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THE STABLE HOMOLOGY OF A FLAT TORUS

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To Werner Fenchel on his 70th birthday

1. Introduction.

It has been known for some time that on compact Kähler manifolds there is a fundamental relationship between the study of the generalized Plateau problem and the question of which homology classes carry complex analytic subvarieties. In fact, some of the recent work of R. Harvey, J. King and others has shown that the crux of this relationship rests on the concept of a *stable* homology class, which can be described roughly as follows. Let M be a compact riemannian manifold and consider a class $\alpha \in H_p(M; \mathsf{Z})$ where homology is defined by singular Lipschitz chains. Each chain $c \in \alpha$ has a naturally defined mass $\boldsymbol{M}(c)$, from which one obtains a length

$$\|\alpha\| = \inf\{\boldsymbol{M}(c) : c \in \alpha\} .$$

This notion of mass can be extended to all (de Rham) currents on M in several natural ways, and each such extension gives a length

$$\|\alpha_{\mathsf{R}}\| = \inf\{\boldsymbol{M}(c) : c \in \alpha_{\mathsf{R}}\}$$

where α_{R} is the homology class of p-currents determined by α. Since $\alpha \subset \alpha_{\mathsf{R}}$ we have $\|\alpha\| \geqq \|\alpha_{\mathsf{R}}\|$, which leads immediately to the question of which classes α satisfy $\|\alpha\| = \|\alpha_{\mathsf{R}}\|$. In order for this question to have some natural content, it is reasonable to require that the extension of $\boldsymbol{M}$ be chosen so that any p-current of finite mass can be $\boldsymbol{M}$-approximated by singular Lipschitz chains with real coefficients. Federer [3] has shown that this requirement uniquely determines the extension (to be the one introduced in [4] and defined below), and that with this extension,

$$\lim_{m\to\infty} (1/m)\|m\alpha\| = \|\alpha_{\mathsf{R}}\|$$

for any integral homology class α. A given homology class α is then called *stable* if and only if there exists a positive integer m such that $(1/m)\|m\alpha\| = \|\alpha_{\mathsf{R}}\|$.

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Examples of Almgren and Federer show that on general compact manifolds unstable classes can exist, even for $p=1$ and $n=3$. However, on Hodge manifolds, as outlined in section 2 below, the presence of stable homology is directly related to the non-triviality of the Chow ring. This, if nothing else, motivates the study of stable classes at least in basic algebraic manifolds. One of the most important collections of such manifolds (algebraically speaking) is the tori. They have also the advantage of having a reasonably simple topology and local geometry while their global geometry can be quite subtle. It is the main purpose of this paper to begin a study of the stable homology on flat tori.

We begin by deriving a general structure theorem for homologically mass-minimizing currents on a torus (section 3). Sections 4 and 5 are devoted to proving the following result for a flat n-torus T^n.

THEOREM. *Let $\alpha \in H_p(T^n; \mathbf{Z})$ and suppose either that α is the Poincaré dual of an element of the form $b^i \cup c^j$ where $b \in H^1(T^n; \mathbf{Z})$ and $c \in H^2(T^n; \mathbf{Z})$, or that α is a sufficiently large multiple of an element of the form $\beta^i \cdot \gamma^j$ where $\beta \in H_1(T^n; \mathbf{Z})$ and $\gamma \in H_2(T^n; \mathbf{Z})$ and where the product is induced by the addition map $T^n \times T^n \to T^n$. Then*

$$\|\alpha\| = \|\alpha_{\mathbf{R}}\| .$$

Furthermore, for any rectifiable current $\mathscr{S} \in \alpha$ of least mass, $\operatorname{supp}(\mathscr{S})$ *is a p-dimensional real analytic subvariety with codimension-2 singularities.*

Consequently, all the homology of dimension 1, 2, $n-1$, and $n-2$ is stable (although by results of Matsusaka [10] it is not always true that $\|\alpha\| = \|\alpha_{\mathbf{R}}\|$ in dimension 2). In particular, all the homology of a flat n-torus for $n \leqq 5$ is stable. This, however, is as far as it goes. In section 6 we show the following.

THEOREM. *For each pair (p,n) with $3 \leqq p \leqq n-3$, there is a flat n-torus T^n and a class $\alpha \in H_p(T^n; \mathbf{Z})$ such that $(1/m)\|m\alpha\| > \|\alpha_{\mathbf{R}}\|$ for all $m \in \mathbf{Z}^+$.*

COROLLARY. *For each integer $n = 2p \geqq 4$, there exists an abelian variety T of complex dimension n and a class $\alpha \in H_{2p}(T; \mathbf{Z})$ such that α is the Poincaré dual of an invariant, positive (p,p)-form; but no multiple of α contains an effective algebraic cycle.*

This last result sharpens a conjecture of Harvey and Knapp [5].

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2. Definitions and preliminary remarks.

Let M be a C^∞ riemannian manifold and denote by $\mathscr{E}^p(M)$ the space of C^∞ p-forms on M with the usual topology. The dual space to $\mathscr{E}^p(M)$ is called the space of *p-dimensional currents* (with compact support) on M and is denoted $\mathscr{E}_p(M)$. For a current $\mathscr{S} \in \mathscr{E}_p(M)$, we define its boundary $d\mathscr{S} \in \mathscr{E}_{p-1}(M)$ by setting $(d\mathscr{S})(\varphi) = \mathscr{S}(d\varphi)$ where $d\varphi$ is the exterior derivative of φ for $\varphi \in \mathscr{E}^{p-1}(M)$.

We recall from Federer [2] the notions of mass and comass norms. Let V be a finite dimensional vector space over $\mathbf{R}$ with inner product $\langle\cdot,\cdot\rangle$. Extend $\langle\cdot,\cdot\rangle$ canonically to the whole tensor algebra of V, and set $|v| = \langle v, v\rangle^{\frac{1}{2}}$ for each v. Now for an element $\xi \in \Lambda^p V$ we define the *mass-norm* of ξ to be

$$\|\xi\| = \inf\{\textstyle\sum_i |\xi_i| : \xi_i \text{ are simple vectors and } \xi = \sum_i \xi_i\} \tag{2.1}$$

For dual elements $\varphi \in \Lambda^p V^* \cong (\Lambda^p V)^*$ we define the *comass norm* to be

$$\|\varphi\|^* = \sup\{\varphi(\xi) : \xi \in \textstyle\bigwedge^p V \text{ and } \|\xi\| = 1\}\,. \tag{2.2}$$

For each $\xi \in \Lambda^p V$, there exist a finite number of simple vectors $\xi_1, \ldots, \xi_N$ such that $\xi = \sum \xi_i$ and $\|\xi\| = \sum |\xi_i|$ (cf. [2, 5]). Hence, the unit ball for $\|\cdot\|$ is the convex hull the unit simple vectors in $\Lambda^p V$, and $\|\xi\| = |\xi|$ if and only if ξ is simple. It follows also that for $\varphi \in \Lambda^p V^*$,

$$\|\varphi\|^* = \sup\{\varphi(\xi) : \xi \in \textstyle\bigwedge^p V,\ \xi \text{ simple and } |\xi| = 1\}\,. \tag{2.3}$$

Of course, the mass and comass norms are dual, and so we have

$$\|\xi\| = \sup\{\varphi(\xi) : \varphi \in \textstyle\bigwedge^p V^* \text{ and } \|\varphi\|^* = 1\}\,. \tag{2.4}$$

These notions carry over to forms and currents. For $\varphi \in \mathscr{E}^p(M)$ we define its *comass* to be

$$\boldsymbol{M}^*(\varphi) = \sup\{\|\varphi_x\|^* : x \in M\}\,.$$

Then for $\mathscr{S} \in \mathscr{E}_p(M)$ we define its *mass* to be

$$\boldsymbol{M}(\mathscr{S}) = \sup\{\mathscr{S}(\varphi) : \boldsymbol{M}^*(\varphi) = 1\}\,.$$

Note that $\boldsymbol{M}(\mathscr{S}) < \infty$ if and only if $\mathscr{S}$ extends to a bounded linear functional on the space of continuous p-forms with the compact-open topology. Thus if $\boldsymbol{M}(\mathscr{S}) < \infty$, we can define a *total variation measure* $\|\mathscr{S}\|$ by setting

$$\|\mathscr{S}\|(F) = \sup\{\mathscr{S}(\varphi) : \|\varphi_x\|^* \leq F(x) \text{ for all } x \in M\}$$

for positive continuous functions F. By the Radon–Nikodym Theorem there will exist an $\|\mathscr{S}\|$-measurable section $\vec{\mathscr{S}}$ of $\Lambda^p T(M)$ with $\|\vec{\mathscr{S}}\|=1$ $\|\mathscr{S}\|$-almost everywhere, such that

$$\mathscr{S}(\varphi) = \int_M \varphi(\vec{\mathscr{S}}_x)\, d\|\vec{\mathscr{S}}\|(x)$$

for all $\varphi \in \mathscr{E}^p(M)$.

We recall now from Federer and Fleming [4] that certain compact families of currents can be used to define homology on M. Let $\mathscr{R}_p(M)$ denote the rectifiable p-currents on M (cf. [2]), and let

$$\hat{\mathscr{R}}_p(M) = \{\mathscr{S} \in \mathscr{R}_p(M) :\ d\mathscr{S} \in \mathscr{R}_{p-1}(M)\}\,.$$

Similarly, let

$$\mathscr{N}_p(M) = \{\mathscr{S} \in \mathscr{E}_p(M) :\ \boldsymbol{M}(\mathscr{S})+\boldsymbol{M}(d\mathscr{S}) < \infty\}\,.$$

The graded groups $\hat{\mathscr{R}}_*(M)$ and $\mathscr{N}_*(M)$ are mapped into themselves by d, and so we have homology groups defined. By fundamental theorems of Federer and Fleming [4] the following is true. There exist natural isomorphisms:

$$H_p(\hat{\mathscr{R}}_*(M)) \cong H_p(M; \mathsf{Z}), \quad H_p(\mathscr{N}_*(M)) \cong H_p(M; \mathsf{R})\,.$$

Furthermore, if M is compact, then for each $\alpha \in H_p(\hat{\mathscr{R}}_*(M))$ (or $H_p(\mathscr{N}_*(M))$, there exists a current $\mathscr{S} \in \alpha$ of *least mass* in α, i.e.,

$$\boldsymbol{M}(\mathscr{S}) \leqq \boldsymbol{M}(\mathscr{S}') \quad \text{for all } \mathscr{S}' \in \alpha\,.$$

We let $\|\alpha\| = \boldsymbol{M}(\mathscr{S})$ denote this infimum. Note that each class $\alpha \in H_p(\hat{\mathscr{R}}_*(M))$ determines (in fact, is contained in) a unique class $\alpha_{\mathsf{R}} \in H_p(\mathscr{N}_*(M))$, and clearly, $\|\alpha\| \geqq \|\alpha_{\mathsf{R}}\|$. In fact, $\|m\alpha\| \geqq \|m\alpha_{\mathsf{R}}\| = m\|\alpha_{\mathsf{R}}\|$ for each positive integer m, and Federer [3] has proven that

$$\lim_{m\to\infty}(1/m)\|m\alpha\| = \|\alpha_{\mathsf{R}}\|\,.$$

This motivates the following.

DEFINITION 2.1. A class $\alpha \in H_p(M; \mathsf{Z})$ on a compact riemannian manifold M is said to be *stable* if there exists a positive integer m such that $(1/m)\|m\alpha\| = \|\alpha_{\mathsf{R}}\|$.

The geometry of stable classes is an interesting subject, particularly because of its relationship to the study of Hodge manifolds. The remainder of this section is concerned with the statement of that relationship.

We begin the discussion by recalling some facts from linear algebra. Let V and $\langle\cdot,\cdot\rangle$ be as above, and suppose $J\colon V \to V$ is an orthogonal transformation with $J^2 = -1$. There is associated to J a non-degenerate 2-form $\omega \in \Lambda^2 V^*$, called the Kähler form, defined by setting $\omega(v_1, v_2) = \langle Jv_1, v_2\rangle$ for $v_1, v_2 \in V$. Observe that the mapping J (and its adjoint $J: V^* \to V^*$) can be extended uniquely to a derivation of $\Lambda^* V$ (and $\Lambda^* V^*$ respectively). The space of *(p,p)-vectors* is then defined as

$$\Lambda^{p,p} V = \{\xi \in \Lambda^{2p} V : J\xi = 0\},$$
$$\Lambda^{p,p} V^* = \{\varphi \in \Lambda^{2p} V^* : J\varphi = 0\}.$$

More generally, a vector $\xi \in (\Lambda^{p+q} V)\otimes \mathbf{C}$ is of type (p,q) iff $J\xi = i(p-q)\xi$. Note that a simple vector is of type (p,p) if and only if it corresponds to a J-invariant subspace of dimension $2p$. Every such subspace has a canonical orientation induced by J; and so we call a simple vector $\xi \in \Lambda^{2p} V$ *positive* if

$$\xi = v_1 \wedge Jv_1 \wedge v_2 \wedge Jv_2 \wedge \ldots \wedge v_p \wedge Jv_p$$

for vectors $v_1, \ldots, v_p \in V$. More generally, a vector $\xi \in \Lambda^{p,p} V$ is called positive if it can be expressed as a finite sum of positive simple vectors. The set $\mathscr{P}^{p,p}(V)$ of positive vectors is a closed cone in $\Lambda^{p,p} V$ having ω^p in its interior, where ω is the Kähler form defined above. The following result is due in this generality to R. Harvey and A. Knapp [5].

THEOREM 2.2. (The Wirtinger inequality) *For any $\xi \in \Lambda^{2p} V$,*

$$(1/p!)\omega^p(\xi) \leqq \|\xi\|$$

with equality if and only if $\xi \in \mathscr{P}^{p,p}(V)$.

PROOF. If ξ is simple, the result goes back to Wirtinger and an elegant, elementary proof has been given by Federer [1]. If ξ is not simple, then there exist simple vectors $\xi_1, \ldots, \xi_N$ such that $\xi = \sum \xi_i$ and $\|\xi\| = \sum |\xi_i|$. Hence,

$$(1/p!)\omega^p(\xi) = (1/p!)\sum \omega^p(\xi_i) \leqq \sum |\xi_i| = \|\xi\|,$$

with equality if and only if $(1/p!)\omega^p(\xi_i) = |\xi_i|$ for each i, that is, if and only if $\xi_i \in \mathscr{P}^{p,p}$ for each i. This completes the proof.

Suppose now that M is a compact, C^∞ riemannian $2n$-manifold with an orthogonal almost complex structure J (i.e., for each $x \in M$, $J_x: T_x(M) \to T_x(M)$ is an orthogonal transformation with $J_x{}^2 = -1$). There is then an associated *Kähler form* $\omega \in \mathscr{E}^2(M)$ defined by setting $\omega(X, Y) = \langle JX, Y\rangle$ for tangent vectors X and Y, and we say that M is an *almost Kähler manifold* if $d\omega = 0$. Some of the important examples of such manifolds are given by the compact complex manifolds of constant holomorphic curvature (e.g. projective space and tori) and their complex submanifolds.

A differential form $\varphi \in \mathscr{E}^{2p}(M)$ is said to be a *positive* (p,p) *form* iff $\varphi_x \in \mathscr{P}^{p,p}(T_x{}^*(M))$ for all $x \in M$. Similarly, a current $\mathscr{S} \in \mathscr{E}_{2p}(M)$ of finite mass is called a *positive* (p,p)-current iff $\vec{\mathscr{S}}_x \in \mathscr{P}^{p,p}(T_x(M))$ for $\|\mathscr{S}\|$-almost all x.

REMARK 2.3. Suppose M is a complex manifold (i.e., suppose J is integrable) and let W be a complex analytic subvariety of dimension p in M. Then the Hausdorff $2p$-measure of W is finite and the current $[W]$ defined by

$$[W](\varphi) = \int_{\substack{\text{regular points}\\ \text{of } W}} \varphi$$

is a positive (p,p)-current with $d[W]=0$. Moreover, by a result of J. King [7], every d-closed, positive (p,p)-current in $\mathscr{R}_{2p}(M)$ is a finite sum of currents of this form. Such currents are called *positive holomorphic cycles.*

REMARK 2.4. If ω is a d-closed positive (p,p)-form on M, then the current $\mathscr{S} \in \mathscr{E}_{2n-2p}(M)$ given by

$$\mathscr{S}(\varphi) = \int_M \varphi \wedge \omega$$

is a d-closed, positive $(n-p,n-p)$-current on M. The current $\mathscr{S}$ is said to be the (Poincaré) *dual* of ω.

REMARK 2.5. From Theorem 2.2 we have that for any current $\mathscr{S} \in \mathscr{E}_{2p}(M)$ of finite mass,

$$\begin{aligned}\mathscr{S}(\omega^p) &= \int_M \omega^p(\vec{\mathscr{S}}_x)\, d\|\mathscr{S}\|(x)\\ &\leqq (p!) \int_M d\|\mathscr{S}\| = (p!)\boldsymbol{M}(\mathscr{S})\end{aligned}$$

with equality if and only if $\mathscr{S}$ is a positive (p,p)-current.

These examples above make the following corollary of the Wirtinger inequality particularly interesting.

THEOREM 2.5. *Let M be a compact almost-Kähler manifold, and let $\mathscr{S} \in \mathscr{E}_{2p}(M)$ be a positive (p,p)-current with $d\mathscr{S}=0$. Then $\mathscr{S}$ is a current of least mass in its homology class. Moreover, any d-closed current $\mathscr{S}' \in \mathscr{E}_{2p}(M)$ which is homologous to $\mathscr{S}$ and has the same mass is also a positive (p,p)-current.*

PROOF. Since $d\omega^p = 0$, we have $\mathscr{S}(\omega^p) = \mathscr{S}'(\omega^p)$ for all $\mathscr{S}'$ in the homology class of $\mathscr{S}$. Hence, by Remark 2.4, we have

$$M(\mathscr{S}) = (1/p!)\mathscr{S}(\omega^p) = (1/p!)\mathscr{S}'(\omega^p) \leqq M(\mathscr{S}')$$

with equality if and only if $\mathscr{S}'$ is positive and type (p,p).

COROLLARY 2.6. *Let M be a compact Kähler manifold and suppose $\alpha \in H_{2p}(M; \mathsf{Z})$ has the property that α_{R} contains the dual of a closed, positive (p,p)-form. Then $\|\alpha\| = \|\alpha_{\mathsf{R}}\|$ if and only if α contains a positive holomorphic cycle.*

This result leads immediately to the following conjecture of Harvey and Knapp [5].

CONJECTURE 2.7. *Let M be a compact Kähler manifold and suppose $\alpha \in H_{2p}(M; \mathsf{Z})$ has the property that α_{R} is the dual of a closed positive (p,p)-form. Then the class α is stable.*

One of the main points of this paper is to prove that this conjecture is false, even for $(2,2)$-classes on a 4-dimensional abelian variety.

Observe now that since ω^{n-p} lies pointwise in the interior of the cone of positive $(n-p,n-p)$-forms, any closed $(n-p,n-p)$-form can be made positive by adding a sufficiently large multiple of ω^{n-p}. If, moreover, ω has integral periods, then the dual of ω^{n-p} is homologous to a rectifiable $2p$-cycle. In fact, if we replace ω by a sufficiently large integral multiple of ω, then from the work of Kodaira we know the following. There exists a projective embedding $M \subsetneq \mathsf{P}^N(\mathsf{C})$ such that for each p, the dual of ω^{n-p} is homologous to the positive holomorphic cycle $[H^p]$ where $H^p = M \cap \mathsf{P}^{N-n+p}(\mathsf{C})$ for any linear subspace $\mathsf{P}^{N-n+p}(\mathsf{C})$ of dimension $N-n+p$. If we now let $h^p \in H_{2p}(M; \mathsf{Z})$ denote the homology class of $[H^p]$ and suppose that $\alpha \in H_{2p}(M; \mathsf{R})$ has the property that α_{R} is dual to a closed $(n-p,n-p)$-form, then Conjecture 2.7 implies that for all sufficiently large integers k, the class $\alpha + kh^p$ is stable. Hence, there exists $m = m_k \in \mathsf{Z}^+$ such that $m\alpha + mkh^p$ contains a positive holomorphic cycle. In particular, some integral multiple of α contains an algebraic (i.e. holomorphic) cycle.

Conversely, let $M \subset \mathsf{P}^N(\mathsf{C})$ be a projective manifold and suppose $\alpha \in H_{2p}(M; \mathsf{Z})$ contains a current of the form $r[V] - s[W]$ where V and W are p-dimensional, algebraic subvarieties of M and $r,s \in \mathsf{Z}^+$. It was pointed out to me by P. Griffiths that every p-dimensional subvariety W of M is a union of some components of a p-dimensional complete intersection. That is, there exists some p-dimensional subvariety $W' \subset M$ such that $[W]+[W'] \sim k_0[H^p]$ for some $k_0 \in \mathsf{Z}^+$, where „$\sim$" means they are homologous in the complex $\hat{\mathscr{R}}_*(M)$. Hence,

$$r[V]-s[W] \sim r[V]+s[W']-sk_0[H^p] ,$$

and so the class $\alpha+kh^p$ contains a positive holomorphic cycle and, in particular, is stable for all $k \geqq k_0$. This proves the following.

THEOREM 2.8. *Let M be a compact Kähler manifold such that the Kähler form ω has integral periods. Let $h^p \in H_{2p}(M; \mathsf{Z})$ be the Poincaré dual of the cohomology class of ω^{n-p}, and suppose $\alpha \in H_{2p}(M; \mathsf{Z})$ is the dual of any closed integral $(n-p, n-p)$-form. Then α contains a rational chain of holomorphic cycles if and only if α is stable.*

3. Stable classes on flat tori.

We now want to study currents of least mass on flat riemannian tori. We begin by fixing our terminology. Every flat n-torus can be presented as a quotient $T^n = \mathsf{R}^n/\mathscr{L}$ where $\mathsf{R}^n = \{(x_1, \ldots, x_n) : x_j \in \mathsf{R}\}$ carries the metric $ds^2 = \sum dx_j \circ dx_j$ and where $\mathscr{L}$ is a subgroup of rank n generated by vectors $v_1, \ldots, v_n \in \mathsf{R}^n$. Let $\langle \cdot, \cdot \rangle$ denote the standard inner product in R^n and identify R^n with its dual space via $\langle \cdot, \cdot \rangle$. Then there exists a *dual lattice* $\mathscr{L}^* \subset \mathsf{R}^n$ generated by vectors $v_1{}^*, \ldots, v_n{}^*$ where $\langle v_i{}^*, v_j \rangle = \delta_{ij}$. It will be convenient to normalize the torus to have volume one. That is, we will assume

$$v_1 \wedge \ldots \wedge v_n = *1 \quad \text{and} \quad v_1{}^* \wedge \ldots \wedge v_1{}^* = *1 .$$

The space R^n can be identified in a natural way with the tangent space to T^n at the identity. In this way every invariant tensor field on T^n is canonically identified with an element of $\otimes_*^* \mathsf{R}^n$. The metric ds^2 corresponds to $\langle \cdot, \cdot \rangle$, and the space of invariant p-forms corresponds to $\Lambda^p \mathsf{R}^n$. Now on any compact Lie group it is easy to see that the cohomology of the invariant forms is isomorphic under inclusion with the deRham cohomology. On T^n invariant forms are closed, in fact, harmonic. Hence, we have a natural isomorphism

$$H^*(T^n; \mathsf{R}) \cong \bigwedge{}^* \mathsf{R}^n . \tag{3.1}$$

Since it is torsion free, we have $H^*(T^n; \mathsf{Z}) \subset H^*(T^n; \mathsf{R})$. Under the above isomorphism we get the correspondence:

$$H^*(T^n; \mathsf{Z}) \cong \bigwedge{}^*{}_{\mathsf{Z}} \mathscr{L}^* \tag{3.2}$$

where $\Lambda^p{}_{\mathbf{Z}}\mathscr{L}^*$ is the subgroup generated by $v_I{}^* = v_{i_1}^* \wedge \ldots \wedge v_{i_p}^*$ for all multi-indices $I = \{i_1, \ldots, i_p\}$ of length $|I| = p$.

Now on T^n we have

$$H_p(T^n; G) \cong \mathrm{Hom}_G(H^p(T^n; G), G)$$

for $G = \mathbf{R}$ and $\mathbf{Z}$. Hence, there are natural isomorphisms:

$$H_*(T^n; \mathbf{R}) \cong \bigwedge{}^* \mathbf{R}^n \tag{3.3}$$

$$H_*(T^n; \mathbf{Z}) \cong \bigwedge{}^*{}_{\mathbf{Z}} \mathscr{L} \tag{3.4}$$

The isomorphism (3.4) is the restriction of (3.3), which can be described as follows. Given $\xi \in \Lambda^p \mathbf{R}^n$, extend ξ to be an invariant field of tangent p-vectors. Then we can define a current $\mathscr{S}_\xi \in \mathscr{E}_p(T^n)$ by setting

$$\mathscr{S}_\xi(\varphi) = \int_{T^n} \langle \varphi, \xi \rangle *1 = \int_{T^n} \varphi \wedge *\xi . \tag{3.5}$$

The current $\mathscr{S}_\xi$ is closed and invariant. If φ is an invariant p-form, then since volume $(T^n) = 1$, we have

$$\mathscr{S}_\xi(\varphi) = \langle \varphi, \xi \rangle .$$

Thus $\mathscr{S}_\xi$ represents the expected element in homology.

Suppose now that $\xi = v_I$ for some multiindex I of length p. Then we define T_I to be the geodesic subtorus

$$T_I = \mathrm{span}_{\mathbf{R}}\{v_{i_1}, \ldots, v_{i_p}\}/\mathscr{L}_I$$

where $\mathscr{L}_I$ is the sublattice generated by $v_{i_1}, \ldots, v_{i_p}$. Note that vol. $(T_I) = |v_I|$ and $v_I/|v_I|$ is the field of unit tangent p-vectors on T_I. Hence, for an invariant p-form φ, we have

$$\begin{aligned} \mathscr{S}_{v_I}(\varphi) = \langle v_I, \varphi \rangle &= \left\langle \frac{v_I}{|v_I|}, \varphi \right\rangle |v_I| \\ &= \int_{T_I} \left\langle \frac{v_I}{|v_I|}, \varphi \right\rangle *1 \\ &= \int_{T_I} \varphi . \end{aligned}$$

It follows that $\mathscr{S}_{v_I}$ is homologous to integration over torus T_I.

Remark 3.1. Note that the exterior product in $\Lambda^*{}_{\mathbf{Z}}\mathscr{L}$ carries over via (3.4) to a multiplication in $H_*(T^n; \mathbf{Z})$. This is the natural multiplication (suitably renormalized) induced by the addition map $T^n \times T^n \to T^n$.

REMARK 3.2. The star operator in the space $\Lambda^* \mathsf{R}^n$ corresponds naturally to Poincare duality. That is, if we consider $\xi \in \Lambda^p \mathsf{R}^n$ as a real homology class via (3.3), then $*\xi$ corresponds to the dual cohomology class via (3.1) (and conversely). This is a direct consequence of equation (3.5). In particular, $*$ maps integral homology classes to integral cohomology classes and conversely. For each multi-index I, we have

$$*v_I = \pm v_{I'}^* , \tag{3.6}$$

where $I' = \{1,\ldots,n\} - I$ is the complementary multi-index and where the sign is determined by the permutation (I,I'). To see this directly, we write $*v_I = \sum_{|J|=p} a_J v_{J'}^*$ and note that modulo signs

$$*\delta_{K,I} = *\langle v_K{}^*, v_I\rangle = v_K{}^* \wedge *v_I = \textstyle\sum_J a_J v_K{}^* \wedge v_{J'}^* = a_K .$$

Our first important observation is the following.

THEOREM 3.3. *For each vector $\xi \in \Lambda^p \mathsf{R}^n$, let $\mathscr{S}_\xi \in \mathscr{E}_p(T^n)$ be the associated current defined by equation* (3.5). *Then $\mathscr{S}_\xi$ is a current of least mass among all deRham currents in its homology class.*

PROOF. We begin by observing that

$$\boldsymbol{M}(\mathscr{S}_\xi) = \|\xi\| \tag{3.7}$$

where

$$\|\xi\| = \sup\{\langle\varphi,\xi\rangle : \varphi \in \textstyle\bigwedge^p \mathsf{R}^n \text{ and } \|\varphi\|^* \leqq 1\}$$

is the mass norm of ξ in $\Lambda^p \mathsf{R}^n$. Indeed, from (3.5) we have that

$$\mathscr{S}_\xi(\tilde\varphi) = \int \langle\tilde\varphi,\xi\rangle *1 \leqq \|\xi\| \int *1 = \|\xi\|$$

for any $\tilde\varphi \in \mathscr{E}^p(T^n)$ with $\boldsymbol{M}(\tilde\varphi) \leqq 1$. However, by compactness there exists an element $\varphi_0 \in \Lambda^p \mathsf{R}^n$ such that $\|\varphi_0\|^* = 1$ and $\langle\varphi_0,\xi\rangle = \|\xi\|$. We extend φ_0 to be an invariant p-form on T^n. Then $\boldsymbol{M}(\varphi_0) = 1$ and $\mathscr{S}_\xi(\varphi_0) = \|\xi\|$. It follows that $\boldsymbol{M}(\mathscr{S}_\xi) = \|\xi\|$ as claimed. Moreover, we have proven that there exists an invariant p-form φ_0 of comass 1 such that

$$\mathscr{S}_\xi(\varphi_0) = \boldsymbol{M}(\mathscr{S}_\xi) .$$

Suppose now that $\mathscr{S} \in \mathscr{E}_p(T^n)$ is any closed current homologous to $\mathscr{S}_\xi$. Since $d\varphi_0 = 0$, we have $\mathscr{S}_\xi(\varphi_0) = \mathscr{S}(\varphi_0)$; and since $\boldsymbol{M}(\varphi_0) = 1$, we have $\mathscr{S}(\varphi_0) \leqq \boldsymbol{M}(\mathscr{S})$. Therefore, $\boldsymbol{M}(\mathscr{S}_\xi) \leqq \boldsymbol{M}(\mathscr{S})$, and the theorem is proved.

Observe that from equation (3.7) and Theorem 3.3, if $\alpha_{\mathsf{R}} \in H_p(T^n; \mathsf{R})$ is the homology class of $\mathscr{S}_\xi$, then $\|\alpha_{\mathsf{R}}\| = \|\xi\|$. This can be stated as follows.

COROLLARY 3.4. *The maps* (3.1) *and* (3.3) *are isometries if* $\Lambda^p \mathsf{R}^n$ *carries the comass and mass norm respectively.*

We now want to study the structure of those currents $\mathscr{S}$ in the homology class of $\mathscr{S}_\xi$ for which $\boldsymbol{M}(\mathscr{S}) = \boldsymbol{M}(\mathscr{S}_\xi)$. From the paragraph above we know that for any such $\mathscr{S}$ we have

$$\begin{aligned} \mathscr{S}(\varphi_0) &= \int_{T^n} \langle \varphi_0, \vec{\mathscr{S}_x} \rangle \, d\|\mathscr{S}\|(x) \\ &= \boldsymbol{M}(\mathscr{S}) = \int_{T^n} d\|\mathscr{S}\|(x) \end{aligned}$$

Hence, $\langle \varphi_0, \vec{\mathscr{S}_x} \rangle = 1 = \|\vec{\mathscr{S}_x}\|$ for $\|\mathscr{S}\|$-almost all x. Furthermore, this condition must hold for all invariant forms φ such that $\|\varphi\|^* = 1$ and $\langle \varphi, \xi \rangle = \|\xi\|$. This leads us to make the following definitions. For $\xi \in \Lambda^p \mathsf{R}^n$ we set

$$\mathscr{F}^*(\xi) = \{\varphi \in \bigwedge{}^p \mathsf{R}^n : \|\varphi\|^* = 1 \text{ and } \langle \varphi, \xi \rangle = \|\xi\|\}$$

and

$$\mathscr{F}(\xi) = \{\xi' \in \bigwedge{}^p \mathsf{R}^n : \|\xi'\| = 1 \text{ and } \langle \varphi, \xi' \rangle = 1 \text{ for all } \varphi \in \mathscr{F}^*(\xi)\}$$

The set $\mathscr{F}(\xi)$ is called the *facet* of ξ, and $\mathscr{F}^*(\xi)$ is called the *dual facet* of ξ. $\mathscr{F}(\xi)$ and $\mathscr{F}^*(\xi)$ are convex linear cells contained in the boundary of the unit ball of the mass-norm and comass-norm respectively. In fact $\mathscr{F}(\xi)$ is characterized as the largest such cell containing $\xi/\|\xi\|$ in its (relative) interior.

Our discussion above has now proved the following fact.

THEOREM 3.5. *Let* ξ *and* $\mathscr{S}_\xi$ *be as in Theorem* 3.3, *and suppose* $\mathscr{S} \in \mathscr{E}_p(T^n)$ *is any closed current homologous to* $\mathscr{S}_\xi$. *Then* $\mathscr{S}$ *is a current of least mass* (i.e., $\boldsymbol{M}(\mathscr{S}) = \boldsymbol{M}(\mathscr{S}_\xi)$) *if and only if*

$$\vec{\mathscr{S}_x} \in \mathscr{F}(\xi) \tag{3.8}$$

for $\|\mathscr{S}\|$*-almost all* x *in* T^n.

If $\mathscr{S}$ is rectifiable, then $\vec{\mathscr{S}}$ is a simple vector $\|\mathscr{S}\|$-a.e. Therefore, if we let G_p denote the set of unit simple vectors in $\Lambda^p \mathsf{R}^n$ (Note that G_p is the Grassmannian of oriented p-planes in R^n), then we can conclude the following.

COROLLARY 3.6. *Suppose* $\alpha \in H_p(T^n; \mathsf{Z})$ *satisfies* $\|\alpha\| = \|\alpha_{\mathsf{R}}\|$ *and let* $\mathscr{S} \in \alpha$ *be a rectifiable current of least mass in* α. *Then*

$$\vec{\mathscr{S}_x} \in \mathscr{F}(\alpha) \cap G_p \tag{3.9}$$

for $\|\mathscr{S}\|$*-almost all* $x \in T^n$.

Note that $G_p \cap \mathscr{F}(\alpha)$ is contained in the boundary of the linear cell $\mathscr{F}(\alpha)$.

It seems reasonable to conjecture that any current $\mathscr{S} \in \mathscr{R}_p(T^n)$ with $d\mathscr{S}=0$, which satisfies (3.9) for some α, is actually a real analytic cycle, i.e., $\operatorname{supp}\mathscr{S}$ is a p-dimensional, real analytic subvariety of T^n. We shall see that in many cases this is true. The remainder of the paper will be devoted to studying such currents $\mathscr{S}$.

4. The cases $p=1$ and $p=n-1$.

The simplest classes $\alpha \in H_p(T^n; \mathsf{Z}) \subsetneq \Lambda^p \mathsf{R}^n$ to study from the point of view of Corollary 3.6 are those for which $F(\alpha)=\{\alpha/\|\alpha\|\}$, i.e., the „extreme classes'' with respect to the mass-norm. It is not difficult to see that the extreme classes are precisely the ones corresponding to simple vectors in $\Lambda^p \mathsf{R}^n$. (In fact, these are the only classes for which $F(\alpha)\cap G_p$ consists of just one element.) This leads to the question of whether a simple vector $\alpha \in H_p(T^n; \mathsf{Z})$ corresponds to integration over a p-dimensional subtorus, or equivalently, whether a vector $\xi \in \Lambda^p{}_{\mathsf{Z}}\mathscr{L}$ which is decomposable (i.e., simple) in R^n also decomposable in $\mathscr{L}$? To see that this is so, we recall the notion of the span of a p-vector. Given $\xi \in \Lambda^p \mathsf{R}^n$, the *span* of ξ is defined as the smallest R-linear subspace $V \subset \mathsf{R}^n$ such that $\xi \in \Lambda^p V \subset \Lambda^p \mathsf{R}^n$.

PROPOSITION 4.1. *For any p-vector* $\xi \in \Lambda^p{}_{\mathsf{Z}}\mathscr{L}$, *the space* span$(\xi)$ *has a basis (as a vector space over* R) *of vectors from* $\mathscr{L}$. *In particular, if* ξ *is simple, then* $\xi = m v_1 \wedge \ldots \wedge v_p$ *for some integer* $m>0$ *and some appropriate choice of basis* $\{v_1,\ldots,v_n\}$ *for the lattice* $\mathscr{L}$.

NOTE. For a simple vector ξ, the integer m above is defined independently of the basis of $\mathscr{L}$. We shall call m the *Pfaffian* of ξ.

PROOF. It is an elementary fact that

$$\operatorname{span}(\xi) = \operatorname{image}\{i_\xi : \textstyle\bigwedge^{p-1} \mathsf{R}^n \to \mathsf{R}^n\}$$

where i_ξ is the operation of contraction with ξ. (Recall that contraction is defined for simple vectors $\xi = x_1\wedge \ldots \wedge x_p$ and $\eta = y_1 \wedge \ldots \wedge y_{p-1}$ by

$$i_\xi(\eta) = \sum (-1)^{i+1}\langle y_1\wedge \ldots \wedge y_{p-1}, x_1 \wedge \ldots \wedge \hat{x}_i \wedge \ldots \wedge x_p\rangle x_i$$

and then extended linearly.) The vectors $\{v_J{}^*\}_{|J|=p-1}$ form a basis of $\Lambda^{p-1}\mathsf{R}^n$ over R and ξ can be expressed as $\xi = \sum_{|I|=p} n_I v_I$ where $n_I \in \mathsf{Z}$ for each I. Therefore, the vectors $i_\xi(v_J{}^*) \in \mathscr{L}$ for $|J|=p-1$ contain a basis for span(ξ) over R.

Suppose now that ξ is a simple vector. Then by the above $\xi = rx_1 \wedge \dots \wedge x_p$ where $x_1, \dots, x_p \in \mathscr{L}$ and r is a rational number which we may assume to be of the form $r = 1/d$ for $d \in \mathsf{Z}$. Each x_i has an expression of the form $x_i = \sum n_{ij} v_j$ for $n_{ij} \in \mathsf{Z}$. Our main observation is that after a change of basis in the lattice $\mathscr{L}$, we can express x_1 as $x_1 = n_{11} v_1$ for some $n_{11} \in \mathsf{Z}$. This is seen as follows. We may assume after a permutation of basis that $n_{11} \neq 0$. Let $l > 1$ be the first integer such that $n_{1l} \neq 0$, and write $n_{11} = mm_{11}$, $n_{1l} = mm_{1l}$ where $m, m_{11}, m_{1l} \in \mathsf{Z}$ and $(m_{11}, m_{1l}) = 1$. We define a new basis for $\mathscr{L}$ by setting

$$v_1' = m_{11} v_1 + m_{1l} v_l, \qquad v_l' = m_{l1} v_1 + m_{ll} v_l$$

and

$$v_i' = v_i \quad \text{for } i \neq 1, l$$

where m_{l1} and m_{ll} are integers such that $m_{11} m_{ll} - m_{1l} m_{l1} = 1$. In this new basis,

$$x_1 = m v_1' + n_{1l+1} v_{l+1}' + \dots + n_{1n} v_n'.$$

Continuing in this way, we eliminate each of the terms $n_{ij} v_j$ for $j > 1$ until x_1 is an integral multiple of a basis element, as claimed.

After such a change of basis we can alter the remaining x_i's so that they only involve the basis elements $v_2, \dots, v_n$. That is, we have

$$d\xi = n_{11} v_1 \wedge y_2 \wedge \dots \wedge y_p$$

where $y_j = x_j - n_{j1} v_1$. Each y_j is in the lattice $\mathscr{L}_1$ generated by $v_2, \dots, v_n$. Therefore, by the above argument we can make a change of basis in the lattice $\mathscr{L}_1$ so that $y_2 = n_{22} v_2$. Then we have

$$d\xi = n_{11} n_{22} v_1 \wedge v_2 \wedge z_3 \wedge \dots \wedge z_p$$

where $z_j = y_j - n_{j2} v_2$ is a combination of the vectors $v_3, \dots, v_n$. After the pth step in this process we see that there is a basis $v_1, \dots, v_n$ of $\mathscr{L}$ such that $\xi = rv_1 \wedge \dots \wedge v_p$ for some rational number r. Since $\xi \in \Lambda^p{}_{\mathsf{Z}} \mathscr{L}$, we must have $r \in \mathsf{Z}$ and the proof is complete.

THEOREM 4.2. *Suppose that $\alpha \in H_p(T^n; \mathsf{Z})$ corresponds to a simple vector ξ under the isomorphism* (3.4) (i.e., *α_{R} is an extreme class with respect to the mass norm*). *Then $\|\alpha\| = \|\alpha_{\mathsf{R}}\|$. Furthermore, every rectifiable current $\mathscr{S}$ of least mass in α is of the form*

$$\mathscr{S} = \textstyle\sum_{i=1}^{n} [T_i{}^p]$$

where $T_1{}^p, \dots, T_m{}^p$ are mutually parallel, totally geodesic subtori of T^n and where m is the Pfaffian of ξ.

NOTE. If $p=1$ or $n-1$, every $\alpha \in H_p(M; \mathbf{Z})$ corresponds to a simple vector.

PROOF. By Proposition 4.1 there is a basis $v_1, \ldots, v_n$ for $\mathscr{L}$ such that $\xi = mv_1 \wedge \ldots \wedge v_p$. Let

$$T_0{}^p = \operatorname{span}_{\mathbf{R}}\{v_1, \ldots, v_p\}/\operatorname{span}_{\mathbf{Z}}\{v_1, \ldots, v_p\}$$

be the corresponding subtorus. Then

$$M(m[T_0{}^p]) = m|v_1 \wedge \ldots \wedge v_p| = \|\xi\| = \|\alpha_{\mathbf{R}}\|$$

where the last equality follows from Corollary 3.4. Since $m[T_0{}^p]$ is rectifiable, $M(m[T_0{}^p]) \geqq \|\alpha\| \geqq \|\alpha_{\mathbf{R}}\|$, and so $\|\alpha\| = \|\alpha_{\mathbf{R}}\|$.

Suppose now that $\mathscr{S} \in \alpha$ is a rectifiable current of least mass. Then by Corollary 3.6 we know that

$$\vec{\mathscr{S}}_x = \xi \quad \|\mathscr{S}\|-\text{a.e.} \tag{4.1}$$

At this point it is convenient to consider $\mathscr{S}$ as an $(n-p)$-form with distribution coefficients. This is done as follows. Fix an orthonormal basis $\{e_1, \ldots, e_n\}$ of $\mathbf{R}^n$ and extend it to T^n by parallel translation. Then we can express $\mathscr{S} = \sum_{|I|=p} u_I(*e_I)$ where $u_I \in \mathscr{E}_0(T^n)$ is given by $u_I(f) = \mathscr{S}(fe_I)$ for $f \in \mathscr{E}^0(T^n) = C^\infty(T^n)$. It follows that for any $\varphi = \sum f_I e_I \in \mathscr{E}^p(T^n)$, we have

$$\mathscr{S}(\varphi) = \textstyle\sum_{|I|=p} u_I(f_I) .$$

A straightforward calculation then shows that

$$d\mathscr{S} = (-1)^p \textstyle\sum_{i=1}^n \sum_{|I|=p} (\nabla_{e_i} u_I) e_I \wedge (*e_I) . \tag{4.2}$$

If we now assume that $e_1, \ldots, e_n$ were chosen so that $\xi = re_1 \wedge \ldots \wedge e_p$ for $r \in \mathbf{R}^+$, then equation (4.1) immediately implies that the above expression reduces to $\mathscr{S} = u(*e_1 \wedge \ldots \wedge e_p)$. Since $d\mathscr{S} = 0$, we have from (4.2) that

$$\nabla_{e_i} u = 0$$

for $i = 1, \ldots, p$. It follows that $\mathscr{S}$ and, therefore, $\operatorname{supp}(\mathscr{S})$ are invariant by translations from $\operatorname{span}(e_1 \wedge \ldots \wedge e_p)$. Hence, if $x \in \operatorname{supp}(\mathscr{S})$, then $\operatorname{supp}(\mathscr{S})$ contains the p-torus through x parallel to $T_0{}^p$. Since $\mathscr{S}$ is an area minimizing closed rectifiable current, the Hausdorff p-measure of $\operatorname{supp}(\mathscr{S})$ is finite. It follows that $\operatorname{supp}(\mathscr{S})$ is a finite union of translates of $T_0{}^p$. By [4] any closed rectifiable p-current supported in a p-dimensional submanifold is given, up to integer multiples, by integration over that submanifold. This completes the proof.

NOTE. The fact that $\|\alpha\| = \|\alpha_{\mathsf{R}}\|$ for classes of dimension 1 is not generally true, even on manifolds diffeomorphic to the torus (cf. [3]).

5. The cases $p = 2$ and $p = n - 2$.

The main purpose of this section is to prove the following result. We refer the reader to the subsequent discussion for the definition of the rank and Pfaffian of elements in $\Lambda^2{}_{\mathsf{Z}}\mathscr{L}$.

THEOREM 5.1. *Suppose either that $\alpha \in H_{n-2}(T^n; \mathsf{Z})$ or that α is an integral multiple of $N^{k-2}\beta$ for some $\beta \in ((n-1)!)\cdot H_2(T^n; \mathsf{Z})$ where k is the rank and N the Pfaffian of β. Then $\|\alpha\| = \|\alpha_{\mathsf{R}}\|$. Furthermore, for every rectifiable current $\mathscr{S} \in \alpha$ of least mass, $\operatorname{supp}(\mathscr{S})$ is a real analytic subvariety (of dimension $n-2$ or 2 respectively) with singularities of codimension 2.*

COROLLARY 5.2. *Every homology class of dimension 2 or $n-2$ on a flat n-torus is stable.*

REMARK 5.3. The factor $(n-1)!$ in dimension 2 is strictly necessary by results of Matsusaka [10] which, together with Corollary 2.6, show for a certain class $\alpha \in H_2(T; \mathsf{Z})$ on a principally polarized abelian variety T, $\|\alpha\| = \|\alpha_{\mathsf{R}}\|$ if and only if T is a Jacobian. However, $\|(n-1)!\alpha\| = (n-1)!\|\alpha_{\mathsf{R}}\|$.

PROOF. We first consider the case of dimension 2. Let $\xi \in \Lambda^2{}_{\mathsf{Z}}\mathscr{L}$ correspond to the class β via (3.4). Then from elementary linear algebra there exists an orthonormal basis $\{e_1, \ldots, e_n\}$ of R^n such that

$$\xi = \sum_{i=1}^{k} \lambda_i e_{2i-1} \wedge e_{ii} \tag{5.1}$$

where $\lambda_i > 0$ for each i. The integer $k \leqq n/2$ is called the *rank* of ξ. Now by Proposition 4.1 $\operatorname{span}(\xi) = \operatorname{span}(e_1 \wedge \ldots \wedge e_{2k})$ contains a sublattice $\mathscr{L}' \subset \mathscr{L}$ of rank $2k$. For the moment we shall restrict our attention to the subtorus $T^{2k} = \operatorname{span}(\xi)/\mathscr{L}'$. To begin we introduce on T^{2k} an invariant complex structure J by setting $J(e_{2i-1}) = e_{2i}$ and $J(e_{2i}) = -e_{2i-1}$. With respect to this structure the current $\mathscr{S}_\xi$ is a positive $(1,1)$ current, and the torus T^{2k} is an abelian variety. (This second fact follows from the integrality of the class $\beta^{k-1} \in H_{2k-2}(T^{2k}; \mathsf{Z}) \cong \Lambda_{\mathsf{Z}}^{2k-2}\mathscr{L}'$.)

By assumption there exists an element $\eta \in \Lambda^2{}_{\mathsf{Z}}\mathscr{L}$ such that $\xi = ((n-1)!)\eta$. It follows directly from Proposition 4.1 that there exists a basis $v_1, \ldots, v_n$ of $\mathscr{L}$ such that $v_1, \ldots, v_{2k}$ is a basis of $\mathscr{L}'$. Then since

$\eta \in \Lambda^2 \operatorname{span}(\xi)$, we have $\eta \in \Lambda^2{}_{\mathbf{Z}}\mathscr{L}'$. Now by the Frobenius Decomposition Theorem (cf. [8]), there exists a basis $\{w_1,\ldots,w_{2k}\}$ of $\mathscr{L}'$ in which η has the form:

$$\eta = \textstyle\sum_{i=1}^k n_i w_{2i-1} \wedge w_{2i}$$

where $n_1 | n_2 | n_3 \ldots$. The integer $N = n_1 \ldots n_k$ (the Pfaffian of η^k) is called the *Pfaffian* of η. We then consider the form

$$\eta^{k-1} = (k-1)! \textstyle\sum_{i=1}^k (N/n_i) w_1 \wedge \ldots \wedge \hat{w}_{2i-1} \wedge \hat{w}_{2i} \wedge \ldots \wedge w_{2k}$$

and set $\Omega = (1/(k-1)!)\eta^{k-1} \in \Lambda_{\mathbf{Z}}^{2k-2}\mathscr{L}'$.

Now it follows from the elementary theory of theta functions (cf. [8], [11]) that the homology class corresponding to Ω contains a positive holomorphic cycle, namely the divisor $[D]$ of a theta function having Riemann matrix $*\Omega \in \Lambda^2{}_{\mathbf{Z}}\mathscr{L}'^* \cong H^2(T^{2k}; \mathbf{Z})$. Consequently, for each $l = 1,\ldots,k-1$, the homology class dual to $(*\Omega)^l$ also contains a positive holomorphic cycle, namely the intersection $[D]_{t_1} \cap \ldots \cap [D]_{t_l}$ of l generic translations of $[D]$ on T^{2k}. (Recall that cup product in $H^*(T^{2k}; \mathbf{Z})$, which corresponds to wedge product in $\Lambda^*{}_{\mathbf{Z}}\mathscr{L}'^*$, is the Poincaré dual of intersection in $H_*(T^{2k}; \mathbf{Z})$, and that Poincaré duality corresponds to the $*$-operation in $\Lambda^* \mathbf{R}^n$ (cf. Remark 3.2).) Thus, for each l, the class $\alpha_l \in H_{2(k-l)}(T^{2k}; \mathbf{Z})$ corresponding to $*[(*\Omega)^l] \in \Lambda_{\mathbf{Z}}^{2(k-l)}\mathscr{L}'$ contains a positive holomorphic cycle and, therefore by Corollary 2.6, satisfies $\|\alpha_l\| = \|(\alpha_l)_{\mathbf{R}}\|$.

Letting $w_1^*,\ldots,w_{2k}^*$ denote the dual basis to $w_1,\ldots,w_{2k}$ in $\operatorname{span}(\xi)$, we have the expression

$$*\Omega = \textstyle\sum_{i=1}^k (N/n_i) w_{2i-1}^* \wedge w_{2i}^* .$$

Consequently,

$$(*\Omega)^{k-1} = (k-1)! \textstyle\sum_{i=1}^k (N^{k-1}n_i/N) w_1^* \wedge \ldots \wedge \hat{w}_{2i-1}^* \wedge \hat{w}_{2i}^* \wedge \ldots \wedge w_{2k}^* ,$$

and so

$$\begin{aligned} *[(*\Omega)^{k-1}] &= (k-1)! N^{k-2} \textstyle\sum_{i=1}^k n_i w_{2i-1} \wedge w_{2i} \\ &= (k-1)! N^{k-2} \eta . \end{aligned}$$

Thus, α is an integral multiple of α_{k-1}. It follows that α contains a positive holomorphic cycle, and $\|\alpha\| = \|\alpha_{\mathbf{R}}\|$.

Suppose now that $\mathscr{S} \in \alpha$ is any rectifiable 2-current with mass $M(\mathscr{S}) = \|\alpha\| = \|\alpha_{\mathbf{R}}\|$, and suppose α corresponds to $\xi \in \Lambda^2 \mathbf{R}^n$ where ξ has the canonical form (5.1). We may assume that n is even, since if it is not, we simply include $T^n \subsetneq T^{n+1} = T^n \times S^1$. We then define a parallel complex structure J on T^n by setting $Je_{2i-1} = e_{2i}$ and $Je_{2i} = -e_{2i-1}$ for each i, and we let $\omega = \sum_{i=1}^{n/2} e_{2i-1} \wedge e_{2i}$ be the corresponding Kähler form. With respect to this structure ξ is clearly a positive $(1,1)$-form. Hence, by

Theorem 2.2, $\langle \omega, \xi \rangle = \|\xi\|$ and $\mathscr{F}(\xi) \subset \mathscr{P}^{1,1}(\mathsf{R}^n)$. It then follows from Theorem 3.5 that $\vec{\mathscr{S}}_x \in \mathscr{P}^{1,1}(\mathsf{R}^n)$ for $\|\mathscr{S}\|$-almost all x; that is, $\mathscr{S}$ is a positive (rectifiable) $(1,1)$-cycle on T^n. Consequently by King's Theorem [7], $\operatorname{supp}(\mathscr{S})$ is a compact holomorphic curve in T^n. This completes the proof for dimension 2.

Note that the same argument applies to any rectifiable current $\mathscr{S} \in \alpha$ of least mass where $\|\alpha\| = \|\alpha_{\mathsf{R}}\|$ and $\alpha = \beta^p$ for some $\beta \in H_2(T^n; \mathsf{Z})$.

Before going on we need to establish the following fact. Let ξ be as above. Then

$$(5.2) \qquad \mathscr{F}(\xi) = \{\xi' \in \mathscr{P}^{1,1}(\mathsf{R}^n) : \|\xi'\| = 1 \text{ and } \operatorname{span}(\xi') \subset \operatorname{span}(\xi)\} .$$

To see this, we use the following.

LEMMA 5.4. *Let* $\varphi \in \Lambda^p \mathsf{R}^n$ *have* $\|\varphi\|^* = 1$ *and suppose* $\xi \in \Lambda^p \mathsf{R}^n$ *satisfies* $\langle \varphi, \xi \rangle = \|\xi\|$. *Then* $\operatorname{span}(\xi) \subset \operatorname{span}(\varphi)$.

PROOF. To begin we suppose ξ to be simple, and for convenience we assume $\|\xi\| = 1$. Let $\pi : \mathsf{R}^n \to \operatorname{span}(\varphi)$ be orthogonal projection and consider the quadratic form $q(v) = \|\pi(v)\|^2$ for $v \in \operatorname{span}(\xi)$. Let $\varepsilon_1, \ldots, \varepsilon_p$ be an orthonormal basis of eigenvectors for q in $\operatorname{span}(\xi)$. Then

$$\varepsilon_i = \cos\theta_i e_i + \sin\theta_i f_i$$

where: $e_1, \ldots, e_p$ are orthonormal vectors in $\operatorname{span}(\varphi)$, $f_1, \ldots, f_p$ are unit vectors in $\operatorname{span}(\varphi)^\perp$, and $0 \leqq \theta_i \leqq \frac{1}{2}\pi$ for each i. Then

$$\langle \varphi, \xi \rangle = \cos\theta_1 \ldots \cos\theta_p = 1 .$$

Hence $\varepsilon_i = e_i$ for all i and $\operatorname{span}(\xi) \subset \operatorname{span}(\varphi)$ as claimed.

For a general p-vector ξ, we write $\xi = \sum \xi_i$ where each ξ_i is simple and $\|\xi\| = \sum \|\xi_i\|$. Then $\langle \varphi, \xi \rangle = \sum \langle \varphi, \xi_i \rangle \leqq \sum \|\xi_i\| = \|\xi\|$. Consequently, $\langle \varphi, \xi_i \rangle = \|\xi_i\|$ and $\operatorname{span}(\xi_i) \subset \operatorname{span}(\varphi)$ for each i. This completes the proof.

We can now prove the equation (5.2). Let $\tilde{\omega} = e_1 \wedge e_2 + \ldots + e_{2k-1} \wedge e_{2k}$. Then $\|\tilde{\omega}\|^* = 1$ (cf. Proposition 6.1), and $\langle \tilde{\omega}, \xi \rangle = \|\xi\|$. Now if $\xi' \in \mathscr{F}(\xi)$, then $\langle \tilde{\omega}, \xi' \rangle = \|\xi'\| = 1$ by definition. Hence, by Lemma 5.4, $\operatorname{span}(\xi') \subset \operatorname{span}(\tilde{\omega}) = \operatorname{span}(\xi)$. Thus, $\mathscr{F}(\xi) \subset \mathscr{P}^{1,1}(\operatorname{span}(\xi))$. However, by Theorem 2.2, each of the vectors $\xi' \in \mathscr{P}^{1,1}(\operatorname{span}(\xi))$ with $\|\xi'\| = 1$ lies in the affine subspace $\{\xi' : \langle \tilde{\omega}, \xi' \rangle = 1\}$. Equation (5.2) then follows from the fact that $\mathscr{F}(\xi)$ is the largest linear cell in $\{\xi' \in \Lambda^p \mathsf{R}^n : \|\xi'\| = 1\}$, having $\xi / \|\xi\|$ in its relative interior.

It follows from (5.2) that each irreducible component of $\operatorname{supp}(\mathscr{S})$, where $\mathscr{S}$ is the current above, is contained in a translate of T^{2k}.

We now consider $\alpha \in H_{n-2}(T^n; \mathbf{Z})$ and we let $\xi \in \Lambda_{\mathbf{Z}}^{n-2}\mathscr{L}$ correspond to α via (3.4). Then there exists an orthonormal basis $\{e_1, \ldots, e_n\}$ of $\mathbf{R}^n$ and a basis $\{v_1{}^*, \ldots, v_n{}^*\}$ of $\mathscr{L}^*$ such that

$$
\begin{aligned}
(5.3) \qquad *\xi &= \textstyle\sum_{i=1}^{k} \lambda_i e_{2i-1} \wedge e_{2i} \\
&= \textstyle\sum_{1 \leq i < j \leq 2k} n_{ij} v_i{}^* \wedge v_j{}^*
\end{aligned}
$$

where $\lambda_i > 0$ and $n_{ij} \in \mathbf{Z}$ for each i, j. It follows that

$$
\begin{aligned}
\xi &= e_{2k+1} \wedge \ldots \wedge e_n \wedge \textstyle\sum_{i=1}^{k} \lambda_i e_1 \wedge \ldots \wedge \hat{e}_{2i-1} \wedge \hat{e}_{2i} \wedge \ldots \wedge e_{2k} \\
&= v_{2k+1} \wedge \ldots \wedge v_n \wedge \textstyle\sum_{1 \leq i < j \leq 2k} n_{ij} v_1 \wedge \ldots \wedge \hat{v}_{2i-1} \wedge \hat{v}_{2i} \wedge \ldots \wedge v_{2k}
\end{aligned}
$$

where $\{v_1, \ldots, v_n\}$ is the dual basis of $\mathscr{L}$.

Recall now that the $*$-operator is an isometry on $\Lambda^* \mathbf{R}^n$ in the L^2-norm $|\cdot|$. Hence, it also preserves the mass and the comass norms. It follows that for any $\xi' \in \Lambda^p \mathbf{R}^n$, we have

$$
(5.4) \qquad \mathscr{F}(\xi') = (-1)^{p(n-p)} * \mathscr{F}(*\xi') \,.
$$

Therefore, from equation (5.2) and the fact that $*\mathscr{P}^{i,i} = \mathscr{P}^{n-i,n-i}$, we have

$$
\mathscr{F}(\xi) = \{e_{2k} \wedge \ldots \wedge e_n \wedge \xi' : \xi' \in \mathscr{P}^{k-1,k-1}(\operatorname{span}(e_1 \wedge \ldots \wedge e_{2k}))\}
$$

where the complex structure on $\operatorname{span}(e_1 \wedge \ldots \wedge e_{2k})$ is given by $Je_{2i-1} = e_{2i}$ and $Je_{2i} = -e_{2i-1}$ for $i = 1, \ldots, k$.

Let us assume for the moment that n is even, and introduce an invariant complex structure J on T^n by setting $Je_{2i-1} = e_{2i}$ and $Je_{2i} = -e_{2i-1}$ for $i = 1, \ldots, n$. By (5.3), T^n with this structure is an abelian variety, and from the elementary theory of theta functions there exists a theta divisor $[D] \subset T^n$ which is dual to $*\xi$. Thus $[D]$ is a positive holomorphic cycle in the class α, and so by Theorem 2.5, $\boldsymbol{M}([D]) = \|\alpha\| = \|\alpha_{\mathbf{R}}\|$. Moreover, if $\mathscr{S} \in \alpha$ is any rectifiable cycle of least mass, then by Theorem 3.5, $\vec{\mathscr{S}}_x \in \mathscr{F}(\xi)$ for $\|\mathscr{S}\|$-almost all x. In particular, $\mathscr{S}$ is a positive, $(n-1, n-1)$-cycle, and by King's Theorem [7], $\operatorname{supp}(\mathscr{S})$ is a compact, complex subvariety of codimension-one in T^n. Furthermore, if $k < n$, then $\operatorname{supp}(\mathscr{S})$ is foliated by translates of the torus

$$
T^{n-2k} = \operatorname{span}_{\mathbf{R}}\{v_{2k+1}, \ldots, v_n\} / \operatorname{span}_{\mathbf{Z}}\{v_{2k+1}, \ldots, v_n\} \,.
$$

Suppose now that n is odd. Then we embed $T^n \to T^{n+1} = T^n \times S^1$ where S^1 has unit length and we consider the class $\alpha' \in H_{n-1}(T^{n+1} : \mathbf{Z})$ corresponding to $\xi' = \xi \wedge e_{n+1}$ where e_{n+1} is the unit vector in the „new" direction. Then $\|\alpha_{\mathsf{R}}\| = \|\xi\| = \|\xi'\| = \|\alpha_{\mathsf{R}}'\| = \|\alpha'\|$. Moreover, from the last paragraph, any current of least mass $\mathscr{S}' \in \alpha'$ can be written as $\mathscr{S}' = \mathscr{S} \times [S^1]$ where $\mathscr{S} \in \mathscr{R}_{n-2}(T^n)$ and $\operatorname{supp}(\mathscr{S})$ is a real analytic variety of dimension $n-2$ with codimension-2 singularities. Since $M(\mathscr{S}') = M(\mathscr{S})$, we have $\|\alpha\| = \|\alpha_{\mathsf{R}}\|$.

Suppose now that $\mathscr{S} \in \alpha$ is a rectifiable cycle of least mass in α. Then $\mathscr{S}' = \mathscr{S} \times [S^1] \in \alpha'$ is a rectifiable cycle of least mass in α'. Hence, by the paragraph above, $\operatorname{supp}(\mathscr{S})$ is a real analytic variety with codimension-2 singularities. This completes the proof.

The above argument easily generalizes to prove the following.

THEOREM 5.5. *Let $\alpha \in H_p(T^n; \mathsf{Z})$ be such that its Poincaré dual $*\alpha \in H^{n-p}(T^n; \mathsf{Z})$ can be written as a product $*\alpha = \beta^r \wedge \gamma^s$ where $\beta \in H^1(T^n; \mathsf{Z})$ and $\gamma \in H^2(T^n; \mathsf{Z})$. Then $\|\alpha\| = \|\alpha_{\mathsf{R}}\|$, and for any rectifiable current $\mathscr{S} \in \alpha$ of least mass in α, $\operatorname{supp} \mathscr{S}$ is a real analytic variety of dimension p with codimension-2 singularities.*

REMARK 5.6. The arguments above also prove that any class of the form $\alpha = \beta^r \wedge \gamma^s$ for $\beta \in H_1(T^n; \mathsf{Z})$ and $\gamma \in H_2(T^n; \mathsf{Z})$ has some integer multiple of the form hypothesized in Theorem 5.5. Thus, every such homology class is stable.

6. The cases $3 \leqq p \leqq n-3$.

Despite the mounting evidence it is not true that every homology class on a flat n-torus is stable. Of course, by the previous results we must have $n \geqq 6$ to find a counter-example. However, we shall show that for every pair of integers (p, n) with $3 \leqq p \leqq n-3$, there is a flat n-torus T^n and a class $\alpha \in H_p(T^n; \mathsf{Z})$ such that

$$(1/m)\|m\alpha\| > \|\alpha_{\mathsf{R}}\|$$

for all positive integers m.

To begin we need the following elementary result.

PROPOSITION 6.1. *Let $\xi', \xi'' \in \Lambda^p \mathsf{R}^n$ be such that $\operatorname{span}(\xi') \perp \operatorname{span}(\xi'')$. Then*

$$\|\xi' + \xi''\|^* = \max\{\|\xi'\|^*, \|\xi''\|^*\}$$

and

$$\|\xi' + \xi''\| = \|\xi'\| + \|\xi''\| .$$

PROOF. Suppose $\eta \in \Lambda^p \mathsf{R}^n$ is a unit simple vector. Let $\pi\colon \mathsf{R}^n \to \operatorname{span}(\xi')$ be orthogonal projection and consider the quadratic forms $q'(v) = \|\pi(v)\|^2$ and $q''(v) = \|v\|^2 - \|\pi(v)\|^2$ on $\operatorname{span}(\eta)$. Then there exists an orthonormal basis $\{\varepsilon_1, \ldots, \varepsilon_p\}$ of $\operatorname{span}(\eta)$ such that ε_i is simultaneously an eigenvector of both q' and q'' for each i. Hence, there exists an orthonormal basis $\{e_1, \ldots, e_n\}$ of R^n, with $\operatorname{span}(\xi') = \operatorname{span}(e_1 \wedge \ldots \wedge e_k)$, such that

$$\varepsilon_i = \cos\theta_i e_i + \sin\theta_i e_{k+i}$$

where $0 \leqq \theta_i \leqq \frac{1}{2}\pi$ for $i = 1, \ldots, p$. Consequently if we set

$$\mu = \max\{\|\xi'\|^*, \|\xi''\|^*\},$$

we have from (2.3) that

$$\begin{aligned} \langle \xi' + \xi'', \eta\rangle &= \cos\theta_1 \ldots \cos\theta_p \langle \xi', e_1 \wedge \ldots \wedge e_p\rangle + \\ &\quad + \sin\theta_1 \ldots \sin\theta_p \langle \xi'', e_{k+1} \wedge \ldots \wedge e_{k+p}\rangle \\ &\leqq (\cos\theta_1 \ldots \cos\theta_p + \sin\theta_1 \ldots \sin\theta_p)\mu . \end{aligned}$$

However, from the Schwarz Inequality,

$$\begin{aligned} (6.1)\quad & \cos\theta_1 \ldots \cos\theta_p + \sin\theta_1 \ldots \sin\theta_p \\ & \leqq [\cos^2\theta_1 + \sin^2\theta_1]^{\frac{1}{2}}[(\cos\theta_2 \ldots \cos\theta_p)^2 + (\sin\theta_1 \ldots \sin\theta_p)^2]^{\frac{1}{2}} \\ & \leqq \cos\theta_2 \ldots \cos\theta_p + \sin\theta_2 \ldots \sin\theta_p \\ & \leqq \text{etc.} \leqq 1 \end{aligned}$$

This shows that $\langle \xi' + \xi'', \eta\rangle \leqq \mu$ for all unit simple vectors η, and so $\|\xi' + \xi''\|^* \leqq \mu$ by (2.3). The reverse inequality is immediate, and this proves the first part of the proposition.

For the second part we choose $\zeta^{(i)} \in \Lambda^p \mathsf{R}^n$ with

$$\operatorname{span}(\zeta^{(i)}) \subset \operatorname{span}(\xi^{(i)}), \|\zeta^{(i)}\|^* = 1$$

and

$$\langle \zeta^{(i)}, \xi^{(i)}\rangle = \|\xi^{(i)}\| \quad \text{for } i = 1, 2 .$$

By the above $\|\zeta' + \zeta''\|^* = 1$, and so

$$\langle \zeta' + \zeta'', \xi' + \xi''\rangle = \|\xi'\| + \|\xi''\| \leqq \|\xi' + \xi''\| .$$

The reverse inequality is immediate. This completes the proof.

Observe that if $p > 2$, then equality holds in (6.1) if and only if

$$\cos\theta_2 \ldots \cos\theta_p = 1 \quad \text{or} \quad \sin\theta_2 \ldots \sin\theta_p = 1 .$$

Repeating the argument with the indices permuted then proves that equality holds if and only if $\cos\theta_1 \ldots \cos\theta_p = 1$ or $\sin\theta_1 \ldots \sin\theta_p = 1$. This gives the following fact. (See also [4, 9.15].)

COROLLARY 6.2. *Let $p \geqq 3$, and let $\xi', \xi'' \in \Lambda^p \mathsf{R}^n$ be such that* $\operatorname{span}(\xi') \perp \operatorname{span}(\xi'')$ *and* $\|\xi'\|^* = \|\xi''\|^* = 1$. *Then for any $\eta \in \Lambda^p \mathsf{R}^n$,*

$$\langle \xi' + \xi'', \eta \rangle \leqq \|\eta\|$$

with equality if and only if $\eta = \eta' + \eta''$ with $\operatorname{span}(\eta^{(i)}) \subset \operatorname{span}(\xi^{(i)})$. *In particular, if η is simple, then equality holds if and only if*

$$\operatorname{span}(\eta) \subset \operatorname{span}(\xi') \quad \text{or} \quad \operatorname{span}(\eta) \subset \operatorname{span}(\xi'') .$$

PROOF. For η simple, the statement follows from the paragraph above. Otherwise $\eta = \sum \eta_i$ where each η_i is simple and $\|\eta\| = \sum \|\eta_i\|$. Then

$$\langle \xi' + \xi'', \eta \rangle = \sum \langle \xi' + \xi'', \eta_i \rangle \leqq \sum \|\eta_i\| = \|\eta\| ,$$

and the result follows immediately.

Suppose now that $3 \leqq p \leqq n/2$ and consider the vector

$$\xi = e_1 \wedge \ldots \wedge e_p + e_{p+1} \wedge \ldots \wedge e_{2p} \tag{6.2}$$

where $\{e_1, \ldots, e_n\}$ is an orthonormal basis of R^n. Then by the above

$$\mathscr{F}(\xi) = \{\alpha_1 e_1 \wedge \ldots \wedge e_p + \alpha_2 e_{p+1} \wedge \ldots \wedge e_{2p} : \alpha_i \geqq 0 \text{ and } \alpha_1 + \alpha_2 = 1\} . \tag{6.3}$$

To see this, note that $\|\xi\|^* = 1$ and $\langle \xi, \xi \rangle = 2 = \|\xi\|$. Hence, $\xi \in \mathscr{F}^*(\xi)$, and therefore $\langle \xi, \xi' \rangle = \|\xi'\| = 1$ for all $\xi' \in \mathscr{F}(\xi)$. It then follows from Corollary 6.2 that any $\xi' \in \mathscr{F}(\xi)$ must be of the form $\alpha_1 e_1 \wedge \ldots \wedge e_p + \alpha_2 e_{p+1} \wedge \ldots \wedge e_{2p}$. Since $\langle \xi, \xi' \rangle = 1$ and $\|\xi'\| = 1$, we get $\alpha_1 + \alpha_2 = 1$ and $|\alpha_1| + |\alpha_2| = 1$. This proves that $\mathscr{F}(\xi)$ is contained in the set on the right of (6.3). Since this set is a linear cell containing ξ in its relative interior, we have equality.

REMARK 6.3. For the case of integers between $n/2$ and $n-3$ we shall consider vectors of the form $*\xi$ where ξ is of the form (6.2). The formula for $\mathscr{F}(*\xi)$ is given directly by (6.3) and (5.4).

Our first main observation is the following.

THEOREM 6.4. *Let $\alpha \in H_p(T^n; \mathsf{Z})$, $3 \leqq p \leqq n/2$, satisfy $\|\alpha\| = \|\alpha_{\mathsf{R}}\|$, and suppose α corresponds via (3.4) to a p-vector $\xi = \xi_1 + \xi_2 \in \Lambda^p{}_{\mathsf{Z}} \mathscr{L}$ where ξ_1 and ξ_2 are simple vectors (not necessarily in $\Lambda^p{}_{\mathsf{Z}} \mathscr{L}$) with* $\operatorname{span}(\xi_1) \perp \operatorname{span}(\xi_2)$. *The any rectifiable current $\mathscr{S} \in \alpha$ of least mass is given locally as a finite* *sum* $\sum n_i[L_i]$ *where* $n_i \in \mathsf{Z}^+$ *and* L_i *is an oriented p-plane parallel to either* $\operatorname{span}(\xi_1)$ *or* $\operatorname{span}(\xi_2)$ *for each i.*

PROOF. Choose an orthonormal basis $\{e_1, \ldots, e_n\}$ of R^n so that

$$\xi_1 = r_1 e_1 \wedge \ldots \wedge e_p \quad \text{and} \quad \xi_2 = r_2 e_{p+1} \wedge \ldots \wedge e_{2p}$$

for $r_1, r_2 \in \mathsf{R}^+$. (Note that we may assume $r_1 \cdot r_2 \neq 0$ since otherwise we are reduced to Theorem 4.2.) Then

$$\mathscr{F}(\xi) = \mathscr{F}(e_1 \wedge \ldots \wedge e_p + e_{p+1} \wedge \ldots \wedge e_{2p}) \, ,$$

and so $\mathscr{F}(\xi)$ is given by equation (6.3). Therefore, it follows from Corollary 3.6 that

$$(6.4) \qquad \vec{\mathscr{S}_x} = e_1 \wedge \ldots \wedge e_p \quad \text{or} \quad e_{p+1} \wedge \ldots \wedge e_{2p} \quad \|\mathscr{S}\|-\text{a.e.}$$

As in the proof of Theorem 4.2 we express $\mathscr{S}$ as an $(n-p)$-form on T^n with distribution coefficients, and observe that from (6.4) this expression reduces to

$$(6.5) \qquad \mathscr{S} = u_1 *(e_1 \wedge \ldots \wedge e_p) + u_2 *(e_{p+1} \wedge \ldots \wedge e_{2p})$$

for $u_1, u_2 \in \mathscr{E}_0(T^n)$. Then since $d\mathscr{S} = 0$, we conclude that

$$\nabla_{e_i} u_1 = 0 \quad \text{and} \quad \nabla_{e_{p+i}} u_2 = 0 \quad \text{for } i = 1, \ldots, p \, .$$

It follows that u_1 is invariant by translations from $\operatorname{span}(e_1 \wedge \ldots \wedge e_p)$, and u_2 is invariant by translations from $\operatorname{span}(e_{p+1} \wedge \ldots \wedge e_{2p})$. Therefore, if $x \in \operatorname{supp}(\mathscr{S})$, then $\operatorname{supp}(\mathscr{S})$ contains a p-plane through x parallel to either $\operatorname{span}(e_1 \wedge \ldots \wedge e_p)$ or $\operatorname{span}(e_{p+1} \wedge \ldots \wedge e_{2p})$. Now since $\mathscr{S}$ is a mass-minimizing, rectifiable current, the Hausdorff p-measure of $\operatorname{supp}(\mathscr{S})$ is locally finite (cf. [2], [9]). It follows that $\operatorname{supp}(\mathscr{S})$ is locally a finite union of such planes. The remainder of the argument goes as in Theorem 4.2. This completes the proof.

COROLLARY 6.5. *Let* $\alpha \in H_p(T^n; \mathsf{Z})$, $3 \leq p \leq \frac{1}{2}n$, *correspond via* (3.4) *to a p-vector* $\xi = \xi_1 + \xi_2 \in \Lambda^p{}_{\mathsf{Z}} \mathscr{L}$ *where* ξ_1 *and* ξ_2 *are simple vectors with* $\operatorname{span}(\xi_1) \perp \operatorname{span}(\xi_2)$. *Then* $\|\alpha\| = \|\alpha_{\mathsf{R}}\|$ *if and only if* $\xi_1, \xi_2 \in \Lambda^p{}_{\mathsf{Z}} \mathscr{L}$.

PROOF. If $\xi_1, \xi_2 \in \Lambda^p{}_{\mathsf{Z}} \mathscr{L}$, then $\|\alpha\| = \|\alpha_{\mathsf{R}}\|$ by Theorem 4.2 and Proposition 6.1. Suppose then that $\|\alpha\| = \|\alpha_{\mathsf{R}}\|$, and let $\mathscr{S} \in \alpha$ be a rectifiable current of least mass. As in equation (6.5) we write

$$\mathscr{S} = \mathscr{S}_1 + \mathscr{S}_2 \quad \text{where } \mathscr{S}_1 = u_1(*e_1 \wedge \ldots \wedge e_p) \, .$$

By Theorem 6.4, $\mathscr{S}_i$ is given locally by integration over a finite number of oriented p-planes parallel to $\operatorname{span}(\xi_i)$ for $i=1,2$. Thus, each $\mathscr{S}_i$ is a closed, rectifiable p-current. It is obvious from evaluation on parallel p-forms that $\mathscr{S}_i$ is homologous to $\mathscr{S}_{\xi_i}$ for each i. Hence, the homology class of $\mathscr{S}_{\xi_i}$ is integral, and by Proposition 4.1, $\xi_i \in \Lambda^p{}_{\mathsf{Z}}\mathscr{L}$. This completes the proof.

NOTE. More geometrically, it is clear that $\operatorname{span}(\xi_i)$ must contain p independent elements from $\mathscr{L}$, for otherwise the closure of $\operatorname{span}(\xi_i)/\operatorname{span}(\xi_i)\cap\mathscr{L}$, a translate of which is contained in $\operatorname{supp}(\mathscr{S})$, would be a subtorus of dimension $>p$.

We are now in a position to construct some „unstable'' homology. Let $n \geqq 2p \geqq 6$ and consider the lattice $\mathscr{L}$ in R^n generated by the vectors:

$$\begin{aligned} v_{2i-1} &= (0,\ldots,0,1,0,\ldots,0) \\ v_{2i} &= (0,\ldots,0,1/\sqrt{2},0,\ldots,0) \end{aligned}$$

for $i=1,\ldots,[(n+1)/2]$, where the non-zero entry of v_j is in the jth position. We then define

$$\begin{aligned} e_i &= v_{2i-1}+\sqrt{2}v_{2i} \\ \bar{e}_i &= v_{2i-1}-\sqrt{2}v_{2i} \end{aligned}$$

for $i=1,\ldots,p$, and set

$$\xi = e_1 \wedge \ldots \wedge e_p + \bar{e}_1 \wedge \ldots \wedge \bar{e}_p\,. \tag{6.6}$$

The vectors $\{e_1,\ldots,e_p,\bar{e}_1,\ldots,\bar{e}_p\}$ are mutually orthogonal, and so ξ satisfies the decomposition hypothèsis of Corollary 6.5. Furthermore, we have $\xi \in \Lambda^p{}_{\mathsf{Z}}\mathscr{L}$. The quickest way to see this is to note that ξ is fixed by the Galois automorphism in the „lattice'' $\Lambda^p{}_{\mathsf{Z}[\sqrt{2}]}\mathscr{L}$. However, no integer multiple of $e_1\wedge\ldots\wedge e_p$ (or of $\bar{e}_1\wedge\ldots\wedge\bar{e}_p$) lies in $\Lambda^p{}_{\mathsf{Z}}\mathscr{L}$ since $e_1\wedge\ldots\wedge e_p$ has irrational coefficients with respect to the basis $\{v_I\}_{|I|=p}$. Therefore, from Corollary 6.5 we conclude the following for the torus $T=\mathsf{R}^n/\mathscr{L}$.

THEOREM 6.6. *Let $\alpha \in H_p(T^n;\mathsf{Z})$ be the class corresponding to the p-vector ξ given in* (6.6). *Then*

$$(1/m)\|m\alpha\| > \|\alpha_{\mathsf{R}}\|$$

for all $m \in \mathsf{Z}^+$.

We now consider the special case where $n=2p=4k$ for an integer $k\geqq 2$, and introduce on T an invariant, orthogonal complex structure J by setting $Jv_i = v_{i+p}$ and $Jv_{i+p} = -v_i$ for $i = 1, \ldots, p$. Hence, $Je_i = e_{i+k}$ and $J\bar{e}_i = \bar{e}_{i+k}$ for $i = 1, \ldots, k$, and therefore we have that

$$\begin{aligned} \xi &= (-1)^{k(k-1)/2}\xi \\ &= e_1 \wedge Je_1 \wedge \ldots \wedge e_k \wedge Je_k + \bar{e}_1 \wedge J\bar{e}_1 \wedge \ldots \wedge \bar{e}_k \wedge J\bar{e}_k \end{aligned} \tag{6.7}$$

is a positive (k,k)-vector. Consequently, ξ determines a homology class $\alpha \in H_{2k}(T; \mathsf{Z})$ which is dual to a cohomology class containing a positive (k,k)-form, namely the form $*\xi = \xi$. However, by the above argument there is no integer m such that $\|m\alpha\| = m\|\alpha_{\mathsf{R}}\|$. It then follows from Theorem 2.5 that there is no integer m such that $m\alpha$ contains a positive holomorphic cycle. Note that the torus T with this structure is an abelian variety, in fact, it is a product of elliptic curves. Putting this together we have the following.

THEOREM 6.7. *For each $k \geqq 2$ the above torus T is an abelian variety of complex dimension $2k$, and the class $\alpha \in H_{2k}(T; \mathsf{Z})$ determined by ξ in* (6.7) *is the Poincaré dual of an invariant, positive (k,k)-form. However, no multiple of α carries an effective* (i.e., *positive*) *algebraic cycle.*

We remark that the classes α in Theorem 6.7 do carry algebraic cycles of mixed sign. This can be verified by direct computation.

It is probably worth noting the forms $*\xi = \xi$ are special in that they lie on the boundary of the positive cone $\mathscr{P}^{k,k}$.

As mentioned in §2, this result sharpens a conjecture of Harvey and Knapp concerning stability and the Hodge Conjecture.

As a final comment we point out that, beginning with Remark 6.3, the above arguments carry through isomorphically for $\frac{1}{2}n \leqq p \leqq n-3$.

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