MATH. SCAND. 36 (1975), 97–108

INVOLUTION-INVARIANT GEODESICS

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To Werner Fenchel on his 70th birthday.

Introduction.

Given an isometry $A : M \to M$ on a compact, connected Riemannian manifold M, the two most natural questions concerning A-invariant geodesics on M seems to be: (1) *Are there any*? and if so (2) *How many*? In [6] and [7], we have developed a theory which gives rather satisfactory answers to question (1). If e.g. A is an involution and M is simply connected, then A has a (non-trivial) invariant geodesic; obvious examples (rotation or reflexion on S^2) shows however that A does not in general have more than one invariant geodesic. The strategy in deriving the existence results in [6] and [7] was to assume, that A has no invariant geodesics. In the same spirit we shall here assume, that A has only finitely many invariant geodesics. We obtain thereby a topological condition which ensures the existence of infinitely many closed A-invariant geodesics, when $A^2 = 1_M$. Besides [6] and [7], the theory developed in Gromoll and Meyer [4] and [5] as well as in Bott [1] will be important for us.

1. Preliminaries.

Throughout the paper M shall denote a connected, compact Riemannian manifold and $A : M \to M$ an isometry on M. Recall that *a geodesic $\gamma : \mathsf{R} \to M$ is said to be A*-invariant *if and only if there is a $\theta \geqq 0$ such that $A(\gamma(t)) = \gamma(t+\theta)$ for all $t \in \mathsf{R}$*. Note that if $p_0 \in M$ belongs to the fixed point set, $\mathrm{Fix}(A)$ of A, then $\gamma(t) = p_0$ for all $t \in \mathsf{R}$ is an A-invariant geodesic; such a geodesic is called a trivial A-invariant geodesic. Whenever we say A-invariant geodesic we think of a non-trivial A-invariant geodesic.

Let $L_1^2(I,M)$ denote the Hilbert manifold consisting of absolutely continuous paths, $\sigma : I = [0,1] \to M$ with square integrable derivative, $\dot\sigma$. The tangent space to $L_1^2(I,M)$ at σ consists of absolutely continuous vectorfields, X along σ with square integrable covariant derivative X'. The Riemannian structure on M induces in a natural way a complete

Recived November 4, 1974.

Riemannian structure on $L_1^2(I,M)$. If X and Y are tangent vectors at σ we define

$$\langle X,Y\rangle_1 = \langle X,Y\rangle_0 + \langle X',Y'\rangle_0 ,$$

where $\langle X,Y\rangle_0 = \int_0^1 \langle X(t),Y(t)\rangle_{\sigma(t)} dt$ is the L^2-inner product (see Flaschel and Klingenberg [3]). In order to study A-invariant geodesics we introduced in [6] the closed submanifold

$$\Lambda_A(M) = \{\sigma \in L_1^2(I,M) \mid \sigma(1) = A(\sigma(0))\} .$$

The reason for this is that the critical points for the energy integral

$$E\colon \Lambda_A(M) \to \mathsf{R} ; \quad \sigma \to \tfrac{1}{2}\|\dot\sigma\|_0^2$$

are exactly the geodesics, $c\colon I \to M$ with $\dot c(1) = A_*\dot c(0)$. Here $A_*\colon TM \to TM$ denotes the induced map of A. Thus, the existence of A-invariant geodesics on M is equivalent to the existence of positive critical E-values. Since $E\colon \Lambda_A(M) \to \mathsf{R}$ satisfies condition (C) of Palais and Smale (see [6]), there is a rich critical point theory available for us.

Note that $\Lambda_A(M)$ can be identified with the set of locally L_1^2-maps, $\tilde\sigma\colon \mathsf{R} \to M$ satisfying $\tilde\sigma(t+1) = A(\tilde\sigma(t))$ for all $t \in \mathsf{R}$. The action of R on the parameter induces in this way a continuous action on $\Lambda_A(M)$ by isometries (see [7]). E is clearly invariant and the orbit of a critical point is an immersed critical submanifold.

Assume from now on that A has finite order i.e. $A^k = 1_M$ for some $k \in \mathsf{N}$. The R-action reduces then to an S^1-action and each critical S^1-orbit is an imbedded critical submanifold of $\Lambda_A(M)$. By the index, $\lambda(c)$ and nullity, $\nu(c)$ of a critical point, c we mean the index and nullity of the corresponding S^1-orbit as a critical submanifold. The Hessian of E at a critical point c is given by

$$H(E)(X,Y) = \langle X',Y'\rangle_0 - \langle R(X,\dot c)\dot c, Y\rangle_0 , \tag{1.1}$$

where R is the Riemannian curvature tensor of M. It follows from this and the fact that the inclusion $L_1^2 \subset L^2$ is compact, that the corresponding selfadjoint operator, S determined by $H(E)(X,Y) = \langle SX,Y\rangle_1$ admits a decomposition $S = \mathrm{Id} + K$, where K is a compact operator (for an explicit expression for K compare Eliasson [2]).

We shall now collect what we need from equivariant degenerate Morse theory. In finite dimensions, Morse defined for any isolated critical point a *local homological invariant*. This was modified and generalized to infinite dimensions by Gromoll and Meyer [4] under the above assumption on S (and condition (C)).

Consider an isolated critical orbit $S^1 \cdot c$. Choose a small tubular neighbourhood of $S^1 \cdot c$, and let E_c denote the energy restricted to the fiber over

c. From the splitting lemma of Gromoll and Meyer [4] (compare also [5]) it follows that E_c satisfies condition (C) and has only c as critical point. Thus we have a well-defined local homological invariant of E_c at c,

$$\mathscr{H}(E,c) = H_*(W_c, W_c^-) ,$$

where (W_c, W_c^-) is a pair of so called admissible regions (see [4, §2]). Although it is not necessary here, we take homology with coefficient in a field of characteristic zero. From this we obtain a well defined local homological invariant associated to the orbit $S^1 \cdot c$ as follows

$$\mathscr{H}(E, S^1 \cdot c) = H_*(W, W^-) ,$$

where $W = S^1 \cdot W_c$ and $W^- = S^1 \cdot W_c^-$. The crucial property of this invariant is contained in the following lemma, which is proved exactly as Lemma 4 in Gromoll and Meyer [5].

LEMMA 1.2. *If b is the only critical value of E in $[b-\varepsilon, b+\varepsilon]$ for some $\varepsilon > 0$ and $S^1 \cdot c_1, \ldots, S^1 \cdot c_n$ the only critical orbits in $E^{-1}(b)$, then*

$$H_*(\Lambda_A(M)^{b+\varepsilon}, \Lambda_A(M)^{b-\varepsilon}) = \oplus_{i=1}^n \mathscr{H}(E, S^1 \cdot c_i) ,$$

where $\Lambda_A(M)^a$ as usual denotes $E^{-1}((-\infty, a])$.

In order to use this lemma we must control $\mathscr{H}(E, S^1 \cdot c)$. The isotropy group at c,

$$S_c^1 = \{z \in S^1 \mid z \cdot c = c\}$$

operates by covering transformations on $S^1 \times (W_c, W_c^-)$ with quotient (W, W^-). Hence

$$\mathscr{H}_k(E, S^1 \cdot c) \subset \mathscr{H}_{k-1}(E,c) \oplus \mathscr{H}_k(E,c) . \tag{1.3}$$

In Gromoll and Meyer [4, §3] there was also introduced a *characteristic invariant*, $\mathscr{H}^0$, which is determined by the degenerate part of the function. This characteristic invariant together with the index, λ of c determines $\mathscr{H}$ completely by the shifting theorem:

$$\mathscr{H}_{k+\lambda}(E,c) = \mathscr{H}_k^0(E,c) . \tag{1.4}$$

We shall sometimes omit E in $\mathscr{H}(E,c)$ as well as in $\mathscr{H}^0(E,c)$.

All in all we can now see from (1.2–1.4) how information about the index and characteristic invariant of the critical points in $\Lambda_A(M)$ can give us information about the homology of $\Lambda_A(M)$. The problem here is of course that for each A-invariant geodesic, we have a whole tower of different critical orbits in $\Lambda_A(M)$. In the next sections we shall see how this problem can be treated when A is an involution.

2. Translated geodesics.

Although some of the constructions in this section can be carried out in general, we assume from now on that $A^2 = 1_M$.

The S^1-action from Section 1 takes the form:

$$\mu: S^1 \times \Lambda_A(M) \to \Lambda_A(M)\,;$$

$$\mu(e^{\pi i\theta}, \sigma)(t) = \begin{cases} \sigma(t+\theta), & t \in [0, 1-\theta] \\ A(\sigma(t+\theta-1)), & t \in [1-\theta, 1] \end{cases}$$

The critical points $c \in \Lambda_A(M)$ are classified according to their isotropy groups $S^1{}_c$ in the following sense. $S^1{}_c = S^1$ if and only if c is a trivial A-invariant geodesic. Otherwise $S^1{}_c = \mathsf{Z}_p$ for some $p \in \mathsf{N}$. Furthermore $p = 2l$ means that c is an l-fold covering of a prime closed geodesic fixed by A. Similarly $p = 2l+1$ means that c contains an l-fold covering of a prime closed geodesic, on which A operates as "the antipodal map" (and l is maximal with this property). There arises in this way to each prime closed A-invariant geodesic a tower of critical orbits in $\Lambda_A(M)$ corresponding to the coverings of the prime closed geodesic. *If an A-invariant geodesic is not fixed by A we say that it is* translated. A translated geodesic is now clearly characterized by the fact that all isotropy groups in the corresponding tower of critical orbits have odd order. Fixed geodesics are characterized analogously.

We shall now in some sense reduce the study of translated geodesics to that of closed geodesics.

Let Z_2 act on M by A and on S^1 by the antipodal map. The product action is a free isometric Z_2-action on $M \times S^1$. Note that the quotient manifold,

$$M \times_A S^1 \equiv M \times S^1/\mathsf{Z}_2$$

is the usual mapping torus of A, $M \times I/(p,0) \sim (A(p), 1)$.
Complete information about $M \times_A S^1$ is contained in the diagram

$$\begin{tikzcd} M \arrow[d] & M \arrow[d] \\ M \times S^1 \arrow[r] \arrow[d, "pr_2"'] & M \times_A S^1 \arrow[d, "\pi"] \\ S^1 \arrow[r] & S^1 \end{tikzcd},$$

where the horizontal maps are Z_2-coverings. In particular we see that the fiber bundle $\pi\colon M \times_A S^1 \to S^1$ is locally isometric to a product. A geodesic in $M \times_A S^1$ is therefore locally a product of a geodesic in M with a geodesic in S^1.

The space of closed curves on $M \times_A S^1$, denoted $\Lambda(M \times_A S^1)$, contains $\Lambda_A(M)$ as a submanifold. More precisely

$$F\colon \Lambda_A(M) \to \Lambda(M \times_A S^1)$$

defined by $F(\sigma)=\bar{\sigma}$, where $\bar{\sigma}(t)=(\sigma(t),t)$ for $t \in I/0\sim 1$, is an isometric imbedding onto the space of $L_1{}^2$-sections of $\pi\colon M \times_A S^1 \to S^1$. In order to describe the towers of critical orbits in $\Lambda_A(M)$ and $\Lambda(M \times_A S^1)$ we define for any positive integer m the iteration maps

$$m\colon \Lambda_A(M) \to \Lambda_A m(M) \qquad ; \quad \sigma \to \sigma^m$$

and

$$m\colon \Lambda(M \times_A S^1) \to \Lambda(M \times_A S^1) \; ; \quad \sigma \to \sigma^m$$

where $\sigma^m(t)=\tilde{\sigma}(m\cdot t)$ for all $t \in I$. Note that $E(\sigma^m)=m^2E(\sigma)$ and that m is an imbedding. For each translated geodesic, γ there exists a critical point $c \in \Lambda_A(M)$ with $S^1{}_c=\mathsf{Z}_1=\{0\}$, such that the collection $\{c^m \mid m \text{ odd}\}$ represents the tower of critical orbits associated with γ.

Lemma 2.1. *For any odd $m \in \mathsf{N}$ we have with the above notation $\lambda(c^m)=\lambda(\bar{c}^m)$ and $\nu(c^m)+1=\nu(\bar{c}^m)$.*

Proof. Since m is odd, c^m (respectively $\bar{c}^m$) is clearly a critical point in $\Lambda_A(M)$ (respectively $\Lambda(M \times_A S^1)$). The tangent space at an arbitrary point $s \in \Lambda(M \times_A S^1)$ admits a splitting

$$T_s\Lambda(M \times_A S^1)=T_s{}^{\mathrm{v}} \oplus T_s{}^{\mathrm{h}}$$

in vertical and horizontal vectors. In this splitting $X=X^{\mathrm{v}}+X^{\mathrm{h}}$, X^{v} is tangent to the fiber M along s and X^{h} is pointwise orthogonal to X^{v}. If s is a critical point in $\Lambda(M \times_A S^1)$ we get (compare(1.1))

$$\begin{aligned} H(E)_s(X,Y) &= \langle X',Y'\rangle_0-\langle R(X,\dot{s})\dot{s},Y\rangle_0 \\ &= \langle X^{\mathrm{v}\prime},Y^{\mathrm{v}\prime}\rangle_0-\langle R^{\mathrm{v}}(X^{\mathrm{v}},\dot{s}^{\mathrm{v}})\dot{s}^{\mathrm{v}},Y^{\mathrm{v}}\rangle_0 \\ &+ \langle X^{\mathrm{h}\prime},Y^{\mathrm{h}\prime}\rangle_0\,, \end{aligned}$$

where R denotes the curvature tensor on $M \times_A S^1$ and R^{v} is the curvature tensor on M (we have used that $M \times_A S^1$ is locally isometric to $M \times S^1$) Vertical and horizontal vectors are therefore orthogonal with respect to the Hessian i.e. $\lambda(s)=\lambda^{\mathrm{v}}(s)+\lambda^{\mathrm{h}}(s)=\lambda^{\mathrm{v}}(s)$ and $\nu(s)=\nu^{\mathrm{v}}(s)+\nu^{\mathrm{h}}(s)=\nu^{\mathrm{v}}(s)+1$.

To finish the proof, we only have to note that the tangent space at c^m is canonically identified with the vertical vectors at $\bar{c}^m$.

Lemma 2.1 reduces the study of the index and nullity of the translated geodesics to the corresponding study of closed geodesics. Thus Lemma 1 and Lemma 2 in [5] give us immediately

LEMMA 2.2. *Either* $\lambda(c^m)=0$ *for all (odd)* m *or there exist numbers* $\varepsilon>0$ *and* $a>0$ *such that*

$$\lambda(c^{m+s})-\lambda(c^m) \geqq s\cdot\varepsilon - a$$

for all (odd) m *and (even)* s.

LEMMA 2.3. *There are positive (odd) integers* $k_1,\ldots,k_s$ *and (odd) sequences* $m_j^i \in \mathsf{N}, i>0, j=1,\ldots,s$ *with* $m_j^1=1$ such that

$$\nu(c^{m_j^i k_j}) = \nu(c^{k_j})$$

and such that any (odd) m *with* $\nu(c^m)\neq 0$ *can be written uniquely as* $m=m_j^i k_j$.

This lemma enables us to get the desired information about the characteristic invariants. Since m_j^i is odd we have the iteration map $m_j^i\colon \Lambda_A(M)\to\Lambda_A(M)$. We can therefore proceed in complete analogy with Lemma 5 and Theorem 3 in Gromoll and Meyer [5], whereby we obtain

LEMMA 2.4. *With* k_j *and* m_j^i *as in Lemma* 2.3 *we have*

$$\mathscr{H}^0(c^{m_j^i k_j}) = \mathscr{H}^0(c^{k_j}) .$$

We see in particular, that there are at most finitely many different characteristic invariants associated to the tower of critical orbits corresponding to a translated geodesic.

3. Fixed geodesics.

In this section c will denote a prime closed geodesic which is fixed by A, that is, c is a critical point in $\Lambda_A(M)$ with isotropy group $S^1{}_c=\mathsf{Z}_2$. The corresponding tower of critical orbits is represented by all the iterates of c. We must therefore consider c^m for all positive integers m. Thus proceeding as in Section 2 we would have to allow also for even values of m_j^i and k_j. This would then give information about the index and nullity of $c^{m_j^i k_j}$ in $\Lambda(M)=\Lambda_{A^2}(M)$ and not as desired in $\Lambda_A(M)$.

To overcome these difficulties recall that $\mathrm{Fix}(A)$ is a disjoint union of closed totally geodesic submanifolds of M (see e.g. [6]). Hence $\Lambda(\mathrm{Fix}(A))$ is a disjoint union of closed totally geodesic submanifolds of $\Lambda_A(M)$. Denote by $\lambda^{\mathsf{T}}(c^m)$ (respectively $\nu^{\mathsf{T}}(c^m)$) the index (respectively nullity) of c^m in $\Lambda(\mathrm{Fix}(A))$. Note that the normal space $T_\sigma\Lambda(\mathrm{Fix}(A))^{\perp}$ of $\Lambda(\mathrm{Fix}(A))$ at $\sigma \in \Lambda(\mathrm{Fix}(A))$, consists of vectors $X \in T_\sigma\Lambda_A(M)$ with X pointwise orthogonal to $\mathrm{Fix}(A)$. Using that $\mathrm{Fix}(A)$ is totally geodesic together with the symmetry properties of the curvature tensor we get from (1.1).

Lemma 3.1. *The above splitting $T_{c^m}\Lambda_A(M) = T^{\mathsf{T}}_{c^m} \oplus T^{\perp}_{c^m}$ is orthogonal with respect to the Hessian at c^m. In particular*

$$\lambda(c^m) = \lambda^{\mathsf{T}}(c^m) + \lambda^{\perp}(c^m)$$

and

$$\nu(c^m) = \nu^{\mathsf{T}}(c^m) + \nu^{\perp}(c^m)\ ,$$

where $\lambda^{\perp}(c^m)$ (respectively $\nu^{\perp}(c^m)$) denotes the index (respectively nullity) of the Hessian restricted to the normal space, $T^{\perp}_{c^m}$ at c^m.

A study of $\lambda^{\mathsf{T}}(c^m)$ (respectively $\nu^{\mathsf{T}}(c^m)$) has as we know from Section 2 already been carried out by Bott [1], and Gromoll and Meyer [5]. It remains to study $\lambda^{\perp}(c^m)$ (respectively $\nu^{\perp}(c^m)$). We observe that the results in Bott [1] can be used here as well. Consider the complexification L of the index operator

$$\mathscr{L}(X) = X'' + R(X, \dot c)\dot c$$

and let $L^{\perp}$(respectively L^{T}) denote the restriction of L to the normal (respectively tangent) space. It now follows that $\lambda^{\perp}(c^m)$ and $\nu^{\perp}(c^m)$ are completely determined by functions $\Lambda^{\perp}\colon S^1 \to \mathbf{Z}^+ \cup \{0\}$ and $N^{\perp}\colon S^1 \to \mathbf{Z}^+ \cup \{0\}$ by the formulas

$$\lambda^{\perp}(c^m) = \textstyle\sum_{z^m=-1} \Lambda^{\perp}(z) \tag{3.2}$$
$$\nu^{\perp}(c^m) = \textstyle\sum_{z^m=-1} N^{\perp}(z)\ .$$

(Note that $X \in T_c{}^{\perp}$ satisfies $X(1) = A_*X(0) = -X(0)$ and compare with Corollary p. 178 in [1]). In the corresponding formulas for λ^{T} and ν^{T} we sum functions Λ^{T} and N^{T} over m-roots of 1. The following quite elementary facts are contained in Proposition 1.3 of Bott [1].

$$\Lambda^{\perp}(z) = \Lambda^{\perp}(\bar z) \quad \text{and} \quad N^{\perp}(z) = N^{\perp}(\bar z)\ .$$

$N^{\perp}(z) = 0$ except for at most $2\cdot\operatorname{codim}\mathrm{Fix}(A)$ points, the Poincaré points of $L^{\perp}$.

(3.3) $\Lambda^{\perp}$ is locally constant except at Poincaré points, where the jump is at most the absolute value of N.

$\lim_{z \to z_0^{\pm}} \Lambda^{\perp}(z) \geqq \Lambda^{\perp}(z_0)$.

As in Lemma 1 of [5] we use these facts together with the corresponding ones for Λ^{T} (derived from the properties of L^{T}) to obtain.

LEMMA 3.4. *Either $\lambda(c^m)=0$ for all m or there are numbers $\varepsilon>0$ and $a>0$ such that*

$$\lambda(c^{m+s})-\lambda(c^m) \geqq s\cdot\varepsilon - a$$

for all m and s.

As far as the index is concerned we have just seen how we could study λ^{T} and $\lambda^{\perp}$ separately. However, in order to study the characteristic invariant, we must study ν^{T} and $\nu^{\perp}$ simultaneously. Denote by $P^{\perp}$ (respectively P^{T}) the Poincaré points for $L^{\perp}$ (respectively L^{T}). These points can be described by means of the geodesic flow in the unit tangent bundle, T_1M of M. The closed orbit in T_1M corresponding to c is simply the velocity vector field of c parametrized by arch length. Let P_c be the differential of the Poincaré map associated to this orbit. In the splitting of TTM in horizontal and vertical subbundles (induced by the Riemannian connection on M), P_c takes the following form:

$$P_c(u,v) = \big(Y(1), Y'(1)\big) ,$$

where Y is the unique Jacobi field orthogonal to c with $\big(Y(0), Y'(0)\big)= (u,v)$. Note also that $A_{**}=(A_*,A_*)$ in the splitting

$$TTM = T^{\mathrm{H}}TM \oplus T^{\mathrm{V}}TM .$$

It is now clear that P_c commutes with A_{**}. Hence P_c preserves the ${}_{(+)}^{\;-}1$-eigenspace of A_{**}. The set of eigenvalues with absolute value 1 of P_c (complexified) restricted to the ${}_{(+)}^{\;-}1$-eigenspace of (A_*,A_*) is now exactly $P^{\perp}$ (respectively P^{T}). The crucial information about the nullity of c^m is contained in

LEMMA 3.5. *There are positive integers $k_1,\ldots,k_s$ and sequences $m_j^{\,i}\in \mathsf{N}$, $i>0, j=1,\ldots,s$ with $m_j^{\,1}=1$ such that*

$$\begin{aligned} \nu(c^{m_j^i k_j}) &= \nu(c^{k_j}), && m_j^{\,i}\ \textit{odd} \\ \nu(c^{m_j^i k_j}) &= \nu^{\mathsf{T}}(c^{m_j^i k_j}) = \nu^{\mathsf{T}}(c^{k_j}), && m_j^{\,i}\ \textit{even} \end{aligned}$$

and such that any m with $\nu(c^m)\neq 0$ can be written uniquely as $m=m_j^{\,i}k_j$.

PROOF. It follows from (3.2) and (3.3) that $\nu^{\perp}(c^m) \neq 0$ if and only if there is a $z = e^{2\pi i \varrho} \in P^{\perp}$ with $z^m = -1$. This implies that ϱ is rational with even denominator. Similarly $\nu^{\mathsf{T}}(c^m) \neq 0$ if and only if there is a $z' = z^{2\pi i \varrho'} \in P^{\mathsf{T}}$ with $z'^m = 1$ (which implies that ϱ' is rational). Let now $Q = Q^{\mathsf{T}} \cup Q^{\perp}$, where

$$Q^{\mathsf{T}} = \{q \in \mathsf{N} \mid \exists z' = \exp(2\pi i p/q) \in P^{\mathsf{T}}, (p,q) = 1\}$$

and

$$Q^{\perp} = \{\bar{q} \in \mathsf{N} \mid \exists z = \exp(2\pi i p/2\bar{q}) \in P^{\perp}, (p, 2\bar{q}) = 1\} .$$

For every $D \subset Q$ let $k(D)$ denote the least common multiple of elements in D. Choose distinct numbers $k_1, \ldots, k_s$ such that for each D $\subset Q$ there is a j with $k_j = k(D)$. For each fixed k_j select from the sequence mk_j, $m > 0$ the greatest subsequence $m_j{}^i k_j$, $i > 0$, with the property that whenever $q \in Q$ and $q | m_j{}^i k_j$ then $q | k_j$. If

$$\nu(c^m) = \nu^{\mathsf{T}}(c^m) + \nu^{\perp}(c^m) \neq 0$$

there is a $q \in Q$ which divides m; Let $D = \{q \in Q \mid q|m\}$ and $k(D) = k_j$, then $m = m_j{}^i k_j$ is unique by construction. It is now easy to see that

$$\{z \in P^{\mathsf{T}} \mid z^{m_j{}^i k_j} = 1\} = \{z \in P^{\mathsf{T}} \mid z^{k_j} = 1\}$$

and therefore by (3.2) and (3.3) (or rather the corresponding formulas for N^{T})

$$\nu^{\mathsf{T}}(c^{m_j{}^i k_j}) = \nu^{\mathsf{T}}(c^{k_j}) ,$$

for all $m_j{}^i$ and k_j. The same statement for $\nu^{\perp}$ is false! However, if $\nu^{\perp}(c^{m_j{}^i k_j}) \neq 0$, we see from (3.2) and (3.3) that $m_j{}^i$ must be odd and furthermore

$$\nu^{\perp}(c^{m_j{}^i k_j}) = \nu^{\perp}(c^{k_j}) .$$

In consequence of this lemma we obtain the needed information about the characteristic invariants.

LEMMA 3.6. *With k_j and $m_j{}^i$ as in Lemma* 3.5 *we have*

$$\textit{for } m_j{}^i \text{ odd}: \quad \mathscr{H}^0(c^{m_j{}^i k_j}) = \mathscr{H}^0(c^{k_j})$$

and

$$\textit{for } m_j{}^i \textit{ even}: \quad \mathscr{H}^0(c^{m_j{}^i k_j}) = \mathscr{H}^0(c^{k_j})^{\mathsf{T}} ,$$

where $\mathscr{H}^0(c^k)^{\mathsf{T}}$ denotes the characteristic invariant of c^{k_j} in the space $\Lambda(\mathrm{Fix}(A))$.

PROOF. for $m_j{}^i$ odd we use Lemma 3.5 and proceed exactly as in Section 2 ($m_j{}^i \colon \Lambda_A(M) \to \Lambda_A(M)$). If $m_j{}^i$ is even $\nu^{\perp}(c^{m_j{}^i k_j}) = 0$ by Lemma

3.5. In other words the null-space of the Hessian at $c^{m_j{}^i k_j}$ is contained in the tangent space to $\Lambda(\mathrm{Fix}(A))$. Furthermore $\operatorname{grad} E$ is tangent to $\Lambda(\mathrm{Fix}(A))$. To see this let $\sigma \in \Lambda(\mathrm{Fix}(A)) \subset \Lambda_A(M)$ and assume without loss of generality that $\tilde{\sigma}$ is C^∞. For $X \in T_\sigma{}^\perp$ we have (see [6] or [3])

$$dE(X) = \langle X', \dot{\sigma} \rangle_0 = -\langle X, \dot{\sigma}' \rangle_0$$

which is equal to zero since $\mathrm{Fix}(A)$ is totally geodesic. Thus Lemma 7 in Gromoll and Meyer [4] tells us that

$$\mathscr{H}^0(E, c^{m_j{}^i k_j}) = \mathscr{H}^0(E_{|\Lambda(\mathrm{Fix}(A))}, c^{m_j{}^i k_j})$$

and therefore by construction (see Section 1)

$$\mathscr{H}^0(c^{m_j{}^i k_j}) = \mathscr{H}^0(c^{m_j{}^i k_j})^\mathsf{T} .$$

To finish the proof, we note that $\nu^\mathsf{T}(c^{m_j{}^i k_j}) = \nu^\mathsf{T}(c^{k_j})$ by Lemma 3.5 and hence

$$\mathscr{H}^0(c^{m_j{}^i k_j})^\mathsf{T} = \mathscr{H}^0(c^{k_j})^\mathsf{T}$$

by Theorem 3 in [5] applied to $c^{k_j} \in \Lambda(\mathrm{Fix}(A))$.

The conclusion of this section is the same as that of Section 2: There are at most finitely many different characteristic invariants associated to the tower of critical orbits corresponding to a fixed geodesic.

4. Existence of infinitely many invariant geodesics.

The finiteness of the characteristic invariants together with the growth estimate of the indices is sufficient information to prove our main theorem:

THEOREM 4.1. *Let M be a compact, 1–conneted Riemannian manifold and A: $M \to M$ an involutive isometry on M. Then A has infinitely many closed invariant geodesics if the sequence of Betti numbers*

$$\beta_k(\Lambda_A(M)) = \dim H_k(\Lambda_A(M)) ,$$

$k \geqq 0$ is unbounded.

PROOF. Suppose A has at most finitely many different (prime closed) invariant geodesics. Each critical orbit in $\Lambda_A(M)$ is then isolated and we can therefore apply our results in the previous sections

Fix a prime closed A–invariant geodesic represented by $c \in \Lambda_A(M)$ ($S^1{}_c = \mathbf{Z}_1$ or $\mathbf{Z}_2$) and consider the coresponding tower of critical orbits in $\Lambda_A(M)$. Set

$$B_k(c^m) = \dim \mathscr{H}_k(c^m) \quad \text{and} \quad B_k{}^0(c^m) = \dim \mathscr{H}_k{}^0(c^m) .$$

From Lemma 2.4 or Lemma 3.6 we can find a $B>0$ such that $B_k{}^0(c^m) \leqq B$ for all k and m. Furthermore by definition $B_k{}^0(c^m)=0$ for $k>2(\dim M-1)$ and all m. Thus from this, (1.3) and (1.4) we see that $B_k(c^m)$ is uniformly bounded by $2B$. Moreover, (1.3) and (1.4) in connection with Lemma 2.2 or Lemma 3.4 shows, that the number of orbits, $S^1\cdot c^m$ with $B_k(c^m) \neq 0$ is bounded by some constant $C>0$ independent of $k \geqq 2\dim M$.

Since there are only finitely many towers $\{S^1\cdot c_i{}^m\}$, $i=1,\ldots,r$ of critical orbits, we may assume without loss of generality that $B_k(c_i{}^m) \leqq 2B$ for all i and m and that the number of orbits, $S^1\cdot c_i{}^m$ with $B_k(c_i{}^m) \neq 0$ is bounded by $r\cdot C$ for all $k \geqq 2 \dim M$. Hence with $K=2B\cdot r\cdot C$ we get from Lemma 1.2 together with an exact sequence argument,

$$\beta_k(\Lambda_A(M)^b, \Lambda_A(M)^a) \leqq K, \quad k \geqq 2\dim M ,$$

where $a,b \in \mathsf{R}^+$ are arbitrary regular E-values. According to Corollary 3.3 in [6] we can choose $a=e$ such that $\mathrm{Fix}(A)$ is a strong deformation retract of $\Lambda_A(M)^e$. With this choice,

$$\beta_k(\Lambda_A(M)^b, \Lambda_A(M)^e) = \beta_k(\Lambda_A(M)^b, \mathrm{Fix}(A))$$

and therefore

$$\beta_k(\Lambda_A(M)^b) = \beta_k(\Lambda_A(M)^b, \mathrm{Fix}(A)) \leqq K$$

for all $k \geqq 2 \dim M$ and all regular values $b \in \mathsf{R}^+$. Fix now $k \geqq 2 \dim M$ and choose b so large that

$$\mathscr{H}_k(s^1.c_i{}^m) = 0 \quad \text{for all } c_i{}^m \in \Lambda_A(M)/\Lambda_A(M)^b .$$

This is possible by (1.3), (1.4) and Lemma 2.2 or Lemma 3.4. For such a b obviously $\beta_k(\Lambda_A(M))=\beta_k(\Lambda_A(M)^b)$ and hence

$$\beta_k(\Lambda_A(M)) \leqq K, \quad \text{when } k \geqq 2\dim M .$$

We conclude that $\beta_k(\Lambda_A(M))$, $k \geqq 0$ is bounded if A has only finitely many closed invariant geodesics.

Remark. As far as the topology of $\Lambda_A(M)$ is concerned we may as well consider

$$C_A{}^0(M) = \{f\colon I \to M \mid f \text{ continuous and } f(1) = A(f(0))\},$$

since $\Lambda_A(M) \subset C_A{}^0(M)$ is a homotopy equivalence by theorem 1.3 of [6]. It is very likely, that the recent De Rham homotopy theory of Sullivan can be used to decide when $\Lambda_A(M)$ has unbounded Betti numbers. However so far the question has not been settled completely even in the case $A=1_M$.

REFERENCES

1. R. Bott, *On the Iteration of Closed Geodesics and the Sturm Intersection Theory*, Comm. Pure Appl. Math. 9 (1956), 171–206.
2. H.I. Eliasson, *Convergence of gradient curves on Hilbert manifolds*, Math. Z. 136 (1974), 107–116.
3. P. Flaschel, W. Klingenberg, *Riemannsche Hilbert mannigfaltigkeiten. Periodische Geodätische*, (Lecture Notes in Mathematics 282), Springer-Verlag, Berlin, Heidelberg, New York, 1970.
4. D. Gromoll, W. Meyer, *On differentiable functions with isolated critical points*, Topology 8 (1969), 361–369.
5. D. Gromoll and W. Meyer, *Periodic geodesics on compact Riemannian manifolds*, J. Differential Geometry 3 (1969), 493–510.
6. K. Grove, *Condition* (C) *for the energy integral on certain path spaces and applications to the theory of geodesics*, J. Differential Geometry 8 (1973), 207–223.
7. K. Grove, *Isometry-invariant geodesics*, Topology 13 (1974) 281–292.

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