MATH. SCAND. 36 (1975), 83–96

STABLE SURFACES IN EUCLIDEAN THREE SPACE

NICOLAAS H. KUIPER

To Werner Fenchel on his 70th birthday.

Summary.

This paper consists of two related parts. In A we present smooth maps of the real projective plane P with the non euclidean metric σ, into euclidean spaces such that we can read various interesting properties from the image. We mention and indicate some proofs of known facts. This part is expository. In B we consider C^∞-stable (in the sense of R. Thom) maps of surfaces in E^3. We call these "stable surfaces" for short. The Gauss curvature as a measure ($\int K\,d\sigma$) then exists although the scalar Gauss curvature K may explode at the C^∞-stable singularities. The infimum of the total absolute curvature $(2\pi)^{-1}\int |K\,d\sigma|$ of a compact surface M equals $4-\chi(M)$. This infimum can be reached for any surface in the class of stable maps, but not for all surfaces in the class of immersions, as we know. Stable surfaces of minimal total absolute curvature (tight) are given for the exceptions: the projective plane with 0 or 1 handles and the Klein-bottle. Recall that tight (closed) surfaces in E^N are also characterized as those that are divided into at most two (connected) parts by any (hyper-)plane.[1)]

A. Images of the real projective plane P.

1. A very nice map of (P,σ) into E^5.

The real projective plane P can be obtained from the unit 2-sphere, with equation

$$(1) \qquad S \quad x^2+y^2+z^2 = 1$$

in euclidean three space with metric

$$ds^2 = dx^2+dy^2+dz^2 ,$$

---

1) A tight immersion of a torus in E^3 need not be an embedding, as I recently found.

Received January 7, 1975.

by identification of diametral points

$$\text{diam}: (x,y,z) \to (-x, -y, -z) \tag{2}$$

$$P = S/\text{diam}$$

The local metric in S is invariant under "diam", and it induces the *non euclidean metric* σ on P. Almost every *quadratic function* φ in x, y and z has six non-degenerate critical points on its restriction on S, and it determines, φ being invariant under diam, a function on P, also denoted φ, with three non-degenerate critical points. We call a differentiable function on a manifold which is non-degenerate and has the minimal possible number of critical points, a *tight* function. Hence our non-degenerate function $\varphi: P \to \mathsf{R}$ with three critical points is tight.

A very nice map $f: P \to V \subsetneq S^4 \subset E^5 \subset \mathsf{R}^6$ is given by

$$(u_1,\ldots,u_6) = (x^2,y^2,z^2,\sqrt{2}yz,\sqrt{2}zx,\sqrt{2}xy) .$$

Its image the *Veronese surface* V lies in the 4-sphere

$$S^4: \textstyle\sum_{i=1}^6 u_i^2 = (x^2+y^2+z^2)^2 = 1 ,$$

in the euclidean hyperplane

$$E^5: u_1+u_2+u_3 = x^2+y^2+z^2 = 1$$

in R^6 with metric $ds^2 = \sum_{j=1}^6 du_j^2$. f is an *embedding onto a real algebraic* smooth *variety* V. Because f is obtained from a basis of the quadratic functions in x, y, z, the orthogonal group SO(3) acting on S and on P as the group of motions (SO(3)) induces a representation as a group (call it $f(\mathrm{SO}(3)) \subset \mathrm{SO}(5)$ of linear transformations in R^6, leaving invariant E^5 and S^4. Then it is a group of rotations in E^5. All motions of (P,σ) carried over to V are so obtained. In other words: f is SO(3)-*equivariant*. Consequently $f: P \to V$ is *isometric* but for some constant factor of multiplication of distances. Because $f(\mathrm{SO}(3))$ acts transitively on V, V is contained in the boundary (∂) of the naturally invariant convex hull ($\mathscr{H}$), of V

$$V \subset \partial\mathscr{H}V .$$

It is easy to see that the image of the real projective line $z=0$ in P, is a (euclidean) circle and so in view of the transitive $f(\mathrm{SO}(3))$-action every straight line in P has as *image a circle* in $V \subset E^5$. Any two such circles meet in a point and there is one circle connecting any two points in V. $\partial\mathscr{H}V$ is clearly topologically a four-sphere. It is contained in the third degree four-dimensional algebraic variety which is the union of all lines that meet V in two points or are tangent to V. In [7], see also [8], $\partial\mathscr{H}V$

is seen to be homeomorphic to the quotient space of $\mathsf{C}P(2)$ by complex conjugation.

A differentiable mapping $f'\colon M \to \mathsf{R}^N$ of a compact manifold M into some number space is called *tight* in case almost every linear function $\xi\colon \mathsf{R}^N \to \mathsf{R}$ composes with f' to a tight function $\xi f'\colon M \to \mathsf{R}$. Our $f\colon P \to \mathsf{R}^6$ composes with any linear function $\xi\colon \mathsf{R}^6 \to \mathsf{R}$ to a quadratic function in x,y,z, with for almost all ξ, three critical points on P. Therefore our f is tight.

We finally mention that $f\colon P \to E^5$ (like any embedding) is C^∞-*stable* in the sense of R. Thom. This means that for any $g\colon P \to E^5$ sufficiently near to zero in the C^∞-topology (involving values and derivatives) there exist diffeomorphisms φ and ψ in a commutative diagram

$$\begin{array}{ccc} P & \xrightarrow{\ f\ } & E^5 \\ \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\psi} \\ P & \longrightarrow & E^5 \end{array}$$

2. Embedding of P in E^3 is not possible.

We next try to find rich (as far as information on (P,σ) is concerned) images in euclidean three space E^3. We are immediately obstructed by the statement in the title of this paragraph. More generally one has the known

LEMMA 3.1. *There is no embedding of a compact non-orientable closed surface in E^3.*

We indicate a proof we owe to D. Sullivan for the smooth case: Let $g\colon M \hookrightarrow E^3$ be a smooth embedding. Take a point $p \in M$ and an ordered pair of tangent vectors in p to define an orientation at p in M. Take a third independent vector v at p which is then transversal to M. Move p along a closed embedded curve c' in M so that it comes back with the other orientation in M at p. Assume v very small and drag it also along c'. The end point of v describes a segment c'' which can be completed with two segments on both sides at p of M from the endpoints to p, to obtain an embedded circle c''' in E^3 which meets $g(M)$ in exactly one point. Deform to get a smooth curve c with the same property. Next move this curve $c = c(0)$ away in E^3 in a "generic" manner. Then the number of intersection points $c(t) \cap g(M)$ is constant or changes by two at any time t $(0 \leqq t \leqq 1)$. (To make this argument precise requires some work). At the end $c(1)$ is disjoint from M and the number of intersection points is zero. As we started with *one* intersection point this yields a contradiction.

3. Every smooth immersion $g: P \to E^3$ has at least one triple point[1]**.**

We recall the construction in [4] of an immersion. First we observe that the tight function $\varphi = xz$: $(S^2 \to)P \to E^3$ (see (1), (2)) has level sets: $\varphi^{-1}(c)$ equal to:

a point for $c = \sqrt{2}$
a circle for $0 < c < \sqrt{2}$
a figure 8 for $c = 0$
a circle for $-\sqrt{2} < c < 0$
a point for $c = -\sqrt{2}$

Arranging these level sets suitably as levels of a height function (one coordinate) in E^3 one can obtain an immersion of P in E^3 as in fig. 1. The self-intersection is an immersed circle with one triple point as we see. This can not be avoided. We prove:

LEMMA 4.1. *If $g: M \to E^3$ is a smooth immersion in E^3 of a closed surface M with odd Euler characteristic $\chi(M)$, then g has at least one triple point $q \in g(M)$: $g^{-1}(q) \subset M$ contains at least three points.*

PROOF. Let $\chi(M)$ be odd, M a closed surface, $g: M \to E^3$ a smooth immersion without triple points. The self-intersection points then form a compact 1-dimensional manifold Σ, that is a union of circles in $g(M) \subset E^3$. Take a point $q \in E^3$ outside $g(M)$. We can for any point p outside $g(M)$ consider curve segments connecting q and p, not meeting the self-intersection $\Sigma \subset g(M)$ and meeting $g(M)$ transversally only. Any such segment can be deformed into any other, such that the only "catastrophies" that occur happen because the curve segment meets Σ. At such a catastrophy the number of intersection points changes by an even number (0, 2 or -2). Therefore $E^3 \setminus g(M)$ can be divided into two parts, namely U^e the set of points p for which this number is even, and U^o the set of points p for which this number is odd. $g(M)$ is near any component Σ_i of Σ a cross-bundle over a circle. Going around the circle once yields a holonomy map of the cross onto itself, which in a suitable representation is a rotation over a multiple of $\pi/2$. However, $\frac{1}{2}\pi$ and $\frac{3}{2}\pi$ mod 2π are excluded because the pair of quadrants filling two parts of

[1] This theorem is known to many people. I do not know a reference.

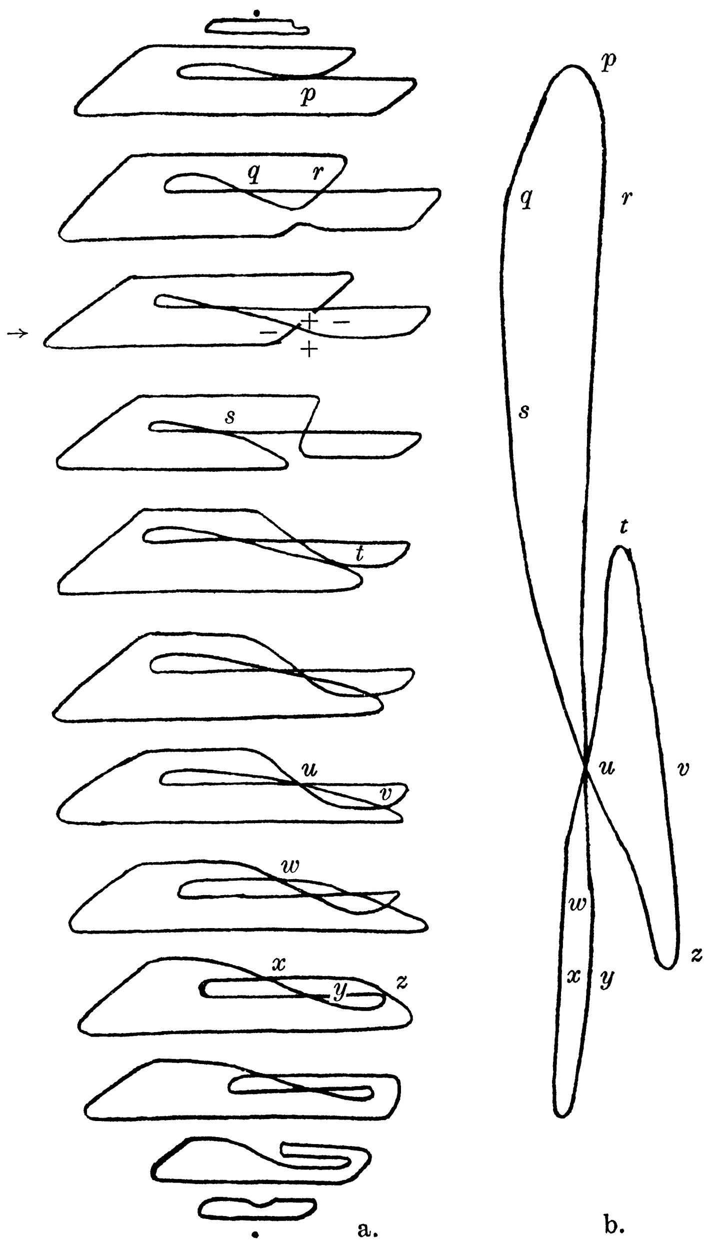

Fig. 1.

the cross, defined by U^e, is invariant. For each circle we can now replace the cross-bundle by something like a hyperbola-bundle (see fig. 2) smoothly fitted to the remaining embedded part of $g(M)$. Then we obtain an *embedded* (!) surface M'. The Euler-characteristic changes by this surgery by an even number. Therefore $\chi(M')$ is odd. Consequently M' is a non-orientable and embedded surface. This is impossible by lemma 3.1.

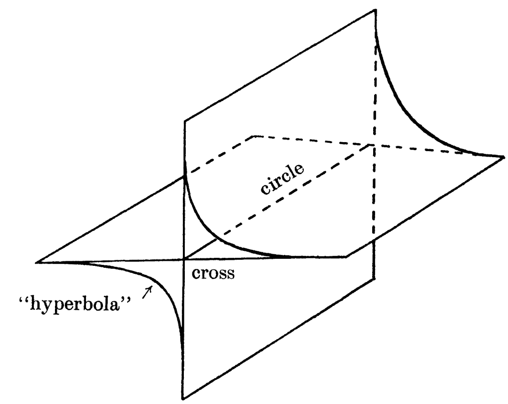

Fig. 2.

4. There is no tight immersion of P in E^3.

For a differentiable immersion of a compact surface M in E^3 tightness is equivalent to $\int_M |K d\sigma|/2\pi$ (K is the Gauss curvature, $d\sigma$ is the area element) being equal to its infimum $4-\chi(M)$. This is proved to be impossible for $M=P$, $\chi(P)=1$ in [3]. Also for topological immersions and a natural tightness definition tightness is impossible.

5. Isometric immersions of (P,σ) into E^3 are possible by the methods of [2] if we only require C^1-*immersions* (with continuous first derivatives). For C^2-*immersions* the intrinsic positive Gauss curvature is reflected in a positive extrinsic Gauss curvature and such closed immersed surfaces are necessarily boundaries of strictly convex bodies. Therefore C^2-isometric immersions of P in E^3 do not exist. Gromov proved that C^∞-isometric immersion of (P,σ) into E^4 is impossible [1].

6. A concrete image of P in E^3 with several good properties.

An example of a nice map $g: P \to E^3$ is defined by

$$((x,y,z) : x^2+y^2+z^2=1) \to (u,v,w) = (x^2-y^2, 2xy, yz) \tag{4}$$

Observe that $u+iv=(x+iy)^2$, so that the projection of $g(P)$ into the u,v-plane is a double covering branched at $(0,0)$ of the unit disc, with a fold along the boundary. The third coordinate w serves to pull the two covers apart.

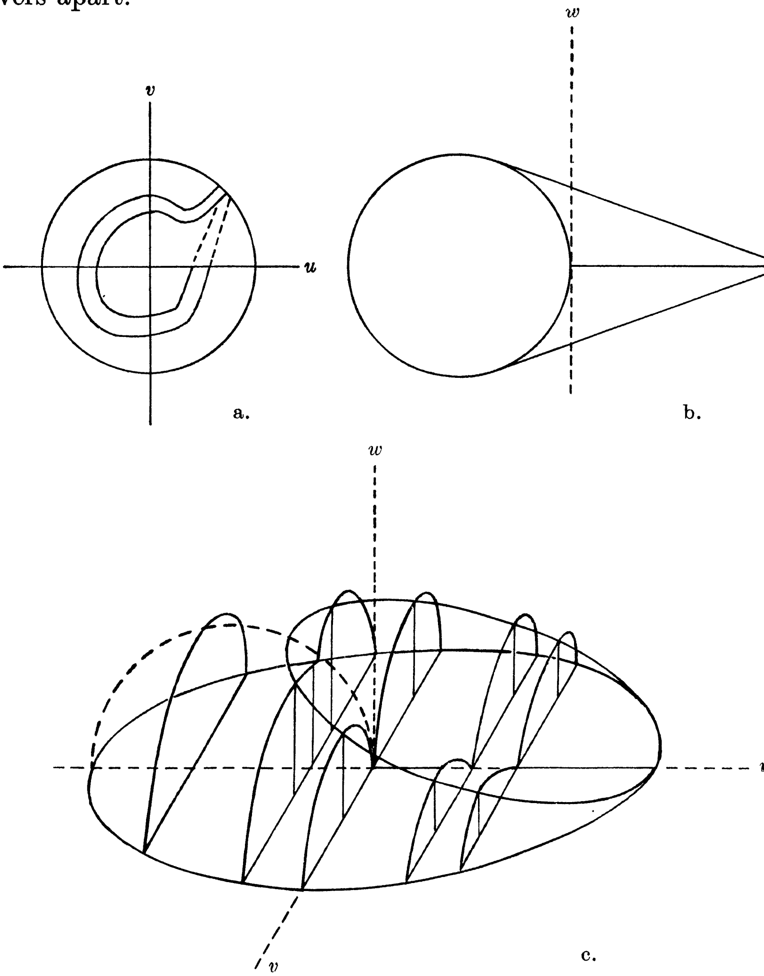

Fig. 3.

The mapping g is *locally an embedding except at the two points* $(x,y,z) = (0,0,1)$ and $(1,0,0)$ in P. It carries a small disc around each of these points onto a cone over a figure 8. Thom [10] proved that these singularities represent the only existing C^∞-stable (see section 2) kind of singularity of maps of R^2 to R^3. We conclude that our mapping g is locally C^∞-*stable* (it is even globally C^∞-*stable*). We also easily see that g has *no triple points*. At $(0,0,1)$ and $(1,0,0)$ respectively we have

$$g(x,y) = (x^2-y^2, 2xy, y\sqrt{1-x^2-y^2})$$

and

$$g(y,z) = (1-2y^2-z^2, 2y\sqrt{1-y^2-z^2}, yz),$$

whose 2-jets have the affine normal forms

$$(x^2-y^2, xy, y) \quad \text{and} \quad (y^2+z, yz, y).$$

Each of these two triples contains one linear coordinate. The remaining two quadratic forms determine a pencil of only indefinite forms in the first case and a pencil containing definite forms in the second case. This distinguishes the two main affine types of 2-jets of this stable singularity.

From the image $g(P)$ we can see that P is non-orientable: an embedded Moebius band is indicated in fig. 3a.

That one can not get rid of a singularity $h\colon (x,y) \to (x^2-y^2, 2xy, y)$ by a small deformation is seen from the rank $(=1$ at $(x,y)=(0,0))$ of the matrix

$$\begin{pmatrix} \partial h/\partial x \\ \partial h/\partial y \end{pmatrix} = \begin{pmatrix} 2x & 2y & 0 \\ -2y & 2x & 1 \end{pmatrix}$$

with determinants $4(x^2+y^2), 2x, 2y$. A small deformation will give two curves in place of $2x=0$ and $2y=0$ which will meet in one point where the rank is again *one*. Because g is given by quadratic functions in (x,y,z) (see (1) (2)), almost every linear function on E^3 induces on P a tight function, and so g itself is a *tight map* of P into E^3.

Next we show that $g(P)$ *is not a complete real algebraic variety in* E^3. An easy way to obtain the conclusion for this special example is as follows. Intersection of $g(P)$ with the line L with equation $v=w=0$ yields the closed interval $0 \leqq u \leqq 1$ of L. This is not the set of zero's of a polynomial equation, hence it is not an algebraic set, and neither is $g(P)$. It follows that any algebraic variety containing $g(P)$ must also contain the line L. If we eliminate x,y,z from the equations for g we get the algebraic variety (*Steiner surface*) with equation

$$v^2(u^2+v^2+2w^2-1)+2w^2(2u+2u^2+v^2+2w^2) = 0 \tag{5}$$

whose zeros form just the set

$$g(P) \cup L.$$

It is not more as we see by looking at the intersection-curves with the planes $u=$ constant. A different way to see that $g(P)$ is not an algebraic variety is by the theorem conjectured by D. Sullivan and first proved by P. Deligne and J. Mather (see [9]), that some neighbourhood of any point of a real algebraic variety is homeomorphic with a cone CX on a simplicial complex X, for which the Euler-characteristic $\chi(X)$ is even. At the point $(1,0,0)$ and $(0,0,1)$ we obtain a cone on X, a figure 8 with $\chi(X) = -1$. For $g(P) \cup L$ we have to add to X one point corresponding to the extra tail on L, to obtain X' with the required even Euler-characteristic $\chi(X') = 0$.[1)]

From (5) we can *read some plane sections* of $g(P)$, which has $v=0$ and $w=0$ as *planes of symmetry.*

$v = 0$: $w^2[(u+\frac{1}{2})^2 + w^2 - \frac{1}{4}] = 0$, line and circle;
$u = 0$: $(v \pm \frac{1}{2})^2 - 2w^2 + \frac{1}{4} = 0$, two ellipses, tangent at $(0,0,0)$;
$u = c$, $0 < c < 1$, yields a fourth degree curve like a figure 8 with main curvature vector directed to the interior.

Because g is tight it follows from the theory of tight maps [3,5] that there must be at least two tangent planes of the convex body $\mathscr{H}(g(P))$ that is the convex hull of $g(P)$, that meet P in an essential one-cycle. In our circumstances these planes are

$$u-1 = \sqrt{2}w \quad \text{and} \quad u-1 = -\sqrt{2}w\,.$$

Each of these planes meets $g(P)$ in a fourth degree curve consisting of double points. Hence this is a *conic section* with multiplicity two, as can be confirmed by calculation. The union of these two ellipses divides $g(P)$ into one part in the boundary of the convex hull with non-negative Gauss curvature, and the other part with non-positive Gauss curvature.

B. Stable surfaces and Gauss-curvature.

7. The Gauss-curvature of locally stable surfaces in E^3.

A surface M together with a locally C^∞-stable map into E^3 will be called a *locally stable surface* in E^3. It is an immersion except for a finite number of singularities, all diffeomorphic to each of the singularities in the example (the map g) of section 6. Outside the two singularities the Gauss curvature K of $g(P)$ is well defined. Near to the point $(1,0,0)$ we easily see from figure 3 that K (as a product of two main curvatures) tends to $+\infty$ along the curve in $v=0$. However, the integrals $\int K|d\sigma|$ and $\int |K d\sigma|$ are still convergent. Indeed the integrals mentioned are volumes of the images by the Gauss-normal map into the unitsphere of

---

1) An easy observation is the following theorem: If $h: M \to E^3$ is a stable smooth map of a compact surface onto an algebraic variety then h is an immersion.

unit vectors. More specifically (see [5]), for any open set $U \subset P$ we can compute

$$(6) \qquad 1/2\pi \int_U K|d\sigma| = 1/4\pi \int_{S^2} \textstyle\sum_{k=0}^{2} (-1)^k \mu_k(\varphi \circ g, U)\, d\tau ,$$

$$1/2\pi \int_U |K\, d\sigma| = 1/4\pi \int_{S^2} \textstyle\sum_{k=0}^{2} \mu_k(\varphi \circ g, U)\, d\tau ,$$

where φ varies over the unit-sphere S^2 with volume element $d\tau$ of normalized linear functions (or their gradient vectors) $\varphi\colon (E^3, x_0) \to (\mathrm{R}, 0)$, and μ_k is the number of non degenerate critical points of index k. But in our special example, as g is tight, we find

$$\mu_k(\varphi \circ g, U) \leqq \mu_k(\varphi \circ g, P) = 1 \text{ for } k = 0, 1, 2, \text{ for almost every } \varphi .$$

Hence the integrals converge for g, and in particular

$$1/2\pi \int_P K|d\sigma| = 1, \; 1/2\pi \int_P |K\, d\sigma| = 3 .$$

Next we consider a *general* locally stable surface in E^3 near a singularity at $x_0' \in U'$

$$g'\colon (U', x_0') \to (E^3, z_0') .$$

We can compare it with g near $x_0 = (0,0,1)$ and we know there exist C^∞-diffeomorphisms φ and ψ in a commutative diagram

$$\begin{array}{ccc} x_0 \in U & \xrightarrow{\;g\;} & E^3 \ni z_0 \\ \downarrow \quad \downarrow \varphi & & \downarrow \psi \quad \downarrow \\ x_0 \in U' & \xrightarrow{\;g'\;} & E^3 \ni z_0' \end{array}$$

for U and U' sufficiently small (that is inside compact parts where φ and ψ are also well defined). φ and ψ as well as their inverses, have values and first and second derivatives that are bounded. Therefore on comparing the volume elements and Gauss curvatures in corresponding points (both expressed in coordinates on U for example) we have

$$|d\sigma'| < \alpha |d\sigma| \quad \text{and} \quad |K'| \leqq \alpha(1 + |K|)$$

for some constant $\alpha > 0$. Then $\int_{U'} K' |d\sigma'|$ and $\int_{U'} |K' d\sigma'|$ converge because $\int_U |K\, d\sigma|$ converges. So we have

THEOREM. *Although the Gauss curvature as a scalar function K can explode near singularities of C^∞-stable surfaces in E^3, the Gauss curvature as a measure $\int K\, d\sigma$ is well defined and well behaved. For any compact C^∞-stable surface M in E^3 we have in particular*

$$(7a) \qquad 1/2\pi \int_M K|d\sigma| = \chi(M)$$

and

$$1/2\pi \int_M |K d\sigma| \geqq 4-\chi(M)\,, \tag{7b}$$

with $\chi(M)$ *the Euler-characteristic of* M.

PROOF. As for immersed surfaces this can be deduced with (6) from the Morse-inequalities for a non-degenerate function ζ on a compact surface M, which imply

$$\textstyle\sum_{k=0}^{2} (-1)^k \mu_k(\zeta) = \sum_{k=0}^{2} (-1)^k \beta_k(M) = \chi(M) \tag{8a}$$

$$\textstyle\sum_{k=0}^{2} \mu_k(\zeta) \geqq \sum_{k=0}^{2} \beta_k(M) = 4-\chi(M) \tag{8b}$$

where $\beta_k = \dim H_k(M, \mathbf{Z}_2)$ is the kth Bettinumber with respect to coefficients $\mathbf{Z}_2$ (chosen in order to cover also the non-orientable surfaces).

8. Tight locally stable surfaces in E^3.

A locally stable compact surface M in E^3 is tight if and only if it has minimal total absolute curvature

$$\int_M |K d\sigma| = 4-\chi(M)\,. \tag{9}$$

Claim equality in (7b); use (6), (8b). Tight embeddings in E^3 exist for the orientable surfaces. Tight immersions exist for the non-orientable surfaces except for the projective plane P and the Klein-bottle. The existence or non existence is not known for the surface P_H with $\chi(P_H) = -1$ obtained from P by attaching a handle. (See [4]). We prove:

THEOREM. *Every smooth compact surface admits a* C^∞*-stable map in* E^3 *which is tight* (*has minimal total absolute curvature*).

PROOF. In section 6 we gave already a tight stable map g of the projective plane into E^3. See fig. 3. We obtain from this map a tight stable map for P_H as follows. First we flatten the surface $g(P)$ near two points of $g(P)$ on the boundary of its convex hull $\partial\mathscr{H}g(P)$ without hurting the convexity of this part. (for example near the points where $v=0$ and $u=-1+0{,}2$ (see fig. 4)). Then we connect by a suitable handle: We connect the two points by a straight segment, bore a tube in the three-dimensional body bounded by $g(P)$ along this segment, smoothen, and keep $K \leqq 0$ in the new part of the boundary of the new body. As $\int_{K>0} K|d\sigma| = 4\pi$ remains unchanged under this surgery, tightness of the resulting surface follows from the equations (6, 8b, 9) with $\chi(M) = -1$.

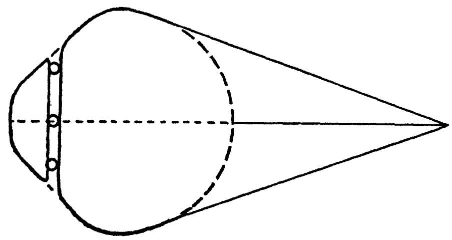

Fig. 4.

There only remains the problem of *the construction of a tight stable map of the Klein-bottle into* E^3.

We again start from the tight map $g\colon P \to E^3$ of the projective plane. We first make a projective transformation so that the plane $u=1$ "goes to infinity". Then the two supporting-ellipses in the planes $u-1=\sqrt{2}w$ and $u-1=-\sqrt{2}w$ are transformed into two parabolas in parallel support planes of the new surface W' (which extends to infinity). The part $u \leqq c^2$ $(c>0)$ of the new surface W' is like the part $u \leqq c'$ $(0<c'<1)$ of $g(P)$, a stable image of a Moebius band. We reflect with respect to the plane $u=c^2$ and then the union is a Klein bottle unfortunately not smooth along the curve in the plane $u=c$. This difficulty is avoided by a little trick in the calculations that follow now. Equivalent to equation (5) $g(P)$) we have

(5b) $$(2w^2+u+u^2+v^2)^2-(1+u)^2(u^2+v^2) = 0$$

or in homogeneous coordinates (u,v,w,s):

(5c) $$(2w^2+us+u^2+v^2)^2-(s+u)^2(u^2+v^2) = 0\,.$$

We want the points of the hyperplane $u=s$ mapped into the infinite hyperplane $s=0$. So we introduce the map defined by

$$(\mathbf{u},\mathbf{v},\mathbf{w},\mathbf{s}) = (u,v,w,s-u)\,.$$

The image of $g(P)$ has the equation

$$(2\mathbf{w}^2+\mathbf{su}+2\mathbf{u}^2+\mathbf{v}^2)-(\mathbf{s}+2\mathbf{u})^2(\mathbf{u}^2+\mathbf{v}^2) = 0\,.$$

We use again the former symbols u in place of $\mathbf{u}$ etc., put $\mathbf{s}=1$, and obtain the equation of the new surface

$$W'\colon (2u^2+u+v^2+2w^2)^2-(2u+1)^2(u^2+v^2) = 0$$

or

(10) $$(v^2+2w^2)^2-(2u+1)(v^2-4uw^2) = 0\,.$$

The supporting planes $w = \pm 1/2\sqrt{2}$ meet W' indeed along parabolas

$$(u - v^2 + \tfrac{1}{4})^2 = 0 .$$

The only (stable) singularity is at (0,0,0).

Next we substitute for u *not just* $c^2 + u$ in order to translate W' so that for the new surface W'' the part $u \leqq 0$ is a Moebius band, *but* we substitute for u instead $c^2 - u^2$ (the trick announced). We obtain our *stable* and as we will see *tight Klein bottle* W with equation:

$$(v^2 + 2w^2) + (2u^2 - 2c^2 - 1)[v^2 + 4(u^2 - c^2)w^2] = 0 . \tag{11}$$

For $c = \frac{1}{2}$ we have an illustration in fig. 5. W is symmetric with respect to each of the coordinate planes $u = 0$, $v = 0$ and $w = 0$, smoothly immersed except at the two stable singularities in the points $(u,v,w) = (\pm c, 0, 0)$ connecting the arc of (all) double points: $v = w = 0$, $-c < u < c$. W has no triple points. There are two exceptionnal support-planes $w = \pm 1/2\sqrt{2}$ which meet W in *circles* with multiplicities 2:

$$[u^2 + v^2 - (c^2 + \tfrac{1}{4})] = 0$$

and the pair of these circles divides W into one part on $\partial\mathscr{H}W$ where $K \geqq 0$ and the remaining part where K is seen to be $\leqq 0$ because almost all points there are saddle points. So W is tight. As in the case of $g(P)$ we must remark that the equation (11) defines a set of points $W \cup L$ including the whole line L: $v = w = 0$, from which suitable parts must be deleted to obtain the stable Klein-bottle surface we denoted by W.

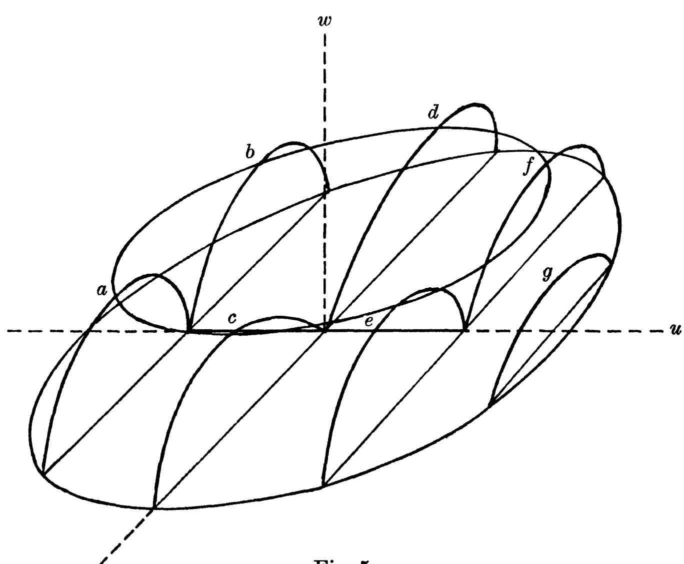

Fig. 5.

REFERENCES

1. Gromov and Rokhlin, *Embeddings and immersions in Riemannian geometry*, Russian mathematical surveys, Vol. 25, no. 5, (1970).
2. N. H. Kuiper, *On C^1-isometric embeddings*, Proc. Amsterdam, Indagationes Math. XVII (1955), 545–556 and 683–689.
3. N. H. Kuiper, *On surfaces in euclidean three-space*, Bulletin Soc. Math. de Belgique XII, Fasc. id 2 (1960), 5–22.
4. N. H. Kuiper, *Convex immersions of closed surfaces in E^3*, Commentarii Mathematici Helvetici 35 (1961), 88.
5. N. H. Kuiper, *Minimal total absolute curvature for immersions*, Invent. Math. 10 (1970), 209–238 (Survey paper).
6. N. H. Kuiper, *Tight topological embeddings of the Moebius band*, J. Differential Geometry 6 (1972), 271–283.
7. N. H. Kuiper, *The quotient space of $\mathbf{C}P(2)$ by complex conjugation is the 4-sphere*, Math. Ann. 208 (1974), 175–177.
8. W. S. Massey, *The quotient space of the complex projective plane under conjugation is a 4-sphere*, Geometrica Dedicata 2 (1973), 371–374.
9. D. Sullivan, *Combinatorial invariants of analytic spaces*, Proc. I, Liverpool Singularities Symposium, Lecture Notes in Mathematics, Springer 192 (1970), 165–168.
10. R. Thom, *Les singularités des applications différentiables*, Ann. Inst. Fourier (Grenoble), 6 (1955–56), 43–87.

INSTITUT DES HAUTES ETUDES SCIENTIFIQUES
BURES-SUR-YVETTE
FRANCE