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SOME REMARKS ON THE BELTRAMI EQUATION

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To Werner Fenchel on his 70th birthday.

1. Introduction.

It is very well known (see [2] or [6]) that if $\mu(z)$ is an L^∞ function on the closed unit disk D with $\|\mu\|_\infty < 1$, there is a unique quasiconformal mapping $w(z)$ of D onto itself that fixes the points $z=0$ and 1 and solves the Beltrami equation

$$w_{\bar{z}} = \mu w_z .$$

In addition (see [1], [3], or [4]), if $\mu(z)$ is a Hölder continuous function in D, the solution $w(z)$ will be a diffeomorphism with Hölder continuous partial derivatives, and if the function μ varies continuously in some appropriate Hölder space, then so does the solution $w(z)$ (see [5]).

In this paper we offer a new proof of the existence of a smooth solution w depending continuously on the parameter μ. As usual we assume that μ is Hölder continuous and close to zero. Our proof relies on the same technical facts as the arguments in [3] or [4]. However we do not directly solve any Beltrami equation. Instead we construct a family of diffeomorphisms of D onto itself. We then use the inverse function theorem in Banach spaces to conclude that our family contains a solution to every Beltrami equation with μ sufficiently close to zero. The advantage of this method is that it yields in one step a smooth solution that maps D diffeomorphically onto itself and depends nicely on μ.

2. A preliminary theorem.

2.1. We illustrate our method in this section by using it to prove a weak form of our main theorem. We need two lemmas.

LEMMA 1. *Let $f(z)$ be a C^1 function in the closed unit disk D. Suppose*

$$\max\{|f_z(z)|+|f_{\bar{z}}(z)| \; ; \; z \in D\} < 1 . \tag{1}$$

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Then $w(z)=z+f(z)$ *is a* C^1 *diffeomorphism of* D *onto its image.*

PROOF. Clearly $w(z)$ is a C^1 function. Moreover,

$$|w_z| = |1+f_z| \geqq 1-|f_z| > |f_{\bar{z}}| = |w_{\bar{z}}| \geqq 0 ,$$

so the Jacobian of $w(z)$ is positive. Finally, (1) implies that $|f(z)-f(\zeta)| < |z-\zeta|$ if $z \neq \zeta$, so w maps D one-to-one onto its image.

2.2. For each integer $n \geqq 0$ and real number α in the open interval $(0,1)$, let $C^{n,\alpha}(D)$ be the space of C^n functions in D whose partial derivatives of order n satisfy a Hölder condition with exponent α. We norm $C^{n,\alpha}(D)$ in the usual way (see [7, p. 8]), making it a Banach space, in fact a Banach algebra.

Fix n and α, and let V be the closed subspace of $C^{n+1,\alpha}(D)$ consisting of the functions $f(z)$ in $C^{n+1,\alpha}(D)$ satisfying

$$\int_{|z|=1} f(z)(z-\zeta)^{-1}\, dz = 0 \quad \text{if } |\zeta|<1 .$$

Our next lemma is well-known.

LEMMA 2. $f \to f_{\bar{z}}$ *is an invertible bounded linear map of* V *onto* $C^{n,\alpha}(D)$.

PROOF. The above map is obviously bounded, linear, and one-to-one. The inverse map is given explicitly by

$$f(\zeta) = Pg(\zeta) = -\pi^{-1} \iint_D g(z)(z-\zeta)^{-1}\, dx\, dy$$

for all ζ in D and g in $C^{n,\alpha}(D)$. For details see [7, p. 56] or [3, Lecture 5].

2.3. Now we are ready to prove our first theorem.

THEOREM 1. *There are neighborhoods of zero* U_0 *in* $C^{n,\alpha}(D)$ *and* V_0 *in* V *such that for each* μ *in* U_0 *there is a unique* f *in* V_0 *such that* $w_{\bar{z}}=\mu w_z$, *where* w *is the diffeomorphism* $w(z)=z+f(z)$. *Further,* f *and* w *depend continuously on* μ.

PROOF. Let V_1 be the open set in V consisting of the f in V that satisfy (1). Define a map φ from V_1 into $C^{n,\alpha}(D)$ by

$$\varphi(f)(z) = f_{\bar{z}}(z)(1+f_z(z))^{-1} \quad \text{for all } z \in D .$$

(Thus, $\varphi(f)=w_{\bar{z}}/w_z$ if $w(z)=z+f(z)$.)

It is clear that φ is a C^1 map in a neighborhood of zero in V_1. Indeed, $\varphi(f)$ is given, for f near zero, by the convergent power series

$$\varphi(f) = \sum_{k=0}^{\infty} f_{\bar z} f_z^{\,k} .$$

The derivative $\varphi'(0)$ is given by

$$\varphi'(0)f(z) = \lim_{t\to 0}\varphi(tf)(z)/t = f_{\bar z}(z) .$$

Lemma 2 says that $\varphi'(0)$ is an invertible map. The inverse function theorem in Banach spaces tells us there are neighborhoods of zero U_0 in $C^{n,\alpha}(D)$ and V_0 in V_1 such that φ maps V_0 homeomorphically onto U_0. That proves the theorem.

We remark that w actually will depend differentiably on μ because $\varphi^{-1}\colon U_0 \to V_0$ is a C^1 map.

The main theorem.

3.1. The diffeomorphism $w(z)$ in Theorem 1 solves the Beltrami equation $w_{\bar z} = \mu w_z$ but does not map D onto itself unless $f(z)=0$ on ∂D. The Riemann mapping theorem gives us a solution mapping D to itself, but to guarantee the continuous dependence of that solution on μ we would need detailed knowledge about the behavior of the Riemann mapping function in variable regions. Although such knowledge is available we prefer a direct approach along the lines of section 2.

3.2. We need to replace $w(z)=z+f(z)$ by a function that maps D to itself. For that purpose we use stereographic projection to identify D with the upper hemisphere. For $w(z)$ we choose an appropriate point on the great circle through z in the direction $f(z)$. To be explicit we prove

LEMMA 3. *Let $f(z)$ be a C^1 function in D. Suppose*

$$\max\{|f(z)| + |f_{\bar z}(z)| + |f_z(z)| \;;\; z \in D\} < \tfrac{1}{2} \tag{2}$$

and

$$\operatorname{Re}(\bar z f(z)) = 0 \text{ whenever } |z|=1 . \tag{3}$$

Then

$$w(z) = [(1+\bar z z)z + f(z)][(1+\bar z z) - \bar z f(z)]^{-1}, \quad z \text{ in } D , \tag{4}$$

is a C^1 diffeomorphism of D onto itself.

PROOF. Easy calculations show that $w(z)$ is a C^1 function in D with positive Jacobian and that $|w(z)|=1$ when $|z|=1$. It follows for topological reasons that w is a diffeomorphism onto D.

3.3. Let W be the closed subspace of $C^{n+1,\alpha}(D)$ consisting of the functions $f(z)$ in $C^{n+1,\alpha}(D)$ satisfying (3) and

$$f(0) = f(1) = 0 . \tag{5}$$

Our main theorem is the following.

THEOREM 2. *There are neighborhoods of zero U_0 in $C^{n,\alpha}(D)$ and W_0 in W such that for each μ in U_0 there is a unique f in W_0 with $w_{\bar z}=\mu w_z$, where $w(z)$ is the diffeomorphism given by* (4). *Further, f and w depend continuously on μ.*

PROOF. Let W_1 be the set of f in W satisfying (2). Define ψ: $W_1 \to C^{n,\alpha}(D)$ by

$$\psi(f)(z) = \frac{(1+\bar z z)^2 f_{\bar z}(z)+f(z)^2}{(1+\bar z z)^2(1+f_z(z))-2\bar z(1+\bar z z)f(z)} .$$

(Thus, $\psi(f)=w_{\bar z}/w_z$ if w is defined by (4).)

We claim that ψ is a C^1 map near $f=0$. To see this we put $g(z)= w(z)-z=\theta(f)(z)$, where $w(z)$ is defined by (4), and we note that

$$\theta(f) = \textstyle\sum_{k=0}^{\infty} h^k f^{k+1} ,$$

with $h(z)=\bar z(1+\bar z z)^{-1}$. Therefore θ is a C^1 map near $f=0$. But $\psi(f)= \varphi(\theta(f))$ where

$$\varphi(g) = g_{\bar z}(1+g_z)^{-1}$$

is C^1 near $g=0$, so ψ is C^1 as we claimed.

The derivative $\psi'(0)$ is again given by

$$\psi'(0)f(z) = \lim_{t\to 0}\psi(tf)(z)/t = f_{\bar z}(z) ,$$

so Theorem 2 once again follows from the inverse function theorem, provided that we prove the following lemma.

LEMMA 4. *The bounded linear map $f \to f_{\bar z}$ from W to $C^{n,\alpha}(D)$ is invertible.*

PROOF. Lemma 4 follows rather easily from the better known Lemma 2. It is again clear that $f \to f_{\bar z}$ is a bounded one-to-one map of W into $C^{n,\alpha}(D)$. To see that it maps onto $C^{n,\alpha}(D)$, choose ν in $C^{n,\alpha}(D)$ and choose g in V with $g_{\bar z}=\nu$. Put

$$g_0(\zeta) = g(\zeta)-g(0)-\zeta(g(1)-g(0)), \quad \zeta \text{ in } D .$$

The required function f is

$$f(\zeta) = g_0(\zeta) - \frac{1}{2\pi i}\int_{|z|=1}\frac{z^2\bar{g}_0(z)}{z-\zeta}\,dz\,.$$

(The line integral defines a function in $C^{n+1,\alpha}(D)$, by [7, p. 21].) The proof of the lemma, and of Theorem 2, is complete.

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