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ON THE PROJECTIVE STRUCTURE OF A REAL HYPERSURFACE IN C_{n+1}

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To Werner Fenchel on his 70th birthday.

Let C_{n+1} be the complex number space of dimension $n+1$ with the coordinates $z^1,\ldots,z^{n+1}$. A real hypersurface M is defined analytically by the equation

$$(1) \qquad r(z^j,\bar{z}^j) = 0, \quad 1 \leqq j \leqq n+1 ,$$

where r is a real-valued function. We suppose r to be smooth with $\operatorname{grad} r \neq 0$. For $n=1$, B. Segre observed that the corresponding equation

$$(2) \qquad r(z^j,a^j) = 0 ,$$

where $\bar{z}^j$ is replaced by arbitrary parameters a^j, defines a two-parameter family of curves in the complex plane whose invariants were first studied by A. Tresse in 1896 [5]. These invariants clearly provide local invariants of the hypersurface M itself. But they do not give a complete system of invariants of M under biholomorphic transformations in C_{n+1}, as remarked by Elie Cartan [1]. The latter problem has been the object of a recent study by J. Moser and the author [3].

The purpose of this note is to carry out Segre's idea for general n and relate it to the invariants given in [3]. Equation (2) defines an $(n+1)$-parameter family of hypersurfaces when M is non-degenerate. Generalizing the work of Tresse, M. Hachtroudi showed that a projective connection can be defined intrinsically in the space of hyperplane elements of C_{n+1} [4]. The definition is a generalization, by no means obvious, of the construction of classical projective geometry from the data of its hyperplanes; cf. also Chern [2], Yen [6] for a further generalization. We will show that the definition of Hachtroudi's connection is closely related to that of the connection in [3]. This study has the advantage that it works only with the variables z^j and their holomorphic functions; the conjugate variables $\bar{z}^j$ are not involved. Could this fact be of significance for the results to play a rôle in abstract algebraic geometry?

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1. The Equivalence Problem.

We put $w=z^{n+1}$. The hypersurfaces (2) can be considered as the integral hypersurfaces of the completely integrable differential system

$$dw-p_\alpha dz^\alpha = 0,\ dp_\alpha - r_{\alpha\beta}dz^\beta = 0\ , \tag{3}$$

where $r_{\alpha\beta}=r_{\beta\alpha}$ are holomorphic functions of z^α, w, p_β. (Throughout this paper small Greek indices will run from 1 to n and the summation convention will be adopted.) The latter variables, i.e., z^α, w, p_β, can be interpreted as the coordinates in the space of hyperplane elements in C_{n+1}.

We allow biholomorphic changes of coordinates defined by

$$\begin{aligned} z^{*\alpha} &= z^{*\alpha}(z^\beta, w)\ , \\ w^* &= w^*(z^\beta, w)\ , \end{aligned} \tag{4}$$

and the transformation on p_α is given by expressing that $dw^*-p_\alpha^* dz^{*\alpha}$ is a multiple of $dw-p_\alpha dz^\alpha$. It follows that the form $dw-p_\alpha dz^\alpha$ is defined up to a multiple and the sets of forms

$$dw-p_\alpha dz^\alpha,\ dz^\beta \tag{5a}$$

and

$$dw-p_\alpha dz^\alpha,\ dp_\alpha - r_{\alpha\beta}dz^\beta \tag{5b}$$

are each defined up to a linear transformation. Following the general procedure in studying equivalence problems, we set

$$\begin{aligned} \omega &= u(dw-p_\alpha dz^\alpha)\ , \\ \omega^\alpha &= u_\beta{}^\alpha dz^\beta + u^\alpha(dw-p_\beta dz^\beta)\ , \\ \omega_\alpha &= v_\alpha(dw-p_\beta dz^\beta)+v_\alpha{}^\beta(dp_\beta - r_{\beta\gamma}dz^\gamma)\ , \end{aligned} \tag{6}$$

where

$$u, u_\alpha{}^\beta, v_\alpha{}^\beta, u^\alpha, v_\alpha \tag{7}$$

are new variables satisfying

$$u \neq 0,\ \det(u_\alpha{}^\beta) \neq 0,\ \det(v_\alpha{}^\beta) \neq 0\ .$$

Then the forms in (6) are invariant in the space of all the variables: the ones in (7), together with z^α, w, p_β. Computing mod ω, we find

$$d\omega \equiv u dz^\alpha \wedge dp_\alpha\ ,$$

$$i\omega^\alpha \wedge \omega_\alpha \equiv i u_\beta{}^\alpha v_\alpha{}^\varrho dz^\beta \wedge (dp_\varrho - r_{\varrho\gamma}dz^\gamma)\ .$$

The condition

$$d\omega \equiv i\omega^\alpha \wedge \omega_\alpha$$

is therefore equivalent to

(8) $$u\delta_\alpha^{\ \beta} = iu_\alpha^{\ \gamma}v_\gamma^{\ \beta}\ .$$

We suppose (8) fulfilled, and set

(9) $$d\omega = i\omega^\alpha \wedge \omega_\alpha + \omega \wedge \varphi\ ,$$

where φ is defined up to the change

(10) $$\varphi \rightarrow \varphi + t\omega\ .$$

We will take t as another new variable. Our variables are now

(11) $$u(\neq 0),\ u_\alpha^{\ \beta},\ u^\alpha,\ v_\alpha,\ t,\ z^\alpha,\ w,\ p_\alpha\ ,$$

which are $(n+2)^2-1$ in number, the $v_\alpha^{\ \beta}$ being determined by (8), and we have the invariant forms ω, ω^α, ω_α, φ. Our purpose is to show that it is possible to introduce

$$(n+2)^2-1-(2n+2) = n^2+2n+1$$

other invariant forms, characterized by intrinsic conditions, so that the totality forms an independent set.

Clearly we can write

(12) $$d\omega^\alpha = \omega^\beta \wedge \varphi_\beta^{\ \alpha} + \omega \wedge \varphi^\alpha\ .$$

LEMMA 1. *The forms $\varphi_\alpha^{\ \beta}$ in* (12) *can be so chosen that*

(13) $$d\omega_\alpha \equiv \varphi_\alpha^{\ \beta} \wedge \omega_\beta + \omega_\alpha \wedge \varphi, \quad \bmod \omega\ .$$

They are then determined up to additive terms in ω.

In fact, exterior differentiation of (9) gives the equation

(14) $$i\omega^\alpha \wedge (-d\omega_\alpha + \varphi_\alpha^{\ \beta} \wedge \omega_\beta + \omega_\alpha \wedge \varphi) + \omega \wedge (-d\varphi + i\varphi^\alpha \wedge \omega_\alpha) = 0\ .$$

It follows that

(14a) $$-d\omega_\alpha + \varphi_\alpha^{\ \beta} \wedge \omega_\beta + \omega_\alpha \wedge \varphi \equiv 0, \quad \bmod \omega, \omega^\beta\ .$$

Since the system

$$\omega = 0,\ \omega_\alpha = 0$$

is completely integrable, the left-hand side of (14a) is also $\equiv 0$, $\bmod \omega, \omega_\beta$. Hence we have

$$-d\omega_\alpha + \varphi_\alpha^{\ \beta} \wedge \omega_\beta + \omega_\alpha \wedge \varphi \equiv a_{\alpha\beta}^{\ \ \gamma}\omega^\beta \wedge \omega_\gamma, \quad \bmod \omega\ .$$

Substituting into (14), we get

$$a_{\alpha\beta}^{\ \ \gamma} = a_{\beta\alpha}^{\ \ \gamma}\ .$$

Writing $\varphi_\alpha{}^\beta$ for $\varphi_\alpha{}^\beta - a_{\alpha\gamma}{}^\beta \omega^\gamma$ fulfills the equation (13) and leaves (12) unchanged. The second statement in the lemma is immediate.

We shall therefore put

$$d\omega_\alpha = \varphi_\alpha{}^\beta \wedge \omega_\beta + \omega_\alpha \wedge \varphi + \omega \wedge \varphi_\alpha \, . \tag{15}$$

Using (14) we let

$$d\varphi = i\omega^\alpha \wedge \varphi_\alpha + i\varphi^\alpha \wedge \omega_\alpha + \omega \wedge \psi \, , \tag{16}$$

where ψ is a new one-form. The forms $\varphi_\alpha{}^\beta$, φ^α, φ_α, ψ are determined up to the transformation

$$\begin{aligned} \varphi_\alpha{}^\beta &= \varphi_\alpha{}^{*\beta} + b_\alpha{}^\beta \omega \, , \\ \varphi^\alpha &= \varphi^{*\alpha} + b_\beta{}^\alpha \omega^\beta + c^\alpha \omega \, , \\ \varphi_\alpha &= \varphi_\alpha{}^* - b_\alpha{}^\beta \omega_\beta + d_\alpha \omega \, , \\ \psi &= \psi^* + i(d_\alpha \omega^\alpha - c^\alpha \omega_\alpha) + e\omega \, . \end{aligned} \tag{17}$$

We shall determine the coefficients $b_\alpha{}^\beta$, c^α, d_α, e by intrinsic conditions imposed on the exterior derivatives of the forms.

For this purpose we take the exterior derivatives of the equations (12) and (15). The resulting equations can be written

$$\begin{aligned} \omega^\beta \wedge \Phi_\beta{}^\alpha + \omega \wedge \Phi^\alpha &= 0 \, , \\ \Phi_\alpha{}^\beta \wedge \omega_\beta - \omega \wedge \Phi_\alpha &= 0 \, , \end{aligned} \tag{18}$$

where we set

$$\begin{aligned} \Phi_\alpha{}^\beta &= d\varphi_\alpha{}^\beta - \varphi_\alpha{}^\gamma \wedge \varphi_\gamma{}^\beta - i\omega_\alpha \wedge \varphi^\beta + i\varphi_\alpha \wedge \omega^\beta + i\delta_\alpha{}^\beta(\varphi_\sigma \wedge \omega^\sigma) + \tfrac{1}{2}\delta_\alpha{}^\beta \psi \wedge \omega \, , \\ \Phi^\alpha &= d\varphi^\alpha - \varphi \wedge \varphi^\alpha - \varphi^\beta \wedge \varphi_\beta{}^\alpha + \tfrac{1}{2}\psi \wedge \omega^\alpha \, , \\ \Phi_\alpha &= d\varphi_\alpha - \varphi_\alpha{}^\beta \wedge \varphi_\beta + \tfrac{1}{2}\psi \wedge \omega_\alpha \, . \end{aligned} \tag{19}$$

From (18) it follows that

$$\Phi_\alpha{}^\beta = S_{\alpha\varrho}{}^{\beta\sigma} \omega^\varrho \wedge \omega_\sigma + \omega \wedge \psi_\alpha{}^\beta \, , \tag{20}$$

where

$$S_{\alpha\varrho}{}^{\beta\sigma} = S_{\varrho\alpha}{}^{\beta\sigma} = S_{\alpha\varrho}{}^{\sigma\beta} \tag{21}$$

and $\psi_\alpha{}^\beta$ is a one-form. Equation (18) then gives

$$\begin{aligned} \Phi^\alpha &= \omega^\beta \wedge \psi_\beta{}^\alpha + \omega \wedge \lambda^\alpha \, , \\ \Phi_\alpha &= \psi_\alpha{}^\beta \wedge \omega_\beta + \omega \wedge \mu_\alpha \, , \end{aligned} \tag{22}$$

where λ^α, μ_α are one-forms.

Applying the transformation (17) and denoting the new coefficients by asterisks, we get

$$S^*_{\alpha\varrho}{}^{\beta\sigma} = S_{\alpha\varrho}{}^{\beta\sigma} - i(\delta_\varrho{}^\sigma b_\alpha{}^\beta + \delta_\alpha{}^\sigma b_\varrho{}^\beta + \delta_\varrho{}^\beta b_\alpha{}^\sigma + \delta_\alpha{}^\beta b_\varrho{}^\sigma) \, . \tag{23}$$

Putting

$$S_\alpha{}^\beta = S_{\alpha\rho}{}^{\beta\rho}, \quad S^{*\beta}_\alpha = S^{*\beta\rho}_{\alpha\rho}, \tag{24}$$

the contraction of (23) gives

$$S^{*\beta}_\alpha = S_\alpha{}^\beta - i(n+2)b_\alpha{}^\beta - i\delta_\alpha{}^\beta b_\rho{}^\rho . \tag{25}$$

Lemma 2. *The forms $\varphi_\alpha{}^\beta$ are determined uniquely by the condition*

$$S_\alpha{}^\beta = 0 . \tag{26}$$

In fact, setting $S^{*\beta}_\alpha = 0$ in (25), we find

$$i(n+2)b_\alpha{}^\beta = S_\alpha{}^\beta - \tfrac{1}{2}(n+1)^{-1}\delta_\alpha{}^\beta S_\rho{}^\rho . \tag{27}$$

From now on we suppose (26) to be fulfilled.
Exterior differentiation of (16) gives

$$-i\omega^\alpha \wedge \Phi_\alpha + i\Phi^\alpha \wedge \omega_\alpha - \omega \wedge \Psi = 0 , \tag{28}$$

where we set

$$\Psi = d\psi - \varphi \wedge \psi - 2i\varphi^\alpha \wedge \varphi_\alpha . \tag{29}$$

Exterior differentiation of the first equation of (19) gives

$$d\Phi_\alpha{}^\beta + \Phi_\alpha{}^\sigma \wedge \varphi_\sigma{}^\beta - \varphi_\alpha{}^\sigma \wedge \Phi_\sigma{}^\beta - i\omega_\alpha \wedge \Phi^\beta - i\Phi_\alpha \wedge \omega^\beta - i\delta_\alpha{}^\beta \Phi_\sigma \wedge \omega^\sigma - \tfrac{1}{2}\delta_\alpha{}^\beta \Psi \wedge \omega = 0 . \tag{30}$$

Contracting, we get

$$d\Phi_\alpha{}^\alpha - i(n+1)\Phi_\sigma \wedge \omega^\sigma - i\Phi^\alpha \wedge \omega_\alpha - \tfrac{1}{2}n\Psi \wedge \omega = 0 . \tag{31}$$

On the other hand, by (20) we have, as a result of (26),

$$\Phi_\alpha{}^\alpha = \omega \wedge \psi_\alpha{}^\alpha .$$

Substitution of this and (22) into (31) gives

$$(n+2)\psi_\alpha{}^\beta + \delta_\alpha{}^\beta \psi_\rho{}^\rho \equiv 0, \quad \mathrm{mod}\, \omega, \omega^\rho, \omega_\sigma ,$$

whence

$$\psi_\alpha{}^\beta \equiv R_\alpha{}^\beta{}_\gamma \omega^\gamma + T_\alpha{}^{\beta\gamma}\omega_\gamma, \quad \mathrm{mod}\, \omega . \tag{32}$$

We can therefore write (20) as

$$\Phi_\alpha{}^\beta = S_{\alpha\rho}{}^{\beta\sigma}\omega^\rho \wedge \omega_\sigma + R_\alpha{}^\beta{}_\gamma \omega \wedge \omega^\gamma + T_\alpha{}^{\beta\gamma}\omega \wedge \omega_\gamma . \tag{33}$$

Lemma 3. *The forms φ^α and φ_α are determined uniquely by the conditions*

$$R_\alpha{}^\alpha{}_\beta = 0, \quad T_\alpha{}^{\alpha\beta} = 0 . \tag{34}$$

When they are fulfilled, we have

$$R_{\alpha}{}^{\beta}{}_{\gamma} = R_{\gamma}{}^{\beta}{}_{\alpha}, \quad T_{\alpha}{}^{\beta\gamma} = T_{\alpha}{}^{\gamma\beta} , \tag{35}$$

and (22) *can be written*

$$\begin{aligned} \Phi^{\alpha} &= T_{\beta}{}^{\alpha\gamma}\omega^{\beta} \wedge \omega_{\gamma} + \omega \wedge \lambda^{\alpha} , \\ \Phi_{\alpha} &= R_{\alpha}{}^{\beta}{}_{\gamma}\omega^{\gamma} \wedge \omega_{\beta} + \omega \wedge \mu_{\alpha} . \end{aligned} \tag{36}$$

In fact applying the transformation (17) to the first equation of (19), noticing that $b_{\alpha}{}^{\beta}=0$ and $\varphi_{\alpha}{}^{\beta}$ are completely determined, we find

$$\begin{aligned} R^{*}{}_{\beta}{}^{\alpha}{}_{\gamma} &= R_{\beta}{}^{\alpha}{}_{\gamma} - i\delta_{\gamma}{}^{\alpha}d_{\beta} - \tfrac{1}{2}i\delta_{\beta}{}^{\alpha}d_{\gamma} , \\ T^{*}{}_{\beta}{}^{\alpha\gamma} &= T_{\beta}{}^{\alpha\gamma} - i\delta_{\beta}{}^{\gamma}c^{\alpha} - \tfrac{1}{2}i\delta_{\beta}{}^{\alpha}c^{\gamma} , \end{aligned}$$

so that

$$\begin{aligned} R^{*}{}_{\alpha}{}^{\alpha}{}_{\gamma} &= R_{\alpha}{}^{\alpha}{}_{\gamma} - i(\tfrac{1}{2}n+1)d_{\gamma} , \\ T^{*}{}_{\alpha}{}^{\alpha\gamma} &= T_{\alpha}{}^{\alpha\gamma} - i(\tfrac{1}{2}n+1)c^{\gamma} . \end{aligned}$$

Hence c^{γ} and d_{γ} can be determined to achieve (34).

With conditions (34) we have, from (33),

$$\Phi_{\alpha}{}^{\alpha} = 0 . \tag{37}$$

Equations (28) and (31) then give

$$\omega^{\alpha} \wedge \Phi_{\alpha} : \omega_{\alpha} \wedge \Phi^{\alpha} : \omega \wedge \Psi = 1 : -1 : -2i . \tag{38}$$

This shows that Ψ is of the form

$$\Psi \equiv Q_{\alpha}{}^{\beta}\omega^{\alpha} \wedge \omega_{\beta}, \quad \operatorname{mod} \omega , \tag{39}$$

and hence

$$\begin{aligned} \omega^{\alpha} \wedge \Phi_{\alpha} &= \tfrac{1}{2}i\omega \wedge Q_{\alpha}{}^{\beta}\omega^{\alpha} \wedge \omega_{\beta} , \\ \omega_{\alpha} \wedge \Phi^{\alpha} &= -\tfrac{1}{2}i\omega \wedge Q_{\alpha}{}^{\beta}\omega^{\alpha} \wedge \omega_{\beta} . \end{aligned} \tag{40}$$

From (22), (32), and (40) it follows that, $\operatorname{mod} \omega$,

$$0 \equiv \omega^{\alpha} \wedge \Phi_{\alpha} \equiv \omega^{\alpha} \wedge \psi_{\alpha}{}^{\beta} \wedge \omega_{\beta} \equiv \omega^{\alpha} \wedge (R_{\alpha}{}^{\beta}{}_{\gamma}\omega^{\gamma} + T_{\alpha}{}^{\beta\gamma}\omega_{\gamma}) \wedge \omega_{\beta} .$$

This relation implies (35), and hence also (36). This establishes Lemma 3.

From (36) and (40) we also get

$$\begin{aligned} \omega \wedge \omega^{\alpha} \wedge (\mu_{\alpha} + \tfrac{1}{2}iQ_{\alpha}{}^{\beta}\omega_{\beta}) &= 0 , \\ \omega \wedge \omega_{\beta} \wedge (\lambda^{\beta} + \tfrac{1}{2}iQ_{\alpha}{}^{\beta}\omega^{\alpha}) &= 0 . \end{aligned}$$

We can therefore set

$$\begin{aligned} \lambda^{\alpha} &= -\tfrac{1}{2}iQ_{\beta}{}^{\alpha}\omega^{\beta} + L^{\alpha\beta}\omega_{\beta} , \\ \mu_{\alpha} &= P_{\alpha\beta}\omega^{\beta} - \tfrac{1}{2}iQ_{\alpha}{}^{\beta}\omega_{\beta} , \end{aligned} \tag{41}$$

with

$$L^{\alpha\beta} = L^{\beta\alpha},\ P_{\alpha\beta} = P_{\beta\alpha}\,. \tag{42}$$

Substituting into (36), we have the expressions

$$\begin{aligned}\Phi^{\alpha} &= T_{\beta}{}^{\alpha\gamma}\omega^{\beta}\wedge\omega_{\gamma} - \tfrac{1}{2}iQ_{\beta}{}^{\alpha}\omega\wedge\omega^{\beta} + L^{\alpha\beta}\omega\wedge\omega_{\beta}\,,\\ \Phi_{\alpha} &= R_{\alpha}{}^{\beta}{}_{\gamma}\omega^{\gamma}\wedge\omega_{\beta} + P_{\alpha\beta}\omega\wedge\omega^{\beta} - \tfrac{1}{2}iQ_{\alpha}{}^{\beta}\omega\wedge\omega_{\beta}\,.\end{aligned} \tag{43}$$

By using the second and third equations of (19) we immediately get the lemma:

LEMMA 4. *The form ψ is completely determined by the condition*

$$Q_{\alpha}{}^{\alpha} = 0\,. \tag{44}$$

To find the expression for Ψ, we write down the equation obtained by exterior differentiation of (29), which is

$$d\Psi - \varphi\wedge\Psi + 2i\Phi^{\alpha}\wedge\varphi_{\alpha} - 2i\varphi^{\alpha}\wedge\Phi_{\alpha} = 0\,. \tag{45}$$

By (39) we set

$$\Psi = Q_{\alpha}{}^{\beta}\omega^{\alpha}\wedge\omega_{\beta} + \omega\wedge\nu\,. \tag{46}$$

Substituting this into (45) and making use of (43), we get

$$\{dQ_{\alpha}{}^{\beta} - Q_{\varrho}{}^{\beta}\varphi_{\alpha}{}^{\varrho} + Q_{\alpha}{}^{\sigma}\varphi_{\sigma}{}^{\beta} - 2Q_{\alpha}{}^{\beta}\varphi + i\delta_{\alpha}{}^{\beta}\nu + 2iT_{\alpha}{}^{\sigma\beta}\varphi_{\sigma} - 2iR_{\sigma}{}^{\beta}{}_{\alpha}\varphi^{\sigma}\}\omega^{\alpha}\wedge\omega_{\beta} \equiv 0,\quad \operatorname{mod}\omega\,.$$

It follows that the expression between the braces is $\equiv 0 \operatorname{mod}\omega, \omega^{\varrho}, \omega_{\sigma}$. Contracting α, β and using (34), (35), (44), we conclude that ν is $\equiv 0 \operatorname{mod}\omega, \omega^{\alpha}, \omega_{\beta}$. We can therefore put

$$\Psi = Q_{\alpha}{}^{\beta}\omega^{\alpha}\wedge\omega_{\beta} + H_{\alpha}\omega\wedge\omega^{\alpha} + K^{\alpha}\omega\wedge\omega_{\alpha}\,. \tag{47}$$

We summarize our results in the following theorem:

THEOREM. *Given in C_{n+1} an $(n+1)$-parameter family of hypersurfaces defined by the completely integrable differential system* (3), *there exist in the space of the variables* (11), *the same number of invariant differential forms*

$$\omega,\ \omega^{\alpha},\ \omega_{\alpha},\ \varphi,\ \varphi_{\alpha}{}^{\beta},\ \varphi^{\alpha},\ \varphi_{\alpha},\ \psi\,, \tag{48}$$

linearly independent, whose exterior derivatives are given by the "structure equations" (9), (12), (15), (16), (19), (29). *The "curvature forms" $\Phi_{\alpha}{}^{\beta}$, Φ^{α}, Φ_{α}, Ψ have expressions given by* (33), (43), (47), *whose coefficients satisfy* *the symmetry relations* (21), (34), (35), (42), (44). *The forms* (48) *are completely determined by these conditions.*

When $n=1$, $S_{\alpha\varrho}{}^{\beta\sigma}$, $R_{\alpha}{}^{\beta}{}_{\gamma}$, $T_{\alpha}{}^{\beta\gamma}$, $Q_{\alpha}{}^{\beta}$ all vanish and the lowest-order invariants are L^{11}, P_{11}.

2. Geometrical Construction.

The tangent space T_z at a point $z \in \mathbf{C}_{n+1}$ is a complex vector space of dimension $n+1$. We consider it as a part of a projective space PT_z by adding to it a hyperplane at infinity. In turn PT_z is considered as the quotient space of $V_z{}^* = (V_{n+2} - \{0\})_z$ by the action on $V_{n+2} - \{0\}$ by the multiplication of a non-zero complex number, where V_{n+2} is the complex vector space of dimension $n+2$. To a point $\xi \in PT_z$ the components of the corresponding points of $V_{n+2} - \{0\}$, defined up to a non-zero factor, are called the homogeneous coordinates of ξ. In particular, we let a point $(y^1, \ldots, y^{n+1}) \in T_z$ to have the homogeneous coordinates $(y^1, \ldots, y^{n+1}, 1)$ and a vector $(v^1, \ldots, v^{n+1})$, which can be considered as the difference of two points, to have the homogeneous coordinates $(v^1, \ldots, v^{n+1}, 0)$.

A projective frame in PT_z consists of an ordered set $Z_0, Z_1, \ldots, Z_{n+1} \in V_z{}^*$, linearly independent and defined up to a common factor. On the other hand, a frame in T_z consists of the origin z and an ordered set of $n+1$ vectors. Using the above convention, to a frame in T_z corresponds a uniquely determined projective frame in PT_z.

In the discussion of the last section the forms ω, ω^α constitute a coframe at $z \in \mathbf{C}^*_{n+1}$. They determine uniquely a dual frame, and hence a projective frame $Z_0, \ldots Z_{n+1}$ in PT_z. As in [3], p. 260, we put

$$(49)\quad \pi_0{}^{n+1} = 2\omega,\ \pi_0{}^\alpha = \omega^\alpha,\ \pi_{n+1}{}^{n+1} - \pi_0{}^0 = \varphi,\ \pi_\alpha{}^0 = -i\varphi_\alpha,\ \pi_\alpha{}^{n+1} = 2i\omega_\alpha,$$

$$\pi_{n+1}{}^\alpha = \tfrac{1}{2}\varphi^\alpha,\ \pi_\alpha{}^\beta - \delta_\alpha{}^\beta \pi_0{}^0 = \varphi_\alpha{}^\beta,\ \pi_{n+1}{}^0 = -\tfrac{1}{4}\psi\,.$$

Then it can be verified that the equations

$$(50)\qquad DZ_A = \pi_A{}^B Z_B,\quad 0 \leqq A, B \leqq n+1\,,$$

define a projective connection. Except in notation this is essentially the one defined by M. Hachtroudi [4].

Finally we wish to make a remark on the relation of this connection with the one defined in [3]. We have chosen the notations so that the structure equations are identically the same. This implies that the projective connection underlies the connection in [3].

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