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THE CUNTZ-PIMSNER EXTENSION AND MAPPING CONE EXACT SEQUENCES

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Abstract

For Cuntz-Pimsner algebras of bi-Hilbertian bimodules with finite Jones-Watatani index satisfying some side conditions, we give an explicit isomorphism between the K-theory exact sequences of the mapping cone of the inclusion of the coefficient algebra into a Cuntz-Pimsner algebra, and the Cuntz-Pimsner exact sequence. In the process we extend some results by the second author and collaborators from finite projective bimodules to certain finite index bimodules, and also clarify some aspects of Pimsner's 'extension of scalars' construction.

1. Introduction

Mapping cones play an important role in studying the properties of KK-theory [10], [18], and have likewise been used to further the study of noncommutative topology and dynamics [9], [20]. The aim of this note is to make explicit, in a specific case, the abstract relationship between extensions of C^*-algebras and mapping cone extensions.

The structure of Cuntz-Pimsner algebras lies very close to their K-theoretic properties. Using KK-theory to bring out these features requires numerous calculations [2], [3] which lay bare the similarities of mapping cones and defining extensions. An important feature of this comparison is that the K-theory class produced in the proof of KK-equivalence (cf. §6.2) also appears as a key ingredient when studying Poincaré duality for these algebras [23].

We investigate the relationship between the defining extension of the Cuntz-Pimsner algebra O_E of a bi-Hilbertian bimodule E over A

$$0 \longrightarrow \mathrm{End}_A^0(F_E) \longrightarrow \mathcal{T}_E \xrightarrow{\pi} O_E \longrightarrow 0, \tag{3.1}$$

(here F_E is the Fock module, $\mathcal{T}_E$ the Toeplitz-Pimsner algebra and End_A^0 denotes the algebra of compact endomorphisms) and the exact sequence of the mapping cone $M(A, O_E)$ of the inclusion of the coefficient algebra A into O_E

$$0 \longrightarrow \mathcal{S}O_E \longrightarrow M(A, O_E) \longrightarrow A \longrightarrow 0,$$

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where $\mathscr{S}O_E$ is the suspension. We show that we can construct an explicit isomorphism of the associated K-theory sequences at the level of unbounded KK-cycles.

Abstractly, the existence of such an isomorphism follows from the fact that the KK-category is a triangulated category whose exact triangles are mapping cone triangles, with isomorphisms given by KK-equivalence (cf. [18]). Indeed, for every semi-split extension with quotient map π, by [10], one has an isomorphism of triangles making the extension triangle equivalent to the mapping cone triangle of π, i.e. one has a commutative diagram of triangles where all "vertical" arrows are KK-equivalences.

More specifically, in the case of Cuntz-Pimsner algebras, when the coefficient algebra A is nuclear, the defining extension is semi-split, and hence one obtains an isomorphism of the extension triangle with the mapping cone triangle for π

$$\mathscr{S}O_E \longrightarrow M(\mathscr{T}_E, O_E) \longrightarrow \mathscr{T}_E \xrightarrow{\pi} O_E. \tag{1.1}$$

Using the KK-equivalence between A and $\mathscr{T}_E$ and the natural Morita equivalence between A and $\mathrm{End}^0_A(\mathscr{F}_E)$, one can show that the mapping cone triangle

$$\mathscr{S}O_E \longrightarrow M(A, O_E) \longrightarrow A \longrightarrow O_E$$

for the inclusion of the coefficient algebra A into O_E is in turn isomorphic to (1.1). This follows from the axioms of a triangulated category which imply that the mapping cone of $A \to O_E$ is unique up to a (non-canonical) isomorphism in KK. Combining the two isomorphisms of triangles, one obtains the isomorphism

$$\begin{tikzcd}
\mathscr{S}O_E \arrow[r] \arrow[d, "="'] & M(A, O_E) \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, "\alpha"] & O_E \arrow[d, "="] \\
\mathscr{S}O_E \arrow[r] & \mathrm{End}^0_A(\mathscr{F}_E) \arrow[r] & \mathscr{T}_E \arrow[r] & O_E
\end{tikzcd}$$

which induces an isomorphism of the corresponding KK-exact sequences.

In this paper we provide the isomorphism between the associated six-term exact sequences explicitly at the level of unbounded KK-cycles. This allows one to exploit these mapping cones in concrete computations. We indicate how this works in the case of C^*-algebras of non-singular graphs.

Many of the constructions we rely on from [12], [22] were proved for finitely generated bimodules over unital algebras.

We relax the unitality requirement to extend these results to handle the more general case of bimodules which are the restriction to A of a finitely generated module over a unitisation A_b. The main reason for this is to handle suspensions.

Our main result is as follows.

THEOREM 1.1. *Let E be a bi-Hilbertian A-bimodule with $E = \widetilde{E} \otimes_{A_b} A$ where $\widetilde{E}$ is a finitely generated bimodule over a unitisation A_b of A.*

Further assume that E satisfies Assumptions 1 and 2 on pages 302 and 303 respectively. Let $(\mathcal{O}_E, \Xi_A, \mathcal{D})$ be the unbounded representative of the defining extension of O_E, and $(M(A, \mathcal{O}_E), \hat{\Xi}_A, \hat{\mathcal{D}})$ the lift to the mapping cone. Then

$$\cdot \otimes_{M(A,O_E)} [(M(A, \mathcal{O}_E), \hat{\Xi}_A, \hat{\mathcal{D}})]\colon K_*(M(A, O_E)) \longrightarrow K_*(A)$$

is an isomorphism that makes diagrams in K-theory commute. If furthermore the algebra A belongs to the bootstrap class, the Kasparov product with the class $[(M(A, \mathcal{O}_E), \hat{\Xi}_A, \hat{\mathcal{D}})] \in KK(M(A, O_E), A)$ is a KK-equivalence.

Together with the identity map, $\cdot \otimes_{M(A,O_E)} [(M(A, \mathcal{O}_E), \hat{\Xi}_A, \hat{\mathcal{D}})]$ induces an isomorphism of KK-theory exact sequences.

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2. Finite index bi-Hilbertian bimodules for non-unital algebras

We will now show how to extends the results of [22] and [12] to non-unital algebras using more refined constructions from [13]. In [22] and [12], the basic data was a unital separable nuclear C^*-algebra A, and a bi-Hilbertian bimodule E over A in the sense of [13, Definition 2.3].

In this paper we will relax the assumption of unitality employed in [22], [12], and consequently also the finitely generated and projective hypotheses on the module E. So we will assume throughout the paper that E is a countably generated bi-Hilbertian bimodule over A, i.e., E carries A-valued inner products ${}_A(\cdot|\cdot)$, $(\cdot|\cdot)_A$ for which the respective actions are injective and adjointable, and for which E is complete for both norms. We write ${}_AE$ for E when we wish to emphasise its left module structure and E_A for E when emphasising the right module structure.

Note that a bi-Hilbertian bimodule is a special case of a C^*-correspondence (E,ϕ) over A, which is a right Hilbert A-module E_A endowed with a $*$-homomorphism $\phi\colon A\to\mathrm{End}_A^*(E)$, where $\mathrm{End}_A^*(E)$ is the algebra of adjointable operators on E. For x and y in E_A, we denote the associated rank-one operator by $\Theta_{x,y}:=x(y|\cdot)_A$. The algebra of compact operators $\mathrm{End}_A^0(E)$ is the closed linear span of the rank-one operators $\Theta_{x,y}$. The algebra $\mathrm{End}_A^*(E)$ is the multiplier algebra $\mathrm{Mult}(\mathrm{End}_A^0(E))$ of the compact endomorphisms $\mathrm{End}_A^0(E)$.

Throughout the paper we will assume that there is a unitisation A_b of A and a bi-Hilbertian A_b-bimodule $\widetilde{E}$ which is finitely generated and projective as both a right and left module, such that $E=\widetilde{E}\otimes_{A_b}A$. Sufficient conditions guaranteeing the existence of such a bimodule $\widetilde{E}$ are discussed in [24]. Later, we will need to be even more restrictive and ask for the unitisation to be the minimal one, $A^\sim$.

Since E is countably generated (as a right module) there are vectors $\{e_j\}_{j\geq 1}\subset E$ such that

$$\sum_{j\geq 1}\Theta_{e_j,e_j}=\mathrm{Id}_E,$$

where the convergence is in the strict topology of $\mathrm{End}_A^*(E)$. Such a collection of vectors is called a frame, and [13, Theorem 2.22] proves that

$$\mathrm{e}^{\beta}:=\sum_{j\geq 1}{}_A(e_j|e_j) \tag{2.1}$$

is a well-defined (central positive) element of the multiplier algebra of A if and only if the left action of A on E is by compact endomorphisms. The injectivity of the left action which we assume ensures that e^{β} is invertible (justifying the notation). Equation (2.1) expresses the finiteness of the right Jones-Watatani index of E, which is then independent of the choice of frame. This finiteness condition seems to be the correct replacement for the finitely generated hypothesis in the non unital case, since a module over a unital algebra with finite right Jones-Watatani index is finitely generated (and so projective). As further evidence for this, and for later use, we record the following result.

PROPOSITION 2.1 (cf. [24]). *Let E be a bi-Hilbertian A-bimodule with finite right Jones-Watatani index* e^{β}. *Define the suspended bi-Hilbertian $\mathcal{S}A$-bimodule $\mathcal{S}E$ over the suspension $\mathcal{S}A:=C_0(\mathbb{R})\otimes A$ as follows. Define $\mathcal{S}E:=C_0(\mathbb{R})\otimes E$, with the operations ($f_j,g_j\in C_0(\mathbb{R})$, $a_j\in A$, $e_j\in E$)*

$$\begin{aligned}(g_1\otimes a_1)\cdot(f\otimes e)\cdot(g_2\otimes a_2)&=g_1fg_2\otimes a_1ea_2\\ (f_1\otimes e_1|f_2\otimes e_2)_{\mathcal{S}A}&:=f_1^*f_2\otimes(e_1|e_2)_A\\ {}_{\mathcal{S}A}(f_1\otimes e_1|f_2\otimes e_2)&=f_1f_2^*\otimes{}_A(e_1|e_2).\end{aligned}$$

Then $\mathcal{S}E$ *has finite right Jones-Watatani index given by* $1 \otimes \mathrm{e}^{\beta}$ *where* $1 \in C_b(\mathbb{R})$ *is the constant function with value* 1 *and* $\mathrm{e}^{\beta} \in \mathrm{Mult}(A)$ *is the right Jones-Watatani index of* E*. If* E_A *is full so too is* $\mathcal{S}E_{\mathcal{S}A}$ *and if the left action of* A *on* E *is injective, so too is the left action of* $\mathcal{S}A$ *on* $\mathcal{S}E$*.*

Proof. The proof that $\mathcal{S}E$ is bi-Hilbertian is a routine check of the conditions, and so too the statements about fullness and injectivity. The right Jones-Watatani index must be finite by [13, Theorem 2.22], since $\mathcal{S}A$ acts by compacts on $\mathcal{S}E$, and so it only remains to determine the value of the index.

We let $\{e_j\}_{j\geq 1}$ be a (countable) frame for E and pick a partition of unity $(\phi_k)_{k\in\mathbb{Z}}$ subordinate to the intervals $(k-\epsilon, 1+k+\epsilon)$ for some fixed $0 < \epsilon < 1$. Then by a direct computation we find that $(\sqrt{\phi_k} \otimes e_j)_{j,k}$ is a frame for $\mathcal{S}E_{\mathcal{S}A}$ and similarly that

$$\sum_{j,k} {}_{\mathcal{S}A}\big(\sqrt{\phi_k} \otimes e_j | \sqrt{\phi_k} \otimes e_j\big) = 1 \otimes \mathrm{e}^{\beta} \in \mathrm{Mult}(\mathcal{S}A) \simeq C_b(\mathbb{R}) \otimes \mathrm{Mult}(A).$$

An important class of examples are the self-Morita equivalence bimodules (SMEBs) over A. A self-Morita equivalence bimodule is a full bi-Hilbertian A-bimodule for which

$${}_A(e|f)g = e(f|g)_A.$$

We do not require this compatibility condition in the definition of bi-Hilbertian bimodule: our notion of Hilbert bimodule is different from the one of [7, Definition 1.8], which was used in [1] in the construction of generalised crossed products. We will see in Proposition 3.2 that, upon changing the algebra of scalars, we can always construct a self-Morita equivalence bimodule over a related algebra out of a bi-Hilbertian bimodule. This implies in particular that the Cuntz-Pimsner algebra of a bi-Hilbertian bimodule can always be interpreted as a generalised crossed product in the sense of [1] for a self-Morita equivalence bimodule over a different algebra.

3. Cuntz-Pimsner algebras and their defining extension

We start from a bi-Hilbertian A-bimodule E with finite right Jones-Watatani index. We assume that the left action of A (which is necessarily by compacts) is also injective, and that the right module E_A is full. Regarding E as a right module with a left A-action by adjointable operators (a correspondence) we can construct the Cuntz-Pimsner algebra O_E. This we do concretely in the Fock representation. The algebraic Fock module is the algebraic direct sum

$$F_E^{\mathrm{alg}} = \bigoplus_{k\geq 0}^{\mathrm{alg}} E^{\otimes_A k} = \bigoplus_{k=0}^{\mathrm{alg}} E^{\otimes k} = A \oplus E \oplus E^{\otimes 2} \oplus \cdots$$

where the copy of A is the trivial A-correspondence. The Fock module F_E is the Hilbert C^*-module completion of F_E^{alg}. For $\nu \in F_E^{\mathrm{alg}}$, we define the creation operator T_ν by the formula

$$T_\nu(e_1 \otimes \cdots \otimes e_k) = \nu \otimes e_1 \otimes \cdots \otimes e_k, \qquad e_j \in E.$$

The expression T_ν extends to an adjointable operator on F_E, whose adjoint T_ν^* acts (when ν is homogenous with $\nu \in E^{\otimes|\nu|}$) by

$$T_\nu^*(e_1 \otimes \cdots \otimes e_k) = \begin{cases} (\nu|e_1 \otimes \cdots \otimes e_{|\nu|})_A \cdot e_{|\nu|+1} \otimes \cdots \otimes e_k, & k \geq |\nu|, \\ 0 & \text{otherwise,} \end{cases}$$

and so is called an annihilation operator. The C^*-algebra generated by the set of creation operators $\{T_e : e \in E\}$ is the Toeplitz-Pimsner algebra $\mathcal{T}_E$. It is straightforward to show that $\mathcal{T}_E$ contains the algebra $\mathrm{End}_A^0(F_E)$ of compact endomorphisms on the Fock module as an ideal. The defining extension for the Cuntz-Pimsner algebra O_E is the short exact sequence

$$0 \longrightarrow \mathrm{End}_A^0(F_E) \longrightarrow \mathcal{T}_E \xrightarrow{\pi} O_E \longrightarrow 0. \tag{3.1}$$

It should be noted that Pimsner [19] in his general construction uses an ideal that in general is smaller than $\mathrm{End}_A^0(F_E)$. In our case, A acts from the left on E_A by compact endomorphisms, ensuring that Pimsner's ideal coincides with $\mathrm{End}_A^0(F_E)$. For $\nu \in F_E^{\mathrm{alg}}$, we let S_ν denote the class of T_ν in O_E. If $\nu \in E^{\otimes k}$ we write $|\nu| := k$.

Since we assume A to be separable and nuclear, by [17, Theorem 2.7] (see also [16, Theorem 7.3]) the algebra O_E is separable and nuclear. By [6, Corollary IV.3.2.5] C^*-algebra extensions with separable and nuclear quotients are semi-split, hence the defining extension (3.1) is semi-split, i.e. it admits a completely positive cross section $s\colon O_E \to \mathcal{T}_E$. As a consequence, the above extension will induce six terms exact sequences in KK-theory.

Using the natural Morita equivalence between $\mathrm{End}_A^0(F_E)$ and A, the KK-equivalence between A and $\mathcal{T}_E$ proved in [19, Theorem 4.4] and [19, Lemma 4.7], the six term exact sequences can be simplified to a great extent. Specialising to the case of K-theory we obtain

$$\begin{tikzcd}
K_0(A) \arrow[r, "1-{[E]}"] & K_0(A) \arrow[r, "\iota_*"] & K_0(O_E) \arrow[d, "\partial"] \\
K_1(O_E) \arrow[u, "\partial"] & K_1(A) \arrow[l, "\iota_*"'] & K_1(A) \arrow[l, "1-{[E]}"']
\end{tikzcd}$$

where $\iota_* := \iota_{A,O_E*}$ is the map induced by the inclusion $\iota_{A,O_E}\colon A \hookrightarrow O_E$ of the coefficient algebra into the Pimsner algebra, and $1-[E]$ denotes the Kasparov product $\cdot \otimes_A ([\mathrm{Id}_{KK(A,A)}] - [E])$.

Similarly, the corresponding six term exact sequence for K-homology reads

$$\begin{tikzcd}
K^0(A) \arrow[d, "\partial"'] & K^0(A) \arrow[l, "1-[E]"'] & K^0(O_E) \arrow[l, "\iota^*"'] \\
K^1(O_E) \arrow[r, "\iota^*"'] & K^1(A) \arrow[r, "1-[E]"'] & K^1(A) \arrow[u, "\partial"']
\end{tikzcd}$$

3.1. Pimsner's extension of scalars

Before tackling the extension (3.1), its KK-class and the relation to mapping cones, we examine the relationship of the Cuntz-Pimsner construction to the generalised crossed product set up of [1]. Pimsner [19] showed that by changing the scalars the completely positive cross section mentioned above can be obtained explicitly, though this is at the expense of changing the exact sequence (3.1) and the coefficient algebra.

We will recall these constructions, and a little background, with a view to proving that Pimsner's extension of scalars realises O_E as the Cuntz-Pimsner algebra of a SMEB. While at least some of the content of this statement is folklore, we could find nothing more explicit than Pimsner's original construction in the literature. We provide both a precise statement and proof below.

The formula

$$z \cdot S_\nu := z^{|\nu|} S_\nu, \qquad \forall\, \nu \in E^{\otimes k},$$

can be seen to extend to a $U(1)$-action on O_E by an $\epsilon/3$-argument. We denote the fixed point algebra for this action by O_E^γ. Averaging over the circle action defines a conditional expectation

$$\rho\colon O_E \to O_E^\gamma, \qquad \rho(x) := \int_{U(1)} z \cdot x \,\mathrm{d}z,$$

where $\mathrm{d}z$ denotes the normalized Haar measure on $U(1)$. The infinitesimal generator of the circle action defines a closed operator N on the completion $X_{O_E^\gamma}$ of O_E as a O_E^γ-Hilbert module in the inner product defined from ρ. Under the spectral subspace assumption (see [8, Definition 2.2]), N is a self-adjoint, regular operator with locally compact resolvent whose commutators with $\{S_\nu : \nu \in F_E^{\mathrm{alg}}\}$ are bounded. In particular,

$$(\mathcal{O}_E, X_{O_E^\gamma}, N) \tag{3.2}$$

defines an unbounded (O_E, O_E^γ)-Kasparov module, where $\mathcal{O}_E$ is the polynomial algebra in the creation and annihilation operators S_e and S_e^*, for $e \in E_A$.

With these reminders in place, we turn to the extension of scalars. First, the self-Morita equivalence bimodule case is precisely when we do not need to extend the scalars, for those C^*-correspondences (E, ϕ) over A for which $O_E^\gamma = A$ can be characterised as follows.

PROPOSITION 3.1 (cf. [16, Proposition 5.18]). *Let (E, ϕ) be a C^*-correspondence over A with left action given by compact operators, and let O_E be the corresponding Pimsner algebra. Then E is a self-Morita equivalence bimodule if and only if the fixed point algebra O_E^γ coincides with the coefficient algebra A.*

In general, O_E^γ is substantially larger than A and the generator of the circle action is insufficient for constructing an unbounded $(\mathcal{O}_E, A)$-Kasparov module representing our original extension (3.1).

The unbounded Kasparov module in (3.2) gives a class in $KK^1(O_E, O_E^\gamma)$, and when E is a self-Morita equivalence bimodule, this class represents the extension (3.1), see [22]. In the more general case when $O_E^\gamma \neq A$, Pimsner considered the right O_E^γ-module $E' := E \otimes_A O_E^\gamma$, [19, pp. 195–196]. Under some additional assumptions this enlargement of the scalars puts us back into the self-Morita equivalence bimodule case, where Cuntz-Pimsner algebras are known to correspond to the generalised crossed-products of [1, Definition 2.4] by [15, Theorem 3.7].

PROPOSITION 3.2. *Given a correspondence (E, ϕ), suppose that the module E_A is full and the left action ϕ is essential, i.e., the linear span of $\phi(A)E_A$ is dense in E_A. Then the module $E' := E \otimes_A O_E^\gamma$ is a bi-Hilbertian bimodule over O_E^γ which is left and right full and which satisfies the compatibility condition*

$$_A(\xi|\eta)\zeta = \xi(\eta|\zeta)_A,$$

hence is a self-Morita equivalence bimodule over O_E^γ. The Cuntz-Pimsner algebra $O_E \cong O_{E'}$ agrees with the generalised crossed product $O_E^\gamma \rtimes_{E'} \mathbb{Z}$.

We again thank Jens Kaad and Bram Mesland for fruitful discussions that lead to the formulation and proof of this result.

PROOF. By its very definition, E' is a right Hilbert O_E^γ-module, with right action and inner product given by the interior tensor product construction. In particular, the right inner O_E^γ-valued product is given by

$$(e_1 \otimes f_1|e_2 \otimes f_2)_{O_E^\gamma} := (f_1|(e_1|e_2)_A f_2)_{O_E^\gamma} = f_1^*(e_1|e_2)_A f_2,$$

for $e_1, e_2 \in E$, $f_1, f_2 \in O_E^\gamma$. If the left action of A on E is essential and the right inner product is full, then E' is a right-full Hilbert O_E^γ-module by the following argument. Using the right fullness of E_A, [21, Lemma 5.53] shows that there exists a sequence $y_j \in E$ such that for all $b \in A$

$$\lim_{k\to\infty} \sum_{j=0}^{k} (y_j|y_j)_A b = b,$$

and thus because the left A action is essential $\lim_{k\to\infty} \sum_{j=0}^{k} (y_j|y_j)_A y = y$ for all $y \in E$. Now let $S_{\mu_1\cdots\mu_n} S^*_{\nu_1\cdots\nu_n} \in O_E^{\gamma}$. We want to show that this element of the fixed point algebra O_E^{γ} can be approximated by inner products. By rewriting the inner product

$$\begin{aligned}(S_{y_j} S_{\nu_1\cdots\nu_n} S^*_{\mu_1\cdots\mu_n} | S_{y_j})_{O_E^{\gamma}}\\ &= S_{\mu_1\cdots\mu_n} S^*_{y_j\nu_1\cdots\nu_n} S_{y_j} = S_{\mu_1\cdots\mu_n} (S^*_{y_j} S_{y_j} S_{\nu_1\cdots\nu_n})^*\\ &= S_{\mu_1\cdots\mu_n} ((y_j|y_j)_A S_{\nu_1\cdots\nu_n})^* = S_{\mu_1\cdots\mu_n} (S_{(y_j|y_j)_A \nu_1\cdots\nu_n})^*,\end{aligned}$$

we see that

$$\begin{aligned}\lim_{k\to\infty} \sum_{j=0}^{k} (S_{y_j} S_{\nu_1\cdots\nu_n} S^*_{\mu_1\cdots\mu_n} | S_{y_j})_{O_E^{\gamma}} &= \lim_{k\to\infty} \sum_{j=0}^{k} S_{\mu_1\cdots\mu_n} (S_{(y_j|y_j)_A \nu_1\cdots\nu_n})^*\\ &= S_{\mu_1\cdots\mu_n} S^*_{\nu_1\cdots\nu_n},\end{aligned}$$

and so $E \otimes_A O_E^{\gamma}$ is right full.

The non-trivial part is the left module structure. We define a left action $\widetilde{\phi}\colon O_E^{\gamma} \to \mathrm{End}^*_{O_E^{\gamma}}(E')$ by using the natural inclusion $E' \hookrightarrow O_E$ given on simple tensors by $e \otimes f \mapsto S_e \cdot f$, $e \in E$ and $f \in O_E^{\gamma}$. The fixed point algebra O_E^{γ}, called the core, is generated by elements of the form $S_\mu S_\nu^*$, with $|\mu| = |\nu| = n$. Such elements act on simple tensors by

$$\widetilde{\phi}(S_\mu S_\nu{}^*)(e \otimes f) = \mu_1 \otimes (S_{\mu_2\cdots\mu_n} S^*_{\nu_2\cdots\nu_n} (\nu_1|e)_A f), \qquad e \in E,\ f \in O_E^{\gamma},$$

since $S_{\mu_2\cdots\mu_n} S^*_{\nu_2\cdots\nu_n}$ is again an element of the fixed point algebra O_E^{γ}.

In order to define a left inner product, we again use the above identification and define

$${}_{O_E^{\gamma}}(e_1 \otimes f_1 | e_2 \otimes f_2) := S_{e_1} f_1 f_2^* S_{e_2}^*.$$

We now show this inner product is left-full. This can be done by choosing a frame $(x_i)_{i=1}^N$ for E_A (N can be infinity).

Then

$$\sum_{i=1}^{N} S_{x_i} S^*_{x_i} S_{\mu_1\cdots\mu_n} S^*_{\nu_1\cdots\nu_n} = S_{\mu_1\cdots\mu_n} S^*_{\nu_1\cdots\nu_n},$$

and at the same time writing $\nu = \nu_1\overline{\nu}$ and $\mu = \mu_1\overline{\mu}$ we have

$$\begin{aligned} S_{x_i}S_{x_i}^*S_{\mu_1\cdots\mu_n}S_{\nu_1\cdots\nu_n}^* &= S_{x_i}(x_i|\mu_1)_A S_{\overline{\mu}}S_{\overline{\nu}}^*S_{\nu_1}^* \\ &= {}_{O_E^\gamma}\big(x_i \otimes (x_i|\mu_1)_A|\nu_1 \otimes S_{\overline{\nu}}S_{\overline{\mu}}^*\big), \end{aligned}$$

which shows that the left inner product is full. We conclude by checking the compatibility condition by computing that

$$\begin{aligned} &\widetilde{\phi}\big({}_{O_E^\gamma}(e_1 \otimes f_1|e_2 \otimes f_2)\big)e_3 \otimes f_3 \\ &\quad = \widetilde{\phi}(S_{e_1}f_1f_2^*S_{e_2}^*)e_3 \otimes f_3 = e_1 \otimes (f_1f_2^*S_{e_2}^*S_{e_3}f_3) \\ &\quad = e_1 \otimes f_1\,(f_2^*S_{e_2}^*S_{e_3}f_3) = e_1 \otimes f_1\,(e_2 \otimes f_2|e_3 \otimes f_3)_{O_E^\gamma}. \end{aligned}$$

3.2. The extension class

First we recall that odd Kasparov modules give rise to extensions, and indeed all semi-split extensions [14] give rise to odd Kasparov modules. The Kasparov modules we will deal with will come from unbounded Kasparov modules via the bounded transform [4].

DEFINITION 3.3. An odd unbounded Kasparov module for the C^*-algebras A, B is a triple $(\mathscr{A}, E_B, \mathscr{D})$ where E_B is a countably generated right C^*-B-module, $\mathscr{A} \subset A$ is a dense $*$-subalgebra which is represented as adjointable operators on E_B, and $\mathscr{D}$ is a self-adjoint regular operator such that $a\,\mathrm{Dom}(\mathscr{D}) \subset \mathrm{Dom}(\mathscr{D})$ for all $a \in \mathscr{A}$, $[\mathscr{D}, a]$ extends to an adjointable operator and $a(1+\mathscr{D}^2)^{-1/2}$ is a compact endomorphism.

A Kasparov module representing the class of the extension (3.1) was constructed in [22], under the assumption that A is unital and E is finitely generated, and a further assumption discussed below. Here we recall the salient points, and extend the discussion to handle the non-unital situation.

Let E be a bi-Hilbertian A-bimodule with finite right Jones-Watatani index, full as a right module and with injective left action of A. We choose a frame $(e_i)_{i\geq 1}$ for E_A. The frame $(e_i)_{i\geq 1}$ induces a frame for $E^{\otimes_A k}$, namely $(e_\rho)_{|\rho|=k}$ where ρ is a multi-index and $e_\rho = e_{\rho_1} \otimes \cdots \otimes e_{\rho_k}$. We define

$$\Phi_k\colon \mathrm{End}_A^{00}(E^{\otimes_A k}) \to A, \qquad \Phi_k(T) = \sum_{|\rho|=k} {}_A(Te_\rho|e_\rho).$$

Here $\mathrm{End}_A^{00}(E^{\otimes_A k})$ denotes the finite rank operators on $E^{\otimes_A k}$. It follows from [13, Lem. 2.16] that Φ_k does not depend on the choice of frame and extends to a norm continuous map on $\mathrm{End}^0_A(E^{\otimes k})$, [13, Corollary 2.24]. By [13, Proposition 2.27], the functionals Φ_k extend to strictly continuous maps $\Phi_k\colon \mathrm{End}^*_A(E^{\otimes k}) \to \mathrm{Mult}(A)$.

In particular, we denote by e^{β_k} the element $\Phi_k(\mathrm{Id}_{E^{\otimes_A k}}) = \sum_{|\rho|=k} {}_A(e_\rho|e_\rho) \in \mathrm{Mult}(A)$. Since Φ_k is independent of the choice of frame, so is e^{β_k}. Note that e^{β_k} is a positive, central, invertible element of $\mathrm{Mult}(A)$, [13, Corollary 2.24, 2.28]. Therefore β_k is a well-defined self-adjoint central element in $\mathrm{Mult}(A)$.

We further extend the functional Φ_k to $\Phi_k\colon \mathrm{End}^*_A(F_E) \to \mathrm{Mult}(A)$ by defining $\Phi_k(T) := \Phi_k(P_k T P_k)$ for $T \in \mathrm{End}^*_A(F_E)$, where $P_k\colon F_E \to E^{\otimes k}$ is the projection. Naively, we would like to define

$$\Phi_\infty(T) \text{ “:=” } \mathrm{res}_{s=1} \sum_{k=0}^{\infty} \Phi_k(T)\mathrm{e}^{-\beta_k}(1+k^2)^{-s/2}, \tag{3.3}$$

for suitable $T \in \mathrm{End}^*_A(F_E)$.

Indeed, $\Phi_k(T)\mathrm{e}^{-\beta_k}$ is easily shown to be bounded, and so it is tempting to try to define Φ_∞ using some ‘generalised residue’ in the sense of generalised limits and Dixmier traces. In general, problems arise since Φ_∞ (if well-defined) is not a numerical functional, but A-valued. Worse still, in the non-unital setting we only have the strict continuity of the Φ_k in general. The lack of norm continuity is handled as follows.

LEMMA 3.4. *Suppose that* $T \in \mathcal{T}_E \subset \mathrm{End}^*_A(F_E)$. *Then for* $k = 0, 1, 2, \ldots$, *the compression* $P_k T P_k$ *is a compact endomorphism on* $E^{\otimes k}$, *and hence* $\Phi_k\colon \mathcal{T}_E \to A$ *is norm continuous.*

PROOF. We approximate $T \in \mathcal{T}_E$ in norm by a finite sum of generators $T_\xi T_\eta^*$ for $\xi, \eta \in F_E$ homogenous. If $|\xi| \neq |\eta|$ then $P_k T_\xi T_\eta^* P_k = 0$, and so we suppose that $|\xi| = |\eta|$.

In that case, for $k < |\xi|$ we again have $P_k T_\xi T_\eta^* P_k = 0$, while for $k \geq |\xi|$ the endomorphism $P_k T_\xi T_\eta^* P_k$ coincides with a compact endomorphism of $E^{\otimes k}$ by [19, Corollary 3.7] and the injectivity of the left action of A. Since $P_k T P_k$ is approximated in norm by finite sums of endomorphisms $P_k T_\xi T_\eta^* P_k$, $P_k T P_k$ is a compact endomorphism of $E^{\otimes k}$.

Thus for $\mathrm{Re}(s) > 1$, since $\|\Phi_k(T)\mathrm{e}^{-\beta_k}\| \leq \|T\|$, the map

$$\mathcal{T}_E \ni T \mapsto \sum_{k=0}^{\infty} \Phi_k(T)\mathrm{e}^{-\beta_k}(1+k^2)^{-s/2}$$

is norm continuous. The only remaining problem with the tentative definition in Equation (3.3) is the existence of the residue. Following [22], we work under the following assumption guaranteeing that the residue exists for $T \in \mathcal{T}_E$.

Assumption 1. We assume that for every $k \in \mathbb{N}$, there is a $\delta > 0$ such that whenever $\nu \in E^{\otimes k}$ there exists a $\widetilde{\nu} \in E^{\otimes k}$ satisfying

$$\|e^{-\beta_n} \nu e^{\beta_{n-k}} - \widetilde{\nu}\| = O(n^{-\delta}), \qquad \text{as} \quad n \to \infty.$$

When Assumption 1 holds, Equation (3.3) defines an A-bilinear functional $\Phi_\infty\colon \mathcal{T}_E \to A$, which is a continuous A-bilinear positive expectation, which in addition vanishes on $\mathrm{End}^0_A(E) \subset \mathcal{T}_E$. Hence Φ_∞ descends to a positive A-bilinear expectation $\Phi_\infty\colon O_E \to A$. The details of this construction can be found in [22, Section 3.2], and the only change in the non-unital case is the norm continuity, which follows from Lemma 3.4. This functional furnishes us with an A-valued inner product $(S_1|S_2)_A := \Phi_\infty(S_1^* S_2)$ on O_E, and the completed module is denoted Ξ_A.

We assume that Assumption 1 holds for the remainder of the paper.

Theorem 3.5. *If the bi-Hilbertian bimodule E satisfies Assumption 1, then the tuple* $(O_E, \Xi_A, 2Q - 1)$ *is an odd Kasparov module representing the class of the extension* (3.1). *The projection Q has range isometrically isomorphic to the Fock module F_E.*

Example 3.6. When E is a self-Morita equivalence bimodule, $\Phi_\infty\colon O_E \to A$ coincides with the expectation $\rho\colon O_E \to O_E^\gamma$ discussed prior to Equation (3.2). Therefore

$$\Xi_A = \bigoplus_{n\in\mathbb{Z}} E^{\otimes n}$$

with the convention that $E^{\otimes(-|n|)} = \overline{E}^{\otimes|n|}$, where $\overline{E}$ is the conjugate module, which agrees with the C^*-algebraic dual of E. In this case we can define the number operator N on the module Ξ_A by $N\rho = n\rho$ for $\rho \in E^{\otimes n}$. Then $(\mathcal{O}_E, \Xi_A, N)$ is an unbounded Kasparov module representing the class of the extension (3.1) in $KK^1(O_E, A)$, by [22, Theorem 3.1].

Theorem 3.5 was extended in [12], where an unbounded representative of the class defined by $(O_E, \Xi_A, 2Q - 1)$ was presented. In order to construct the unbounded representative $\mathcal{D}$ we need an additional assumption on the bimodule. Under Assumption 1, we can define the operator $\mathfrak{q}_k\colon E^{\otimes k} \to E^{\otimes k}$ by

$$\mathfrak{q}_k \nu := \widetilde{\nu} = \lim_{n\to\infty} e^{-\beta_n} \nu e^{\beta_{n-k}}.$$

By [12, Lemma 2.2], each $\mathfrak{q}_k$ is adjointable for both module structures, a bimodule map and positive. Then in order to construct $\mathcal{D}$ we need to assume

Assumption 2. For any k, we can write $\mathfrak{q}_k = c_k R_k = R_k c_k$ where $R_k \in \operatorname{End}^*_A(E^{\otimes k})$ is a projection and c_k is given by left-multiplication by an element in $\operatorname{Mult}(A)$.

Both Assumptions 1 and 2 hold for a wide variety of examples, as shown in [22] and [12].

When A is unital and Assumption 2 holds, [12, Theorem 2.10] proves that the module Ξ_A decomposes as a direct sum of bi-Hilbertian A-bimodules $\Xi_{n,r}$ of finite right Jones-Watatani index.

The direct sum decomposition holds just as in the unital case. To check that the summands have finite right Jones-Watatani index in the non-unital case, we need to compute the index directly in terms of the frame for Ξ_A presented in [12, Lem. 2.8, 2.9]. The construction of the frame begins with a frame $\{e_j\}_{j\geq 1}$ for E_A and a frame $\{f_k\}_{k\geq 1}$ for ${}_AE$, and produces a frame $\{W_{e_\rho, c_{|\sigma|}^{-1/2} f_\sigma}\}_{\rho,\sigma} \subset \Xi_A$ for multi-indices ρ, σ, and where $c_{|\sigma|}$ is as in Assumption 2. For fixed values of $|\rho|$, $|\sigma|$ we have

$$\begin{aligned}
&\sum_{|\rho|=r, |\sigma|=s} {}_A\big(W_{e_\rho, c_s^{-1/2} f_\sigma} \big| W_{e_\rho, c_s^{-1/2} f_\sigma}\big) \\
&\quad = \sum_{|\rho|=r, |\sigma|=s} \Phi_\infty(S_{e_\rho} S^*_{c_s^{-1/2} f_\sigma} S_{c_s^{-1/2} f_\sigma} S^*_{e_\rho}) \\
&\quad = \sum_{|\rho|=r, |\sigma|=s} \Phi_\infty(S_{e_\rho} (c_s^{-1/2} f_\sigma | c_s^{-1/2} f_\sigma)_A S^*_{e_\rho}) \\
&\quad \leq \|c_s^{-1}\| \sum_{|\rho|=r, |\sigma|=s} \Phi_\infty(S_{e_\rho} (f_\sigma | f_\sigma)_A S^*_{e_\rho}) \\
&\quad \leq \|c_s^{-1}\| \sum_{|\rho|=r} \Phi_\infty(S_{e_\rho} \ell_s S^*_{e_\rho}) \leq \|c_s\|^{-1} \ell_s \sum_{|\rho|=r} \Phi_\infty(S_{e_\rho} S^*_{e_\rho}) \\
&\quad = \|c_s\|^{-1}\, \ell_s\, \mathrm{e}^{\beta_r},
\end{aligned}$$

where ℓ_s is the *left numerical* Jones-Watatani index of $E^{\otimes s}$, which is finite by [13, Theorem 4.8]. This computation shows (in particular) that the summands $\Xi_{n,r}$ in the decomposition

$$\Xi_A = \bigoplus_{n\in\mathbb{Z}, r\geq \max\{0,n\}} \Xi_{n,r} \tag{3.4}$$

are bi-Hilbertian A-bimodules of finite right Jones-Watatani index, and we denote the projections onto these sub-modules by $P_{n,r}$.

Then one defines $\mathcal{D} = \sum_{n,r} \psi(n,r) P_{n,r}$ where ψ is a suitable function, [12, Definition 2.12]. By [12, Lem. 2.14] the projection Q appearing in Theorem 3.5

has the form

$$Q = \sum_{n=0}^{\infty} P_{n,n},$$

with respect to the above decomposition (3.4).

REMARK 3.7. Note that in the self-Morita equivalence bimodule case, a suitable choice of the operator $\mathscr{D}$ along with the decomposition of the module Ξ_A gives the number operator and the decomposition described in Example 3.6.

We assume that Assumption 2 holds for the remainder of the paper.

THEOREM 3.8. *If the bi-Hilbertian A-bimodule E satisfies Assumptions 1 and 2, then the triple $(O_E, \Xi_A, \mathscr{D})$ is an odd unbounded Kasparov module representing the class of the extension* (3.1). *The spectrum of $\mathscr{D}$ can be chosen to consist of integers with bi-Hilbertian A-bimodule eigenspaces of finite right Jones-Watatani index, and non-negative spectral projection Q.*

The only difference arising from Theorem 2.16 of [12] in the non-unital case is that the resolvent of $\mathscr{D}$ is not compact, but only locally compact. This follows since, just as in Lemma 3.4, the compression $P_{m,s}SP_{n,r}$ of $S \in O_E$ is a compact endomorphism. Since the eigenvalues of $\mathscr{D}$ are chosen to have ±infinity as their only limit points, we find that $S(1 + \mathscr{D}^2)^{-1/2}$ is a norm convergent sum of compacts.

Our last task before turning to the mapping cone exact sequence is to show that the class of modules we consider is stable under suspension. Proposition 2.1 gives us most of what we want, and we just need to check that if E satisfies Assumptions 1 and 2 then so too does $\mathscr{S}E$.

PROPOSITION 3.9. *Let E be a bi-Hilbertian A-bimodule with $E = E_b \otimes_{A_b} A$ where E_b is a finitely generated bimodule over the unitisation A_b of A. If E satisfies Assumptions 1 and 2, then the suspended module $\mathscr{S}E := (C(S^1) \otimes E_b) \otimes_{C(S^1, A_b)} \mathscr{S}A$ is a bi-Hilbertian $\mathscr{S}A$-bimodule and satisfies Assumptions 1 and 2.*

PROOF. This follows from Proposition 2.1 and the fact that the right Jones-Watatani index of $(\mathscr{S}E)^{\otimes k}$ is $1 \otimes e^{\beta_k}$ where e^{β_k} is the right Jones-Watatani index of $E^{\otimes k}$.

4. Comparing the mapping cone and Cuntz-Pimsner exact sequences

In addition to the defining exact sequence for O_E, we can look at the mapping cone extension for the inclusion $\iota_{A,O_E}\colon A \hookrightarrow O_E$ of the scalars into the Cuntz-Pimsner algebra. Recall that the mapping cone $M(A, O_E)$ of the inclusion

ι_{A,O_E} is the C^*-algebra

$$M(A, O_E) := \{f \in C([0,\infty), O_E) : f(0) \in A,\, f(\infty) = 0,\, f \text{ continuous}\}$$

We will frequently abbreviate $M(A, O_E)$ to M. The algebra M fits into a short exact sequence involving the suspension, for which we use the convention $\mathcal{S}O_E \simeq C_0((0,\infty), O_E)$.

The mapping cone extension

$$0 \longrightarrow \mathcal{S}O_E \xrightarrow{j_*} M(A, O_E) \xrightarrow{\text{ev}} A \longrightarrow 0,$$

with $\text{ev}(f) = f(0)$ and $j(g \otimes a)(t) = g(t)a$, is semi-split and induces six term exact sequences in KK-theory.

Specialising to K-theory yields the exact sequence

$$\begin{tikzcd}
K_0(A) \arrow[r, "\partial'"] & K_0(O_E) \arrow[r, "j_*"] & K_1(M) \arrow[d, "\text{ev}_*"] \\
K_0(M) \arrow[u, "\text{ev}_*"] & K_1(O_E) \arrow[l, "j_*"'] & K_1(A) \arrow[l, "\partial'"']
\end{tikzcd}$$

By [9, Lemma 3.1] the boundary map $\partial' \colon K_j(A) \to K_j(O_E)$ is given, up to the Bott map Bott: $K_j(O_E) \to K_{j+1}(\mathcal{S}O_E)$, by minus the inclusion of A in O_E, i.e. $\partial' = -\text{Bott} \circ \iota_{A,O_E*}$. Similar considerations hold for the K-homology exact sequence.

We now compare the defining short exact sequence for O_E and the mapping cone sequence for the inclusion $\iota_{A,O_E} \colon A \hookrightarrow O_E$. To do so, we use the identification Bott: $K_j(O_E) \to K_{j+1}(\mathcal{S}O_E)$ to define a map $j_*^B \colon K_i(O_E) \to K_{i+1}(M)$ given by $j_* \circ \text{Bott}$. Then we have the partial comparison with two out of three maps given by the identity:

$$\begin{tikzcd}
\cdots \arrow[r] & K_*(O_E) \arrow[r, "j_*^B"] \arrow[d, "="] & K_{*+1}(M) \arrow[r, "\text{ev}_*"] \arrow[d, "?"] & K_{*+1}(A) \arrow[r, "\iota_*"] \arrow[d, "="] & K_{*+1}(O_E) \arrow[r] \arrow[d, "="] & \cdots \\
\cdots \arrow[r] & K_*(O_E) \arrow[r, "\partial"] & K_{*+1}(A) \arrow[r, "1-{[E]}"] & K_{*+1}(A) \arrow[r, "\iota_*"] & K_{*+1}(O_E) \arrow[r] & \cdots
\end{tikzcd}$$

Thus the question we seek to address is whether there is a map that can be put in place of ? which makes the diagram commute (and so provides an isomorphism of six-term sequences).

Remark 4.1. As pointed out in the introduction, the existence of an isomorphism between the two exact sequences follows from the fact that the KK-category is triangulated, with exact triangles the mapping cone triangles.

The missing map can be easily constructed as a Kasparov product with the class

$$[\widetilde{\alpha}] \otimes_{M(\mathscr{T}_E, O_E)} [u] \otimes_{\mathrm{End}^0_A(\mathscr{F}_E)} [\mathscr{F}_E] \in KK(M, A), \tag{4.1}$$

where $\widetilde{\alpha} \colon M(A, O_E) \to M(\mathscr{T}_E, O_E)$ is the inclusion of mapping cones induced by the natural inclusion $\alpha \colon A \to \mathscr{T}_E$, $[\mathscr{F}_E] \in KK(\mathrm{End}^0_A(\mathscr{F}_E), A)$ is the class of the Morita equivalence, and $[u] \in KK(M(\mathscr{T}_E, O_E), \mathrm{End}^0_A(\mathscr{F}_E))$ is the KK-equivalence given by [10, Corollary 2.4].

In the following we will provide an unbounded representative for a class that makes diagrams in K-theory commute, by lifting the unbounded representative of the extension class to the mapping cone, as we describe below. The axioms of triangulated categories do not guarantee the uniqueness of such a class, hence we leave it as an open problem to verify that our unbounded Kasparov module is a representative for the class in (4.1).

The map ∂ is implemented by the Kasparov product with the class of the defining extension. Now we are working under Assumptions 1 and 2, and so we have an explicit unbounded representative $(\mathscr{O}_E, \Xi_A, \mathscr{D})$ for the defining extension. As noted earlier, $\mathscr{D}$ has discrete spectrum and commutes with the left action of A, hence we have $\iota^*_{A,O_E}[(\mathscr{O}_E, \Xi_A, \mathscr{D})] = 0$. In particular there is a class $[\hat{\mathscr{D}}] \in KK(M(A, O_E), A)$ such that $j^{B*}[\hat{\mathscr{D}}] = [(\mathscr{O}_E, \Xi_A, \mathscr{D})]$. As the notation suggests, there is an explicit unbounded representative for the class $[\hat{\mathscr{D}}]$, provided by the main result of [9].

Subject to some further hypotheses, the class $[\hat{\mathscr{D}}]$ can be used to help compute index pairings, [9, Theorem 5.1], because of the explicit unbounded representative. The even unbounded Kasparov module representing the class $[\hat{\mathscr{D}}]$ is denoted

$$\big(M(A, O_E), \widehat{\Xi}_A = X \oplus X^{\sim}, \hat{\mathscr{D}}\big). \tag{4.2}$$

The module X is a completion of $L^2([0, \infty)) \otimes \Xi_A$ while $X^{\sim}$ also contains functions with a limit at infinity. The operator is given by

$$\hat{\mathscr{D}} = \begin{pmatrix} 0 & -\partial_t + \mathscr{D} \\ \partial_t + \mathscr{D} & 0 \end{pmatrix},$$

together with suitable APS-type boundary conditions, [9, Section 4.1]. The details will not influence the following discussion, but we stress that the operator is concrete, and so index pairings are explicitly computable.

Trying $\cdot \hat{\otimes}_M \hat{\mathscr{D}}$ in place of ? we find that the squares to the left of each instance of $\hat{\mathscr{D}}$ in the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & K_*(O_E) \arrow[r, "j_*^B"] \arrow[d, "="] & K_{*+1}(M) \arrow[r, "\mathrm{ev}_*"] \arrow[d, "\cdot\otimes[\hat{\mathscr{D}}]"] & K_{*+1}(A) \arrow[r, "\iota_*"] \arrow[d, "="] & K_{*+1}(O_E) \arrow[r] \arrow[d, "="] & \cdots \\
\cdots \arrow[r] & K_*(O_E) \arrow[r, "\partial"] & K_{*+1}(A) \arrow[r, "1-[E]"] & K_{*+1}(A) \arrow[r, "\iota_*"] & K_{*+1}(O_E) \arrow[r] & \cdots
\end{tikzcd} \tag{4.3}$$

commute. Now what about the squares to the right? We tackle this question in the next Section.

5. The K-theory of the mapping cone of a Cuntz-Pimsner algebra

We use the characterisation of the K-theory group $K_0(M)$ due to [20]. Classes in $K_0(M)$ can be realised as (stable homotopy classes of) partial isometries $v \in M_k(O_E^\sim)$ with range and source projections vv^*, $v^*v \in M_k(A^\sim)$. Here we adjoin a unit, when A and O_E are non-unital.

In the usual projection picture, the class of the partial isometry v corresponds to the class [9, Section 5]

$$[e_v] - \left[\begin{pmatrix} 1_k & 0 \\ 0 & 0 \end{pmatrix}\right], \qquad e_v(t) = \begin{pmatrix} 1_k - \frac{1}{1+t^2}vv^* & \frac{-it}{1+t^2}v \\ \frac{it}{1+t^2}v^* & \frac{1}{1+t^2}v^*v \end{pmatrix}.$$

Returning to the exact sequence, we again let v be a partial isometry over O_E, say $v \in M_k(O_E^\sim)$, with v^*v and vv^* projections over $\iota_{A,O_E}(A^\sim)$. Then by [20, Lemma 2.3] we have $\mathrm{ev}_*([v]) = [v^*v] - [vv^*]$. In the other direction, we need to evaluate the product $[v] \otimes_{O_E} [\hat{\mathscr{D}}] \otimes_A ([\mathrm{Id}_{KK(A,A)}] - [E])$.

Our strategy is to use [9, Theorem 5.1], to find that the latter product equals

$$-\operatorname{Index}(Q_k v Q_k \colon v^*vF_E^k \to vv^*F_E^k) \otimes_A ([\mathrm{Id}_{KK(A,A)}] - [E]), \tag{5.1}$$

where $Q_k = Q \otimes 1_k$, and $Q\Xi_A = F_E$, the Fock module. Here $[E]$ is short hand for the class in $KK(A, A)$ of $(A, E_A, 0)$, and similarly $[\mathrm{Id}_{KK(A,A)}]$ can be represented by $(A, A_A, 0)$.

In order to be able to use this formula, we need to check the hypotheses of [9, Theorem 5.1], and then actually compute the product in Equation (5.1). The precise statement of [9, Theorem 5.1] in our case is

THEOREM 5.1. *Let* $(\mathscr{O}_E, \Xi_A, \mathscr{D})$ *be the unbounded Kasparov module for the* C^**-algebras* O_E *and* A *representing the extension class. Let* $(M, \widehat{\Xi}_A, \hat{\mathscr{D}})$ *be the unbounded Kasparov* $M(A, O_E)$*-*A *module of Equation* (4.2). *Then for any unitary* $u \in M_k(O_E^\sim)$ *such that* Q_k *and the projection* $(\ker \mathscr{D}) \otimes \mathrm{Id}_k$ *both* *commute with* $u(\mathscr{D}\otimes \mathrm{Id}_k)u^* =: u\mathscr{D}_k u^*$ *and* $u^*\mathscr{D}_k u$ *we have the following equality of index pairings with values in* $K_0(A)$*:*

$$\begin{aligned}\langle [u], [(\mathscr{O}_E, \Xi_A, \mathscr{D})]\rangle &:= \mathrm{Index}(Q_k u^* Q_k)\\ &= \mathrm{Index}(e_u(\hat{\mathscr{D}}_{k+1,+})e_u) - \mathrm{Index}(\hat{\mathscr{D}}_{k+1,+})\\ &=: \left\langle [e_u] - \left[\begin{pmatrix} 1_k & 0\\ 0 & 0\end{pmatrix}\right], [(M, \widehat{\Xi}_A, \hat{\mathscr{D}})]\right\rangle \in K_0(A).\end{aligned}$$

Moreover, if v *is a partial isometry,* $v \in M_k(\mathscr{O}_E^\sim)$*, with* $vv^*, v^*v \in M_k(A^\sim)$ *and such that* Q_k *and* $(\ker \mathscr{D}_k)$ *both commute with* $v\mathscr{D}_k v^*$ *and* $v^*\mathscr{D}_k v$ *we have*

$$\left\langle [e_v] - \left[\begin{pmatrix} 1_k & 0\\ 0 & 0\end{pmatrix}\right], [(M, \widehat{\Xi}_A, \hat{\mathscr{D}})]\right\rangle = -\mathrm{Index}\big(Q_k v Q_k \colon v^*vF_E^k \to vv^*F_E^k\big).$$

It is important to observe that when we consider $v\mathscr{D}_k v^*$, we are suppressing the representation of v on Ξ_A^k, but it makes a difference in what follows. For this reason we temporarily introduce the notation $\varphi\colon O_E \to \mathrm{End}_A^*(\Xi_A)$ for the representation. This representation naturally extends to a representation $\varphi_k\colon M_k(O_E^\sim) \to \mathrm{End}_A^*(\oplus_{i=1}^k \Xi_A)$.

The next result, that we state here in the particular case of Cuntz-Pimsner algebras of bimodules satisfying Assumptions 1 and 2, holds in general for any representation of an algebra on a bimodule, for which there exists a decomposition of the type in Equation (3.4).

Lemma 5.2. *Given* $v \in M_k(O_E^\sim)$*, define*

$$v_{m,s} := \sum_{n\in\mathbb{Z}, r\geq \max\{0,n\}} P_{n+m,r+s}\varphi_k(v)P_{n,r}.$$

Then $\varphi_k(v) = \sum_{m,s} v_{m,s}$ *where the sum converges strictly. If* v *is a partial isometry with range and source projections in* A*, the* $v_{n,r}$ *are partial isometries with* $v_{n,r}^* v_{m,s} = \delta_{n,m}\delta_{r,s} v_{n,r}^* v_{n,r}$ *and* $v_{n,r}v_{m,s}^* = \delta_{n,m}\delta_{r,s}v_{n,r}v_{n,r}^*$*. Hence the projections* $v_{n,r}^* v_{n,r}$ *are pairwise orthogonal, and likewise the projections* $v_{n,r}v_{n,r}^*$ *are pairwise orthogonal.*

Proof. The first statement follows from the definition of $v_{m,s}$, the orthogonal decomposition $\Xi_A = \oplus \Xi_{n,r}$, together with the fact that $\sum_{n,r} P_{n.r}$ converges to Id_{Ξ_A} strictly.

Now suppose that we have $v \in O_E^\sim$ a partial isometry with range and source projections in $A^\sim$ (the following argument adapts to partial isometries in matrix algebras). Then we have

$$\varphi(v) = \sum v_{m,s}.$$

Since vv^* and v^*v are in $A^\sim$, we see, in particular, that they commute with $\mathscr{D}$. Hence

$$vv^* = \sum v_{m,s}v_{n,r}^* \in A^\sim \Rightarrow v_{m,s}v_{n,r}^* = \delta_{m,n}\delta_{s,r}v_{m,s}v_{m,s}^*.$$

Similarly $v_{m,s}^*v_{n,r} = \delta_{m,n}\delta_{s,r}v_{m,s}^*v_{n,r}$. Now we recall that vv^* is a projection and consider

$$\begin{aligned} vv^* &= \sum v_{m,s}v_{m,s}^* = (vv^*)^2 = \Big(\sum v_{m,s}v_{m,s}^*\Big)^2 \\ &= \sum v_{m,s}v_{m,s}^*v_{n,r}v_{n,r}^* = \sum (v_{m,s}v_{m,s}^*)^2 \end{aligned}$$

where the last equality follows from $v_{m,s}^*v_{n,r} = \delta_{m,n}\delta_{s,r}v_{m,s}v_{m,s}^*$. Since we also have $v_{m,s}v_{m,s}^*v_{n,r}v_{n,r}^* = 0$ for $(m, s) \neq (n, r)$, we see that each $v_{m,s}v_{m,s}^*$ is a projection in $A^\sim$, and the various $v_{m,s}v_{m,s}^*$ are mutually orthogonal. Similarly, the $v_{m,s}^*v_{m,s}$ form a set of mutually orthogonal projections.

We deduce the commutation relation $v_{m,s}P_{\ell,t} = P_{\ell+m,t+s}v_{m,s}$ for all $\ell, m \in \mathbb{Z}$, $t \geq \max\{0, \ell\}$, $s \geq \max\{0, m\}$. This seems surprising given the more complicated commutation relation of [12, Lem. 2.15], but they are reconciled by the following observation (proved in the Lemma below). If $\mu \in F_E$ is homogenous of degree $|\mu|$ then for n sufficiently large and positive

$$P_{n+|\mu|,n+|\mu|}S_\mu P_{n,n} \neq 0.$$

Hence $S_\mu = \sum_{j=0}^{|\mu|}(S_\mu)_{|\mu|,j}$ has $S_{\mu,|\mu|} \neq 0$, so the decomposition in the Lemma uses much more information than just the degree given by the gauge action.

LEMMA 5.3. *Suppose that* $S \in O_E^\sim$ *satisfies* $S_{n,r} \neq 0$ *for some* $n \in \mathbb{Z}$ *and* $r \geq \max\{0, n\}$. *Then* $S_{n,n} \neq 0$.

PROOF. We approximate S by a finite sum of monomials $S_\alpha S_\beta^*$. Then $S_{n,r}$ is approximated by monomials $S_\alpha S_\beta^*$ with $|\alpha| = r$ and $|\alpha| - |\beta| = n$.

For such monomials, and $m > |\beta|$, we have the inclusion $S_\alpha S_\beta^* P_{m,m}\Xi_A \subset P_{m+|\alpha|-|\beta|,m+|\alpha|-|\beta|}S_\alpha S_\beta^*\Xi_A$, and by considering $[S_\beta S_\gamma] \in \Xi_A$ we also see that $S_\alpha S_\beta^* P_{m,m} \neq 0$. Hence $P_{m+|\alpha|-|\beta|,m+|\alpha|-|\beta|}S_\alpha S_\beta^* P_{m,m} \neq 0$ for $m > |\beta|$. Hence $(S_\alpha S_\beta^*)_{|\alpha|-|\beta|,|\alpha|-|\beta|} = (S_\alpha S_\beta^*)_{n,n} \neq 0$ and so also $S_{n,n} \neq 0$.

LEMMA 5.4. *Let* $v \in O_E^\sim$ *be a partial isometry with range and source projections in* $A^\sim$. *Then* $\varphi(v)$ *is a finite sum of 'homogenous' components* $v_{m,s}$.

REMARK 5.5. We are effectively repeating the argument of [8, Lem. 4.4 and 4.5] for modular unitaries and partial isometries.

We know that $\mathrm{e}^{it\mathscr{D}}\varphi(S)\mathrm{e}^{-it\mathscr{D}} \in \varphi(O_E)$ for partial isometries S with range and source in A, and that is all we will need.

PROOF. First suppose that v is unitary, and define $w_t = \varphi(v^*)\mathrm{e}^{it\mathscr{D}}\varphi(v)\mathrm{e}^{-it\mathscr{D}}$. It follows from Lemma 5.2 that w_t commutes (strongly) with $\mathscr{D}$ for all t. Then

$$\begin{aligned} w_{t+s} &= \varphi(v^*)\mathrm{e}^{i(t+s)\mathscr{D}}\varphi(v)\mathrm{e}^{-i(t+s)\mathscr{D}} \\ &= \varphi(v^*)\mathrm{e}^{it\mathscr{D}}\varphi(v)\mathrm{e}^{-it\mathscr{D}}\mathrm{e}^{it\mathscr{D}}\varphi(v^*)\mathrm{e}^{-it\mathscr{D}}\mathrm{e}^{i(t+s)\mathscr{D}}\varphi(v)\mathrm{e}^{-i(t+s)\mathscr{D}} \\ &= w_t\mathrm{e}^{it\mathscr{D}}\big(\varphi(v^*)\mathrm{e}^{is\mathscr{D}}\varphi(v)\mathrm{e}^{-is\mathscr{D}}\big)\mathrm{e}^{-it\mathscr{D}} \\ &= w_t\mathrm{e}^{it\mathscr{D}}w_s\mathrm{e}^{-it\mathscr{D}} \\ &= w_t w_s. \end{aligned}$$

Hence w_t is a norm continuous path of unitaries in $A^\sim$, whence $w_t = \mathrm{e}^{ita}$ for some $a = a^* \in A^\sim$. Thus $\mathrm{e}^{it\mathscr{D}}\varphi(v)\mathrm{e}^{-it\mathscr{D}} = \varphi(v)\mathrm{e}^{ita}$. Recall now that we can choose $\mathscr{D}$ to have only integral eigenvalues, and so $\varphi(v) = \varphi(v)\mathrm{e}^{i2\pi a}$. Hence a has spectrum a finite subset of $\mathbb{Z}$, and we then easily see that only finitely many components $v_{m,s}$ can be non-zero. In the general case we replace v by the unitary

$$\begin{pmatrix} 1 - v^*v & v^* \\ v & 1 - vv^* \end{pmatrix}$$

and argue as above.

LEMMA 5.6. *For any partial isometry v over $O_E^\sim$ with range and source projections in $A^\sim$, the operators $\varphi(v)\mathscr{D}\varphi(v^*)$ and $\varphi(v^*)\mathscr{D}\varphi(v)$ commute with both the kernel projection of $\mathscr{D}$ and the non-negative spectral projection of $\mathscr{D}$, given by Q.*

PROOF. We assume for simplicity that $v \in O_E^\sim$. Using Lemmas 5.2 and 5.4, we see that the following computation is justified and yields the first claim:

$$\begin{aligned} v\mathscr{D}v^* &= \sum_{m,s} vP_{m,s}\psi(m,s)v^* = \sum_{m,s,n,r} v_{n,r}P_{m,s}\psi(m,s)v^* \\ &= \sum_{m,s,n,r} P_{m+n,s+r}\psi(m,s)v_{n,r}v^* = \sum_{m,s,n,r} P_{m+n,s+r}\psi(m,s)v_{n,r}v_{n,r}^*. \end{aligned}$$

The claims about $v^*\mathscr{D}v$ follow in the same way.

6. The isomorphism $K_*(M(A, O_E)) \to K_*(A)$ and the KK-equivalence

To prove that $\cdot \otimes_M \hat{\mathscr{D}}\colon K_*(M(A, O_E)) \to K_*(A)$ gives an isomorphism we need only show that $\cdot \otimes_M \hat{\mathscr{D}}$ makes the diagram commute, since it then follows from the five lemma of homological algebra that taking the Kasparov product with $\hat{\mathscr{D}}$ is an isomorphism.

To prove that $\cdot \otimes_M \hat{\mathscr{D}}\colon K_*(M(A, O_E)) \to K_*(A)$ yields a KK-equivalence requires much more in general, but follows relatively easily in the bootstrap case. We go further, and provide an explicit inverse when it exists, and conjecture that in fact it is an inverse in all generality.

6.1. The isomorphism in K-theory

We now know enough to prove the commutation of the diagram in Equation (4.3). We first consider $\cdot \otimes_M [\hat{\mathscr{D}}]\colon K_0(M(A, O_E)) \to K_0(A)$, and in this situation begin by considering $v = v_{m,s}$ ‘homogenous’.

By [12, Lemma 2.14], the range of Q is the range of $\sum_{n\geq 0} P_{n,n}$, hence $QvQ = 0$ unless $s = m$. For $s = m$ we have

$$\begin{aligned}
&\mathrm{Index}(QvQ\colon v^*vF_E \to vv^*F_E) \otimes_A (\mathrm{Id}_{KK(A,A)} - [E])\\
&\quad= \begin{cases} \left([\oplus_{j=0}^{-m-1} v^*vE^{\otimes j}]\right) \otimes_A ([\mathrm{Id}_{KK(A,A)}] - [E]), & m < 0,\\ \left(-[\oplus_{j=0}^{m-1} vv^*E^{\otimes j}]\right) \otimes_A ([\mathrm{Id}_{KK(A,A)}] - [E]), & m \geq 0, \end{cases}\\
&\quad= \begin{cases} [v^*vA] - [v^*vE^{\otimes -m}], & m < 0,\\ -[vv^*A] + [vv^*E^{\otimes m}], & m \geq 0, \end{cases}
\end{aligned}$$

where the last equality follows from a telescopic argument. So to prove that

$$\begin{aligned}
\mathrm{Index}(QvQ\colon v^*vF_E \to vv^*F_E) \otimes_A (\mathrm{Id}_{KK(A,A)} - [E]) &= \mathrm{ev}_*([v])\\
&= [v^*v] - [vv^*]
\end{aligned}$$

we are reduced to proving the isomorphisms of A-modules

$$vv^*E^{\otimes m} \simeq v^*vA \ \text{ for } \ m > 0 \quad \text{and} \quad v^*vE^{\otimes |m|} \simeq vv^*A \ \text{ for } \ m < 0.$$

This is straightforward though, by the following argument.

For $m < 0$, the map $v\colon v^*vE^{\otimes |m|} \to vE^{\otimes |m|} \subset A$ is a one-to-one A-module map, which is onto its image, which is contained in vv^*A. Hence $v^*vE^{\otimes |m|}$ and $vE^{\otimes |m|}$ are isomorphic.

For $m > 0$, the map $v^*\colon vv^*E^{\otimes |m|} \to v^*E^{\otimes |m|} \subset A$ is a one-to-one A-module map, which is onto its image, which is contained in v^*vA. Hence $vv^*E^{\otimes |m|}$ and $v^*E^{\otimes |m|}$ are isomorphic.

Thus the result is true for homogenous partial isometries, and likewise for direct sums of homogenous partial isometries, and by Lemma 5.4 this is enough. This gives commutativity of the diagram

$$\begin{tikzcd}
K_0(M) \arrow[r, "\mathrm{ev}_*"] \arrow[d, "\cdot\otimes[\hat{\mathscr{D}}]"] & K_0(A) \arrow[d, "="] \\
K_0(A) \arrow[r, "\cdot\otimes(\mathrm{Id}_A - [E])"'] & K_0(A)
\end{tikzcd}$$

and hence an isomorphism $\cdot \otimes_M [\hat{\mathscr{D}}]\colon K_0(M(A, O_E)) \to K_0(A)$. To complete the argument, we need to consider suspensions.

If $f \in \mathscr{S}M(A, O_E)$ we let $f(t) = g_t$ with $g_t \in M(A, O_E)$ for all $t \in \mathbb{R}$. Then define

$$\Psi\colon \mathscr{S}M(A, O_E) \to M(\mathscr{S}A, \mathscr{S}O_E), \quad (\Psi(f)(s))(t) = g_t(s),$$

where $s \in [0, \infty)$, $t \in \mathbb{R}$, and check that Ψ is an isomorphism. Hence, in particular, $K_1(M(A, \mathcal{O}_E)) \cong K_0(M(\mathscr{S}A, \mathscr{S}O_E))$.

Next we observe that $O_{\mathscr{S}E} \cong \mathscr{S}O_E$. The isomorphism is defined on generators by $\varphi(S_{f\otimes e}) = f \otimes S_e$, and using the gauge invariant uniqueness theorem, as in [16, Theorem 6.4], we see that the map is injective, and then since the range contains the generators of $\mathscr{S}O_E$, it is an isomorphism.

The unitary isomorphism $(\mathscr{S}A, \mathscr{S}E_{\mathscr{S}A}, 0) = (C_0(\mathbb{R}), C_0(\mathbb{R})_{C_0(\mathbb{R})}, 0) \otimes_{\mathbb{C}} (A, E_A, 0)$ of Kasparov modules shows that the suspension of the map $\cdot \otimes_A \big((A, A_A, 0) - (A, E_A, 0)\big)$ is the map $\cdot \otimes_{\mathscr{S}A} \big((\mathscr{S}A, \mathscr{S}A_{\mathscr{S}A}, 0) - (\mathscr{S}A, \mathscr{S}E_{\mathscr{S}A}, 0)\big)$. A similar but easier statement holds for the suspension of the evaluation map, and so combining these various facts we find that the diagram

$$\begin{tikzcd}
K_1(M(A, O_E)) \arrow[r, "\mathscr{S}\mathrm{ev}_*"] \arrow[d, "\cdot\otimes[\hat{\mathscr{D}}]"] & K_1(A) \arrow[d, "="] \\
K_1(A) \arrow[r, "\mathscr{S}\otimes(\mathrm{Id}_A - [E])"'] & K_1(A)
\end{tikzcd}$$

is given by

$$\begin{tikzcd}
K_0(M(\mathscr{S}A, \mathscr{S}O_E)) \arrow[r, "\mathrm{ev}_*"] \arrow[d, "\cdot\otimes[\hat{\mathscr{D}}]"] & K_0(\mathscr{S}A) \arrow[d, "="] \\
K_0(\mathscr{S}A) \arrow[r, "\otimes(\mathrm{Id}_{\mathscr{S}A} - [\mathscr{S}E])"'] & K_0(\mathscr{S}A)
\end{tikzcd}$$

where now ev_* is the evaluation map corresponding to the inclusion of $\mathscr{S}A$ into $\mathscr{S}O_E$. Finally, it is straightforward to verify that the analogue of the class $[\hat{\mathscr{D}}]$ for $\mathscr{S}E$ is given by $\mathrm{Id}_{KK(\mathscr{S},\mathscr{S})} \otimes [\hat{\mathscr{D}}]$. Now by Propositions 2.1 and 3.9, $\mathscr{S}E$ satisfies all the assumptions that E does. Thus combining our proof that the 'even part' of the diagram commutes with the Künneth theorem (in the form of [25, Remark 7.11]), shows that the 'odd part' of the diagram commutes.

6.2. The KK-equivalence and the main theorem

We conclude by showing that the class of $[\hat{\mathscr{D}}]$ not only implements an isomorphism in K-theory, but an actual KK-equivalence when A is in the bootstrap class.

First observe that, by [5, Proposition 23.10.1], if A and B are two C^*-algebras in the bootstrap class, then $\alpha \in KK(A, B)$ is a KK-equivalence if and only if the induced map $\cdot \otimes_A \alpha\colon K_*(A) \to K_*(B)$ is invertible. This follows from the Universal Coefficient Theorem of [25].

Next, whenever the coefficient algebra A of the correspondence E_A belongs to the bootstrap class, so does the algebra O_E (cf. [16, Proposition 8.8]), and we obtain a KK-equivalence in this case.

Hence, provided that the coefficient algebra is contained in the bootstrap class, the class $[\hat{\mathscr{D}}] \in KK(O_E, A)$ is a KK-equivalence. The problem with this abstract approach is two-fold. First we need to know or verify that the coefficient algebra is in the bootstrap class. While this is often possible, knowing that $\hat{\mathscr{D}}$ is a KK-equivalence does not provide a representative of the other half of the equivalence.

We ameliorate both these problems by providing an explicit representative for the other half of the KK-equivalence in a special case. To obtain this representative, we need to make the additional assumption that the module $E = E_b \otimes_{A_b} A$ is the restriction of a finitely generated bi-Hilbertian bimodule E_b over the unitisation A_b.

Let $(x_j)_{j=1}^k$ be a frame for E_b: when A is unital we can simply take x_j to be a frame for E. Then we can define the matrix over O_{E_b} by

$$w = \begin{pmatrix} S_{x_1}^* & 0 & \cdots & 0 \\ S_{x_2}^* & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \vdots \\ S_{x_k}^* & 0 & \cdots & 0 \end{pmatrix}.$$

Then $w^*w = \mathrm{Id}_{O_{E_b}} \oplus 0_{k-1} = \iota_{A_b, O_{E_b}}(\mathrm{Id}_{A_b}) \oplus 0_{k-1}$ and

$$\begin{aligned} ww^* &= \begin{pmatrix} (x_1|x_1)_{A_b} & (x_1|x_2)_{A_b} & \cdots & (x_1|x_k)_{A_b} \\ (x_2|x_1)_{A_b} & (x_2|x_2)_{A_b} & \cdots & (x_2|x_k)_{A_b} \\ \vdots & \vdots & \ddots & \vdots \\ (x_k|x_1)_{A_b} & (x_k|x_2)_{A_b} & \cdots & (x_k|x_k)_{A_b} \end{pmatrix} \\ &= (x_i|x_j)_{i,j\geq 1} =: q \in M_k(A_b). \end{aligned}$$

The projection q realises $E_b \cong qA_b^k$ as a finite projective module over A_b, with isomorphism given by $e \mapsto ((x_1|e)_{A_b}, \dots, (x_k|e)_{A_b})^T$. We can explicitly realise $[w]$ as a difference of classes of projections over the minimal unitisation $M(A_b, O_{E_b})^{\sim}$ of the mapping cone $M(A_b, O_{E_b})$. (As usual, the equality of the classes of $e_w(\infty)$ and 1_k gives us classes in the *KK* groups for $M(A_b, O_{E_b})$. See [14, Corollary 1, Section 7].) Using [20], we have an identification of classes $[w] = [e_w] - [1_k]$, where

$$\begin{aligned} e_w(t) &= \begin{pmatrix} 1_k - \frac{1}{1+t^2} q & \frac{-it}{1+t^2} w \\ \frac{it}{1+t^2} w^* & \frac{1}{1+t^2} \mathrm{Id}_{O_{E_b}} \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{1+t^2}(1_k - q) + \frac{t^2}{1+t^2} 1_k & \frac{-it}{1+t^2} w \\ \frac{it}{1+t^2} w^* & \frac{1}{1+t^2} \mathrm{Id}_{O_{E_b}} \end{pmatrix}. \end{aligned}$$

To proceed we need a lemma.

LEMMA 6.1. *Let E be an A-A correspondence with left action both essential and by compacts. Let $A \hookrightarrow A_b$ be a unitisation, and let $E = E_b \otimes_{A_b} A$ where E_b is finitely generated projective right A_b module. Then $A \cdot E_b \subset E$ and O_E is a two-sided ideal in O_{E_b}.*

PROOF. Since $a \in A$ acts compactly on E, for any $\epsilon > 0$ there are finitely many vectors $e_j, f_j \in E \subset E_b$ such that $\left\| a - \sum_j \Theta_{e_j, f_j} \right\| < \epsilon$. Then for $g \in E_b$ we find that

$$\left\| ag - \sum_j \Theta_{e_j, f_j} g \right\| = \left\| ag - \sum_j e_j (f_j|g)_{A_b} \right\| < \epsilon,$$

and since E is closed, we see that $ag \in E$.

Using the fact that the left action is essential, together with Cohen factorisation, we can approximate any element in E by finite sums of elements aeb with $a, b \in A$ and $e \in E$. So for $g \in E_b$ we have the following relations in $O_E \cdot O_{E_b}$

$$S_{aeb} S_g = S_{ae} S_{bg}, \qquad S_{aeb} S_g^* = S_{ae} S_{gb^*}^*.$$

From this the final statement follows by approximation of elements of O_{E_b}.

Direct calculation shows that e_w is a projection. Following the same reasoning as in [20, Sect. 2], we obtain $[e_w] - [1_k] \in K_0(M(A_b, O_{E_b}))$. Let $\phi\colon A_b \to \mathrm{End}_{A_b}(E_b)$ be the homomorphism defining the left action. Then

$$\varphi(a) := \big((x_i \mid \phi(a) x_j)_{A_b} \big)_{i,j}$$

defines a left action of A_b on $q(A_b)^k$.

Since $w(\phi(a) \oplus 0_{k-1})w^* = \varphi(a)$ and $w^*\varphi(a)w = \phi(a) \oplus 0_{k-1}$, it is straightforward to check that for all $t \in [0, \infty)$

$$e_w(t)\begin{pmatrix} ((x_i \mid \phi(a)x_j)_{A_b})_{i,j} & 0 \\ 0 & \phi(a) \end{pmatrix} = \begin{pmatrix} ((x_i \mid \phi(a)x_j)_{A_b})_{i,j} & 0 \\ 0 & \phi(a) \end{pmatrix} e_w(t)$$

as operators on $O_{E_b}^{k+1}$ (or $(A_b)^{k+1}$ for $t = 0$). We define $M(A, O_E)_b$ to be the unitisation of the mapping cone $M(A, O_E)$ given by

$$\begin{aligned} &M(A, O_E)_b \\ &\quad = \big\{ f \in \mathrm{Mult}(C([0, \infty), O_E)) : f(0) \in A, \lim_{t\to\infty} f(t) \in A_b, f \text{ cts.} \big\}. \end{aligned}$$

We let A_b act as $\varphi \oplus \phi$ on $e_w M(A, O_E)_b^{k+1}$ and as φ on $M(A, O_E)_b^k$. As a consequence the Kasparov module 'at infinity'

$$(A_b, A_b^k \oplus A_b^k, 0) = (A_b, e_w(\infty)A_b^{k+1} \oplus A_b^k, 0)$$

is homotopic to a degenerate module, and so defines the zero class.

Corollary 6.2. *Let E be an A-A correspondence with left action both non-degenerate and by compacts. Let $A \hookrightarrow A_b$ be a unitisation, and let $E = E_b \otimes_{A_b} A$ where E_b is finitely generated projective right A_b module. Then*

$$W = \left(A, \begin{pmatrix} e_w M(A, O_E)_b^{k+1} \\ M(A, O_E)_b^k \end{pmatrix}, 0 \right)$$

is a Kasparov module with class in $KK(A, M(A, O_E))$. Similarly

$$[W_b] = \left[\left(A_b, \begin{pmatrix} e_w (M(A, O_E)_b)^{k+1} \\ (M(A, O_E)_b)^k \end{pmatrix}, 0 \right) \right] \in KK(A_b, M(A, O_E)),$$

and $[W] = \iota_{A,A_b} \otimes_{A_b} [W_b]$ where ι_{A,A_b} is the class of the inclusion, and $[w] = \iota_{\mathbb{C},A_b} \otimes_{A_b} [W_b]$.

Proof. Since $e_w \in M(A_b, O_{E_b})^\sim \subset M(A_b, O_{E_b})_b$, A is an ideal in A_b, and O_E is an ideal in O_{E_b} by Lemma 6.1, $e_w M(A, O_E)_b^{k+1} \subset M(A, O_E)_b^{k+1}$. Identifying $a \in A_b$ with the constant function with value $a \in A_b$, we see that $A_b \subset M(A, O_E)_b$, which is the compacts on $M(A, O_E)_b$ as a right $M(A, O_E)_b$-module.

Finally, the observed degeneracy of the module at ∞ and excision in K-theory guarantees that $[W_b]$ defines a class in $KK(A_b, M(A, O_E))$.

Lemma 6.3. *Let* $[\mathscr{D}] \in KK^1(O_E, A) = KK(\mathscr{S}O_E, A)$ *be the class of the defining extension for* O_E, $(M(A, O_E), \hat{\Xi}_A, \hat{\mathscr{D}})$ *the lift to the mapping cone, and* $[W] \in KK(A, M(A, O_E))$ *the class defined above. Then*

$$[W] \otimes_{M(A,O_E)} [\widehat{\mathscr{D}}] = -\operatorname{Id}_{KK(A,A)}\,.$$

Proof. Applying [9, Theorem 5.1] gives

$$[w] \otimes_{M(A,O_E)} [\widehat{\mathscr{D}}] = -\operatorname{Index}\big(P \otimes 1_k w P \otimes 1_k \colon w^*w(\Xi)^k \to ww^*(\Xi)^k\big)$$

where P is the non-negative spectral projection of $\mathscr{D}$. Since the non-negative spectral projection of $\mathscr{D}$ is the projection onto a copy of the Fock space, we have

$$\ker(P \otimes 1_k w P \otimes 1_k) = A_A = E_A^{\otimes 0}, \qquad \ker(P \otimes 1_k w^* P \otimes 1_k) = \{0\}.$$

We can interpret the index not just as a difference of right A-modules, but as a difference of A-bimodules. This works because the left action of A commutes with $\mathscr{D}$ and so P. Hence

$$[W] \otimes_M [\widehat{\mathscr{D}}] = -[(A, A_A, 0)] = -\operatorname{Id}_{KK(A,A)}$$

as was to be shown.

From Lemma 6.3, we know that $-[\widehat{\mathscr{D}}] \otimes_A [W] \in KK(M(A, O_E), M(A, O_E))$ is an idempotent element. *In particular,* $[\widehat{\mathscr{D}}] \otimes_A \cdot$ *is always injective and* $[W] \otimes_M \cdot$ *is always surjective and injective on the image of* $[\widehat{\mathscr{D}}] \otimes_A \cdot$. *Thus as soon as* $[\widehat{\mathscr{D}}] \otimes_A \cdot$ *is surjective,* $[\widehat{\mathscr{D}}]$ *is a KK-equivalence.*

One approach to showing that $[W]$ is in fact an inverse for $[\widehat{\mathscr{D}}]$ would be to show that the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & K_*(O_E) \arrow[r, "j_*^B"] \arrow[d, "="'] & K_{*+1}(M) \arrow[r, "\mathrm{ev}_*"] \arrow[d, "\cdot\otimes[W]"'] & K_{*+1}(A) \arrow[r, "\iota_*"] \arrow[d, "="'] & K_{*+1}(O_E) \arrow[r] \arrow[d, "="'] & \cdots \\
\cdots \arrow[r] & K_*(O_E) \arrow[r, "\partial"] & K_{*+1}(A) \arrow[r, "1-[E]"] & K_{*+1}(A) \arrow[r, "\iota_*"] & K_{*+1}(O_E) \arrow[r] & \cdots
\end{tikzcd}$$

commutes. The composition $[W] \otimes_{M(A,O_E)} [\mathrm{ev}]$ is just the module

$$\left[\left(A, \begin{pmatrix} (1_k - q)A^k \\ A \\ A^k \end{pmatrix}, 0\right)\right]$$

with grading $(1_k - q) \oplus 1 \oplus -1_k$. So $[W] \otimes_{M(A,O_E)} [\mathrm{ev}] = [A] - [qA^k] = [A] - [E]$. Thus $-[W]$ makes one square commute, and we could try to show that the square to the left of W commutes (up to sign) as well. This means showing that $-[\mathscr{D}]\otimes_A[W] = [j] \in KK(\mathscr{S}O_E, M(A, O_E))$, which is implied by the stronger condition $-[\widehat{\mathscr{D}}]\otimes_A[W] = \mathrm{Id}_{KK(M,M)} \in KK(M(A, O_E), M(A, O_E))$. We have not been able to prove this equality in general, and leave it as an open problem.

If A is in the bootstrap class, then so too are O_E and $M(A, O_E)$. In this case $\cdot \otimes_M [\hat{\mathscr{D}}]$ is an isomorphism, hence the map $\cdot \otimes_A [W]$, which is always injective on the range of $\cdot \otimes_M [\hat{\mathscr{D}}]$, is an isomorphism as well: the inverse of $\cdot \otimes_M [\hat{\mathscr{D}}]$.

We have now proved our main result, Theorem 1.1.

We conclude with an application from the theory of graph algebras. Let $G = (G^0, G^1, r, s)$ be a locally finite directed graph (G^0 = vertices, G^1 = edges) with uniformly bounded in- and out-degrees and no sources nor sinks. Applying Theorem 1.1 to the graph C^*-algebra of G yields a well-known exact sequence for computing the K-theory. Let $A = C_0(G^0)$ and E the completion of $C_c(G^1)$ with respect to the norm coming from the inner product

$$(e|f)(v) = \sum_{s(g)=v} \overline{e(g)} f(g).$$

By [13, Example 6.5], E can be made into a bi-Hilbertian bimodule with finite right index. The module E is the restriction of $C_b(G^1)$ with the same inner product but over the algebra $C_b(G^0)$.

The module $C_b(G^1)$ is finite projective over $C_b(G^0)$. To see this, let $N = \sup_{v\in G^0}\{\text{in-degree}(v)\}$. Choose a partition

$$G^1 = A_1 \sqcup A_2 \sqcup \cdots \sqcup A_N$$

of the edges so that for all $v \in G^0$ there is at most one edge g with $r(g) = v$ in each A_j. Set $e_j = \chi_{A_j} \in C_b(G^1)$. Then a simple computation shows that the set $\{e_1, \dots, e_N\}$ is a frame for $C_b(G^1)$, and so we are done.

Then, as is well-known, we have $O_E = C^*(G)$, the graph C^*-algebra. We have the mapping cone exact sequence.

$$0 \longrightarrow K_1(C^*(G)) \longrightarrow K_0(M(A, C^*(G))) \xrightarrow{\mathrm{ev}_*} K_0(A) \longrightarrow K_0(C^*(G)) \longrightarrow 0.$$

Using $K_0(A) = \oplus^{|G^0|}\mathbb{Z}$ and the isomorphism $K_0(M(A, C^*(G))) \cong K_0(A)$ given by $\hat{\mathscr{D}}$, gives

$$0 \longrightarrow K_1(C^*(G)) \longrightarrow \bigoplus^{|G^0|} \mathbb{Z} \xrightarrow{1-V^T} \bigoplus^{|G^0|} \mathbb{Z} \longrightarrow K_0(C^*(G)) \longrightarrow 0.$$

where V is the vertex matrix of the graph G, given by

$$V(i,j) := |\{e \in G^1 : s(e) = v_i,\, r(e) = v_j\}|.$$

Similarly, since $A = C_0(G^0)$ is in the bootstrap class, in K-homology we find

$$0 \longrightarrow K^0(C^*(G)) \longrightarrow \prod^{|G^0|} \mathbb{Z} \xrightarrow{1-V} \prod^{|G^0|} \mathbb{Z} \longrightarrow K^1(C^*(G)) \longrightarrow 0,$$

and these results recapture the results of [11] for non-singular graphs.

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