{n-1}T_y'(x)[1+(n-2)Q(x)]f(x)\,d\omega(x) . \tag{4.4}$$

Substitution from (2.7) proves Theorem 1.

4.5. We proceed to the proof of Theorem 2. The first step is to show that every integral of the form (3.4), or equivalently (4.4), belongs to $\mathscr{F}$. This will be so if

$$h_i(y) = (1-|y|^2)|T_y'(x)|^{n-1}T_y'(x)_{ik} \tag{4.5}$$

satisfies $P^*Ph=0$ for every fixed x and k. This can be verified directly, but the following worksaving exercise will be needed later anyway.

If we assume that $|x|<1$ the formulas of Sec. 2 allow us to express (4.5) in terms of $T_x y$ and $T_x'(y)$. In fact, one finds rather easily that

$$(1-|x|^2)^n h_i(y) = (1-|T_x y|^2)^{n+1}(T_x'(y)^{-1})_{ik} . \tag{4.6}$$

On writing

$$w_i(y) = (1-|y|^2)^{n+1}\delta_{ik} \tag{4.7}$$

this can be read as $(1-|x|^2)^n h = w_{T_x}$. It follows by the invariance of P and P^* that

$$(1-|x|^2)^n Ph = (Pw)_{T_x} \tag{4.8}$$

$$(1-|x|^2)^n P^*Ph = (P^*Pw)_{T_x} .$$

Almost without computation

$$(4.9)\qquad \begin{aligned} Pw(y)_{ij} &= -(n+1)(\delta_{ik}y_j+\delta_{jk}y_i-2n^{-1}\delta_{ij}y_k) \\ P^*Pw(y)_i &= c_n'\delta_{ik}(1-|y|^2)^{n+2} \end{aligned}$$

with an unimportant constant c_n'. By use of the transformation rules (1.5), (1.8) and formulas from Section 2.2 one finds that $(Pw)_{T_x}(y)$ contains the factor $(1-|x|^2)^n$ and $(P^*Pw)_{T_x}(y)$ the factor $(1-|x|^2)^{n+1}$. Although the formulas obtained from (4.8) by dropping the factors $(1-|x|^2)^n$ have been proved only for $|x|<1$ it is clear that they remain valid for $x\in S(1)$. It follows that $P^*Ph=0$ when $|x|=1$, and we have proved that the integral (3.4) is in $\mathscr{F}$.

For future reference we record the explicit formulas that result from (4.8):

$$(4.10)\qquad Ph(y)_{ij} = -\frac{n+1}{[x,y]^{2n}}\left(\Delta(x,y)_{ik}\Delta(x,y)_{j\alpha}+\Delta(x,y)_{jk}\Delta(x,y)_{i\alpha}-\delta_{ij}\delta_{k\alpha}\right)(T_xy)_\alpha$$

and

$$(4.11)\qquad P^*Ph(y)_i = c_n'[x,y]^{-(2n+2)}(1-|x|^2)(1-|y|^2)^{n+2}\Delta(x,y)_{ik}.$$

For $|x| = 1$ (4.10) reduces to

$$(4.10')\qquad Ph(y)_{ij} = (n+1)|x-y|^{-2n} \left(\Delta(x,y)_{ik}\Delta(x,y)_{j\alpha}+\Delta(x,y)_{jk}\Delta(x,y)_{i\alpha}-\delta_{ij}\delta_{k\alpha}\right)x_\alpha.$$

4.6. We show next that (3.4) implies (3.3). The vector field

$$f_i(x) = (n-2)\delta_{ik}-n(2x_ix_k-\delta_{ik}|x|^2)$$

with fixed k satisfies $Sf=0$. Therefore Theorem 1 is applicable. It yields

$$(n-2)\delta_{ik}-n(2y_iy_k-\delta_{ik}|y|^2) = 2(n-1)c_n\int_{S(1)}\frac{(1-|y|^2)^{n+1}}{|y-x|^{2n}}\,[1-2Q(y-x)]_{ik}\,d\omega(x)$$

and after passing to the trace

$$\int_{S(1)}\frac{d\omega(x)}{|y-x|^{2n}} = \frac{1+|y|^2}{(1-|y|^2)^{n+1}}\cdot\omega_n$$

for $|y|<1$. With this identity (3.3) is an immediate consequence of (3.4).

4.7. It remains to show that (3.3) implies (3.4). This does not follow as easily as the analogous property of harmonic functions, the reason being that $f(rx)$ with $r < 1$ is not in $\mathcal{F}$ and therefore cannot be expressed through (3.1). Instead we must repeat the proof of Theorem 1 under weaker conditions.

Equation (4.3) can be rewritten as

$$f(0) = c_n \int_{S(1)} [1+(n-2)Q(x)]f(rx)\,d\omega(x) .$$

Exactly as in Section 4.4 we apply this to $f_{T_y^{-1}}$ and replace x by $T_y x$ as integration variable. The result is a rather complicated formula

$$f(y) = \int_{S(1)} K(x,y,r) f[T_y^{-1}(rT_y x)]\,d\omega(x) \tag{4.12}$$

with

$$K(x,y,r) = c_n(1-|y|^2)|T_y'(x)|^{n-1}[1+(n-2)Q(T_y x)][(T_y^{-1})'(rT_y x)]^{-1} .$$

It is readily seen that $K(x,y,r) \to K(x,y)$ as $r \to 1$, uniformly for fixed y.

Now we replace (4.12) by a weighted average

$$f(y) = \left(\log \frac{1}{1-\varrho}\right)^{-1} \int_{S(1)\times(0,\varrho)} K(x,y,r) f[T_y^{-1}(rT_y x)]\,d\omega(x) \frac{dr}{1-r} \tag{4.13}$$

and change integration variables by means of the mapping

$$(x,r) \mapsto u = T_y^{-1}(rT_y x) . \tag{4.14}$$

The Jacobian determinant of (4.14) is

$$|(T_y^{-1})'(rT_y x)|^n |T_y'(x)|^{n-1} = |T_y'(x)|^{-1}(1+o(1))$$

and

$$u = x-(1-r)T_y'(x)^{-1}T_y x + o(1-r) .$$

By use of (2.21) the development of u simplifies to

$$u = [1-|T_y'(x)|^{-1}(1-r)]x + o(1-r) ,$$

and this implies

$$1-|u| = |T_y'(x)|^{-1}(1-r)(1+o(1)) .$$

With all these estimates (4.13) yields

$$f(y) = \left(\log \frac{1}{1-\varrho}\right)^{-1} \int_{T_y^{-1}B(\varrho)} K(u,y) f(u)(1+o(1)) \frac{du}{1-|u|} . \tag{4.15}$$

If ϱ_1, ϱ_2 denote the minimum and maximum of $|u|$ on $T_y^{-1}S(\varrho)$ trivial estimates show that

$$|\log(1-\varrho)/(1-\varrho_1)| \quad \text{and} \quad |\log(1-\varrho)/(1-\varrho_2)|$$

are uniformly bounded for fixed y. By use of the condition (3.3) it then follows from (4.15) that

$$(4.16)\qquad f(y) = \left(\log\frac{1}{1-\varrho}\right)^{-1}\int_{B(\varrho)} K(x,y)f(x)\frac{dx}{1-|x|} + \varepsilon(\varrho)$$

where $\varepsilon(\varrho) \to 0$ for $\varrho \to 1$.

Define a measure μ_ϱ on $S(1)$ by setting

$$(4.17)\qquad \mu_\varrho(\sigma) = \left(\log\frac{1}{1-\varrho}\right)^{-1}\int_{\sigma\times(0,\varrho)} f(rx)\,d\omega(x)\frac{dr}{1-r}$$

for any Borel set $\sigma \subset S(1)$. In terms of this measure (4.16) becomes

$$(4.18)\qquad f(y) = \int_{S(1)} K(x,y)f(x)\,d\mu_\varrho(x) + \varepsilon(\varrho)\,.$$

It follows from the hypothesis (3.3) that the vectorvalued measures μ_ϱ are bounded. Hence there exists a sequence $\varrho_n \to 1$ such that the μ_{ϱ_n} tend to a limit μ in the sense of weak convergence. When ϱ runs through this sequence (4.18) yields

$$f(y) = \int_{S(1)} K(x,y)f(x)\,d\mu(x)\,,$$

thereby completing the proof of Theorem 2a.

We remark that according to (4.17) the limit measure depends only on the values of f in the immediate neighborhood of $S(1)$. In particular, if $xf \to 0$ for $|x| \to 1$, then $xd\mu = 0$ on $S(1)$.

4.8. To prove Theorem 2b we note first that Theorem 1 with $f_i = \delta_{ik}$ yields

$$\int_{S(1)} K(x,y)\,d\omega(x) = 1$$

as a matrix identity. Therefore b) is trivially true if μ is ω multiplying a constant vector. As a consequence, when proving that

$$(4.19)\qquad I(y) = \int_{S(1)} K(x,y)\,d\mu(x)$$

has nontangential limit $\mu'(x_0)$ for $y \to x_0$ we are free to assume that $\mu'(x_0) = 0$. If we write

$$m(t) = |\mu\{|x-x_0| < t\}|$$

this means that $m(t) < \varepsilon t^{n-1}$ for any given ε and $t < t_0$, say.

The part of the integral (4.19) that corresponds to $|x-x_0| \geqq t_0$ tends to zero as $y \to y_0$. On the other hand, it is easy to see that the remaining part is majorized by a constant multiple of

$$(4.20)\qquad \int_0^{t_0} \frac{(1-|y|)^{n+1}}{[(1-|y|)^2+t^2]^n}\,dm(t)$$

provided that y stays in a Stolz cone with vertex x_0. Integration by parts shows, in familiar manner, that an integral of the type (4.20) increases if $m(t)$ is replaced by a larger increasing function. Therefore (4.20) is dominated by

$$\varepsilon \int_0^{t_0} \frac{(1-|y|)^{n+1}}{[(1-|y|)^2+t^2]^n} t^{n-2}\, dt < \varepsilon \int_0^{\infty} \frac{s^{n-2}}{(1+s^2)^n}\, ds ,$$

the last integral being convergent. This proves property b).

4.9. By deriving formula (4.10) we have essentially already proved Theorem 3. In fact, it is clear that (4.4) remains valid with $f d\omega$ replaced by $d\mu$. Since we are assuming that $x d\mu = 0$ the factor $1+(n-2)Q$ can be omitted. When applying (4.10) to obtain $Pf(y)$ we observe that $T_x y = -x$, $[x,y] = |x-y|$, and $\delta_{ij}\delta_{k\alpha}x_\alpha d\mu_k = 0$. Formula (3.6) of Theorem 3 follows on renaming the indices.

4.10. We shall now prove Theorem 4. Our first remark is that the theorem is almost trivial if $f \in \overline{\mathscr{F}}_0$. In that case (3.6) with $y=0$ takes the form

$$Sf(0)_{ij} = (n+1)c_n \int_{S(1)} (x_i f_j + x_j f_i)\, d\omega . \tag{4.21}$$

The integral can be transformed into a volume integral, and because $xf = 0$ on $S(1)$ one obtains

$$Sf(0) = 2(n+1)c_n \int_{B(1)} Sf(x)\, dx . \tag{4.22}$$

Formula (3.7) follows on applying (4.22) to $f_{T_y^{-1}}$, exactly as in the proof of Theorem 1.

The proof of (4.22) when f is in $\mathscr{F}_0$ and satisfies condition (3.6′) will be based on differentiation at $y=0$ of formula (4.12) which we write more explicitly as

$$f_i(y) = \int_{S(1)} K_{ik}(x,y,r) f_k[T_y^{-1}(rT_y x)]\, d\omega(x) . \tag{4.23}$$

We remark first that $(\partial/\partial y_j)K(x,0,r) \to (\partial/\partial y_j)K(x,0)$ as $r \to 1$. Secondly, one finds by asymptotic development of (2.4) and the corresponding formula for $T_y^{-1} = T_{-y}$ that

$$T_y^{-1} r T_y x = rx + 2r(1-r)(xy)x + (1-r)^2 y + o(|y|)$$

and hence

$$(\partial/\partial y_j) f_k[T_y^{-1} r T_y x]_{y=0} = 2r(1-r) f'_{k\alpha}(rx) x_j x_\alpha + (1-r)^2 f'_{kj}(rx) .$$

It should be recalled, further, that $K(x,0) = c_n(1+(n-2)Q(x))$. With all this in mind we obtain from (4.23)

$$\frac{\partial f_i}{\partial y_j}(0) = \lim_{r\to 1}\left[\int_{S(1)} \frac{\partial K_{ik}(x,0)}{\partial y_j} f_k(rx)\,d\omega(x) + \right.$$
$$\left. + c_n \int_{S(r)} (1+(n-2)Q(x))_{ik}(2r(1-r)f'_{k\alpha}(rx)x_j x_\alpha + (1-r)^2 f'_{kj}(rx))\,d\omega(x)\right].$$

When forming $Sf(0)_{ij}$ the two integrals above can be treated separately. The first follows the same pattern as when $f \in \overline{\mathscr{F}}_0$ and leads to an integral with the same limit as

$$(n+1)c_n \int_{S(1)} (x_i f_j(rx) + x_j f_i(rx))\,d\omega .$$

Obviously, this limit is the right hand side of (4.22). Therefore, what remains to be shown is that the second integral leads to a term with the limit zero.

For this purpose we introduce the notations

$$L(r)_{ij} = \int_{S(1)} f'_{ij}(rx)\,d\omega(x)$$
$$M(r)_{ij} = \int_{S(1)} f'_{i\alpha}(rx)x_\alpha x_j\,d\omega(x)$$
$$N(r)_{ij} = \int_{S(1)} f'_{\alpha\beta}(rx)x_\alpha x_\beta x_i x_j\,d\omega(x)$$

and the abbreviations

$$SL(r)_{ij} = \tfrac{1}{2}[L(r)_{ij} + L(r)_{ji}] - n^{-1}\delta_{ij}L(r)_{kk} ,$$

etc. We need to show that the expression

$$(1-r)E(r) = (1-r)[(1-r)SL(r) + 2rSM(r) + 2(n-2)rSN(r)]$$

tends to zero for $r \to 1$. The limit is known to exist. If it is different from zero the integral $\int_0^1 E(r)\,dr$ cannot converge. Therefore, it suffices to show convergence.

By hypothesis

$$\int_0^\varrho r^{n-1} SL(r)_{ij}\,dr = \int_{B(\varrho)} Sf_{ij}\,dx$$

has a finite limit for $\varrho \to 1$. This easily implies the convergence of $\int_0^1 (1-r)SL(r)\,dr$. Secondly, by Stokes' theorem

$$\int_0^\varrho M(r)_{ij} r\,dr = \int_{B(\varrho)} |x|^{-n} f'_{i\alpha}(x)x_\alpha x_j\,dx$$
$$= \varrho^{1-n}\int_{S(\varrho)} f_i x_j\,d\sigma - \int_0^\varrho r^{-n}\,dr \int_{S(r)} f_i x_j\,d\sigma .$$

Here

$$\int_{S(r)} f_i x_j\,d\sigma = r \int_{B(r)} f'_{ij}\,dx$$

so that, by changing indices,

$$\int_0^\varrho SM(r)_{ij} r\,dr = \varrho^{-n+2}\int_{B(\varrho)} Sf_{ij}\,dx - \int_0^\varrho r^{1-n}\,dr \int_{B(r)} Sf_{ij}\,dx .$$

The convergence of $\int_0^1 SM(r)_{ij} r\,dr$ is now an obvious consequence of (3.6′). Thirdly,

$$\begin{aligned}\int_0^\varrho N(r)_{ij} r\,dr &= \int_{B(\varrho)} |x|^{-n-2} f'_{\alpha\beta}(x) x_\alpha x_\beta x_i x_j\,dx \\ &= \varrho^{-n-1} \int_{S(\varrho)} f_\alpha x_\alpha x_i x_j\,d\sigma - \int_{B(\varrho)} |x|^{-n-2} f_\alpha x_\alpha x_i x_j\,dx\,.\end{aligned}$$

Because $f_\alpha x_\alpha \to 0$ the integral is bounded, and $\int_0^1 SN(r)\,dr$ converges. We have proved (4.22), and as already pointed out, application to $f_{T_y^{-1}}$ proves Theorem 4.

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