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LOCAL INVARIANTS OF A PSEUDO-RIEMANNIAN MANIFOLD

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To Werner Fenchel on his 70th birthday.

Introduction.

Let M be a smooth manifold of dimension m and let TM be the tangent bundle of M. If G is a non-degenerate symmetric bilinear form on TM, the pair (M,G) is called a pseudo-Riemannian manifold. G is an indefinite metric for M; if G is positive definite, then (M,G) is a Riemannian manifold. In this paper, we will study the local invariants of (M,G). Let $\Lambda^s(T^*M)$ be the bundle of s-forms over M and let $\Gamma(\Lambda^s(T^*M))$ denote the space of smooth sections. The invariants we will consider will be s-form valued polynomials P in the derivatives of the metric G; $P(G)(x) \in \Gamma(\Lambda^s(T^*M))$. In the first section we will define the order n of an s-form valued polynomial invariant P. We will show that if $n<s$, then $P=0$. If $n=s$, then $P(G)$ is a Pontrjagin form. This will generalize some earlier results for positive definite metrics to the case of indefinite metrics. If P is s-form valued and if $dP(G)(x)=0$ for all G,x, then $P(G)$ is always a closed s-form. Consequently, $P(G)$ represents a cohomology class $\{P(G)\}$. If $\{P(G)\}$ is independent of G, we will prove that $\{P(G)\}$ is a Pontrjagin class.

If $s=0$, then $P(G)$ is a function valued invariant. Let

$$P(G)(M) = \int_M P(G)|d\,\mathrm{vol}|$$

where M is compact and $|d\,\mathrm{vol}|$ is the measure induced on M by G. The Euler class E_m is such an invariant if m is even. We will give an axiomatic characterization of the Euler class similar to that given in [3] for positive definite metrics. Let $\chi(M)$ be the Euler characteristic of M. I. M. Singer conjectured that if $P(G)(M)$ is independent of the metric G, then there is a constant c such that $P(G)(M)=c\chi(M)$. We proved this conjecture earlier for positive definite metrics; in this paper we extend this result to indefinite metrics. This proves that the only diffeomor-

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phism invariants in the category of unoriented pseudo-Riemannian compact manifolds which are obtained by integrating local formulas in the derivatives of the metric are just multiples of the Euler characteristic.

Section 1.

Let M be a smooth manifold of dimension m and let G be a non-degenerate symmetric bilinear form on TM. The pair (M,G) is a pseudo-Riemannian manifold; G is a pseudo-Riemannian metric for M. Let $X=(x_1,\ldots,x_m)$ be a system of local coordinates defined in a neighborhood of some point $x_0 \in M$. Let

$$g_{ij}(X,G) = G(\partial/\partial x_i, \partial/\partial x_j)\ .$$

Since G is non-degenerate, we can make a linear change of coordinates to diagonalize G at x_0. X is said to be G-normalized at x_0 iff

$$g_{ij}(X,G) = \begin{cases} 0 \text{ for } i \neq j \\ 1 \text{ for } i = j \leqq p \\ -1 \text{ for } i = j > p \end{cases}$$

Let $q=m-p$. If $q=0$, G is positive definite and is a Riemannian metric. We will say in general that G is of type (p,q) for $0 \leqq p \leqq m$, $0 \leqq q \leqq m$, and $p+q=m$. There are topological restrictions on the set of manifolds which admit metrics of type (p,q). For example, a compact manifold M admits a metric of type $(m-1,1)$ if and only if the Euler characteristic of M vanishes.

Let $\{e_1,\ldots,e_m\}$ be the usual basis for R^m and let Q be the bilinear form

$$Q(e_i,e_j) = \begin{cases} 0 \text{ for } i \neq j \\ 1 \text{ for } i = j \leqq p \\ -1 \text{ for } i = j > p \end{cases}$$

Let $O(p,q)$ be the group of all real $m\times m$ matrices which preserve Q; let $SO(p,q)$ be the subgroup of orientation preserving matrices. Let $\alpha=(\alpha(1),\ldots,\alpha(m))$ be a multi-index; let $\operatorname{ord}(\alpha)=\alpha(1)+\ldots+\alpha(m)$. If X is a coordinate system, let

$$d_{X,\alpha} = (\partial/\partial x_1)^{\alpha(1)} \ldots (\partial/\partial x_m)^{\alpha(m)}\ .$$

We say that α corresponds to $\{i_1{}^\alpha,\ldots,i_s{}^\alpha\}$ for $1 \leqq i_j{}^\alpha \leqq m$ iff $s=\operatorname{ord}(\alpha)$ and if $\alpha(i)=\delta_{i,i_1{}^\alpha}+\ldots+\delta_{i,i_s{}^\alpha}$. δ_{ij} is the Kronecker index. Introduce formal variables: $g=(-1^q\det(g_{ij}))^{\frac{1}{2}}$ and $g_{ij/\alpha}=g_{ij/i_1{}^\alpha\ldots i_s{}^\alpha}$. If $\operatorname{ord}(\alpha)=0$, $g_{ij/\alpha}=g_{ij}$. Let $\mathscr{P}$ be the polynomial algebra in the $\{g_{ij/\alpha}, g, g^{-1}\}$ variables subject to the relation $g^2=-1^q\det(g_{ij})$. If X is a coordinate system, we evaluate $g, g_{ij/\alpha}$ on (X,G) by:

$$g(X,G)(x_0) = \left(-1^q \det(G(\partial/\partial x_i, \partial/\partial x_j))(x_0)\right)^{\frac{1}{2}}$$
$$g_{ij/\alpha}(X,G)(x_0) = d_{X,\alpha}(G(\partial/\partial x_i, \partial/\partial x_j))(x_0) .$$

We extend this evaluation to the polynomial algebra to define $P(X,G)(x_0)$ for any $P \in \mathscr{P}$.

Let $A = g^k g_{i_1 j_1/\alpha_1} \dots g_{i_r j_r/\alpha_r}$ be a monic monomial and let $P \in \mathscr{P}$. Define:

$$\operatorname{ord}(A) = \operatorname{ord}(\alpha_1) + \dots + \operatorname{ord}(\alpha_s)$$
$$c(A,P) = \text{the coefficient of the monomial } A \text{ in } P .$$

We say that A is a monomial of P if $c(A,P) \neq 0$. P is homogeneous of order n if all the monomials A of P are of order n. Let $I = (i_1, \dots, i_s)$ with $1 \leqq i_1 < \dots < i_s \leqq m$. Let $|I| = s$ and let $dx_I = dx_{i_1} \wedge \dots \wedge dx_{i_s} \in \Lambda^s(T^*M)$. If P has the form:

$$P = \textstyle\sum_{|I|=s} P_I dx_I \quad \text{for } P_I \in \mathscr{P} ,$$

then P is an s-form valued polynomial in the derivatives of G. If all the P_I are homogeneous of order n, then P is of order n. For such a P, we define:

$$P(X,G)(x_0) = \textstyle\sum_{|I|=s} P_I(X,G)(x_0) dx_{i_1} \wedge \dots \wedge dx_{i_s} \in \Lambda^s(T^*M) .$$

If $P(X,G)(x_0) = P(Y,G)(x_0)$ for any two coordinate systems X and Y, we will say that P is $O(p,q)$-invariant. If $P(X,G)(x_0) = P(Y,G)(x_0)$ for any two coordinate systems X and Y which induce the same local orientation of M, then we will say that P is $SO(p,q)$ invariant. Let $\nu = O(p,q)$ or $SO(p,q)$ and let $\mathscr{P}_{n,\nu,s}$ be the vector space of all s-form valued ν-invariant polynomials of order n in the derivatives of G. Clearly $\mathscr{P}_{n,O(p,q),s} \subset \mathscr{P}_{n,SO(p,q),s}$. If $P \in \mathscr{P}_{n,O(p,q),s}$, let $P(G)(x_0) = P(X,G)(x_0)$; this is independent of the choice of X by hypothesis. If $P \in \mathscr{P}_{n,SO(p,q),s}$ and if orn is a local orientation of M, let $P(G,\mathrm{orn})(x_0) = P(X,G)(x_0)$ for any coordinate system X inducing the orientation orn.

We will need the following characterization of the order of an ν-invariant polynomial in later sections:

LEMMA 1.1. *Let $P \in \mathscr{P}_{n,\nu,s}$ and $c > 0$. $H = c^2 G$ is another pseudo-metric for M of type (p,q). Then $P(c^2 G) = c^{s-n} P(G)$. Consequently, if P is ν-invariant, we can decompose $P = P_0 + \dots + P_r$ where the P_j are homogeneous of order j and ν-invariant separately.*

PROOF. Let X be a ν-coordinate system and let $Y = cX$. Then $\partial/\partial y_i = c^{-1}\partial/\partial x_i$ and $dy_i = c\,dx_i$. Since $g_{ij}(Y,H) = g_{ij}(X,G)$, $g_{ij/\alpha}(Y,G) = c^{-\operatorname{ord}(\alpha)} g_{ij/\alpha}(X,G)$. Let

$$A = g^k g_{i_1 j_1/\alpha_1} \cdots g_{i_r j_r/\alpha_r} dx_I$$

be a monomial of P. Then $A(Y,H) = c^{s-\operatorname{ord} A} A(X,H) = c^{s-n} A(X,H)$. Therefore, $P(Y,H) = c^{s-n} P(X,G)$; since P is ν-invariant this proves the lemma.

The Pontrjagin and Euler classes are invariant polynomials in the derivatives of the metric. They are constructed as follows: let ∇ be the Levi–Civita connection induced by the bilinear form G. Let $\nabla_i = \nabla_{\partial/\partial x_i}$ for notational convenience. ∇ is defined by the identities:

$$\nabla_i(\partial/\partial x_j) = \nabla_j(\partial/\partial x_i) \quad \text{(torsion free)}$$

$$G(\nabla_i \partial/\partial x_j, \partial/\partial x_k) + G(\partial/\partial x_j, \nabla_i \partial/\partial x_k) = g_{jk/i} \quad \text{(Riemannian)}\,.$$

We use these two identities to conclude that:

$$G(\nabla_i \partial/\partial x_j, \partial/\partial x_k) = (g_{jk/i} - g_{ij/k} + g_{ik/j})/2\,.$$

Since G is non-degenerate, this defines the connection ∇ uniquely. Let Ω be the curvature tensor of the connection: Ω is the 2-form valued endomorphism of TM defined by:

$$\Omega(\partial/\partial x_k) = \textstyle\sum_{i<j} (\nabla_i \nabla_j - \nabla_j \nabla_i)(\partial/\partial x_k) \otimes \partial x_i \wedge \partial x_j\,.$$

Let

$$p(\Omega) = \det(I - (1/2\pi)\Omega)$$

be the total Pontrjagin class of the connection. We can express

$$p(\Omega) = 1 + p_1(\Omega) + p_2(\Omega) + \ldots\,.$$

$p_i(\Omega)$ is a $4i$-form valued invariant which is homogeneous of order $4i$ in the derivatives of G. Therefore $p_i(\Omega) \in \mathscr{P}_{4i,O(p,q),4i}$.

If Q is a combination of the $P_i(\Omega)$, Q is said to be a Pontrjagin form.

THEOREM 1.2. *Let* $P \in \mathscr{P}_{n,O(p,q),s}$ *then*

(a) *If* $n < s$, $P = 0$

(b) *If* $n = s$, *P is a Pontrjagin form. If* $P \neq 0$, *n is divisible by* 4.

The case $q = 0$ has been considered previously by the author [3] and by Atiyah, Bott, and Patodi [1]. In section 2, we use the methods of [3] to prove 1.2 for all values of q. Theorem 1.2 has also been proved independently by Stredder [7]. His proof uses H. Weyl's theorem on the invariants of $O(p,q)$ together with some results of Epstein [2].

Since G is non-degenerate, we define the non-zero measure $|d\,\mathrm{vol}| =$

$g|dx_1 \ldots dx_m|$ exactly as in the Riemannian case. Let orn be a local orientation of M, and let $*$ be the Hodge operator induced by the bilinear form G and the orientation orn. Let $\{e_1, \ldots, e_m\}$ be an oriented frame for TM which diagonalizes G. Let $\Omega(e_i) = \Omega_{ij} e_j$. We adopt the convention of summing over repeated indices unless otherwise indicated. Ω_{ij} is a matrix of 2-forms. Let $m = 2k$ be even; define the Euler class E_m by:

$$E_m = *\big(\textstyle\sum_\tau \operatorname{sign}(\tau)\Omega_{\tau(1)\tau(2)} \ldots \Omega_{\tau(m-1)\tau(m)}\big)\big((-1)^{k-1}/2^m \pi^k k!\big) .$$

The sum ranges over all permutations τ of the integers 1 thru m. E_m is independent of the local orientation orn and the frame field $\{e_1, \ldots, e_m\}$. Therefore $E_m \in \mathscr{P}_{m, O(p,q), 0}$.

It is clear that $\mathscr{P}_{m, O(p,q), 0}$ is not 1 dimensional for m even, $m \geqq 4$. We will need some other property of the Euler class E_m to characterize E_m axiomatically. We define linear maps:

$$r^{*+}\colon \mathscr{P}_{n, O(p,q), s} \to \mathscr{P}_{n, O(p-1,q), s} \quad \text{for } s < m,\ p > 0$$

$$r^{*-}\colon \mathscr{P}_{n, O(p,q), s} \to \mathscr{P}_{n, O(p,q-1), s} \quad \text{for } s < m,\ q > 0$$

as follows. First suppose $p > 0$. Let N be an $m-1$ dimensional manifold with a non-degenerate bilinear form G_0 on N of type $(p-1,q)$. Let $M = N \times S^1$ with the product bilinear form $G = G_0 \otimes 1$ on $TM = TN \otimes TS^1$. The bilinear form G is non-degenerate and of type (p,q). Let t be a fixed point of S^1 and let $i(x) = (x,t)\colon N \to M$. Define

$$r^{*+}(P)(G_0)(x) = i^*\big(P(G_0 \times 1)(x,t)\big) \in \Lambda^s(T^*N) .$$

We define r^{*-} similarly. If $G_0 \otimes 1 = G$, let $e_1, \ldots, e_{m-1}$ be a frame for TN and e_m for TS^1. Since the metric is flat in the S^1 direction, $\Omega_{km} = 0$. Thus $E_m(G_0 \times 1) = 0$ implying $r^{*+}(E_m) = 0$. Similarly, $r^{*-}(E_m) = 0$. If $q = 0$, let $r^* = r^{*+}$; otherwise let $r^* = r^{*-}$.

Theorem 1.3. *Let $P \in \mathscr{P}_{n, O(p,q), 0}$ with $r^* P = 0$.*

(a) *If $n < m$, $P = 0$*
(b) *If $n = m$, there is a constant c such that $P = cE_m$.*

This theorem provides a useful characterization of the Euler class. If $q = 0$, G is positive definite and this result was proved in [3]. In section 2, we will generalize the methods of [3] to prove 1.3 for arbitrary q.

Let X be coordinates which are G-normalized at x_0. Then $g_{ij}(X,G)(x_0) = \pm\delta_{ij}$ and $g(X,G)(x_0) = 1$; δ_{ij} is the Kronecker index. If we temporarily restrict to such coordinates, we can assume that the polynomials involve the $g_{ij/\alpha}$ variables with $\operatorname{ord}(\alpha) > 0$. Let $M = \{x \in \mathbf{R}^m : \|x\| < \varepsilon\}$ for some

$\varepsilon > 0$. Let X be the usual coordinates on R^m. We can find a metric G on M of type (p,q) so $g_{ij}(X,G)(0) = \pm\delta_{ij}$, $g(X,G)(0) = 1$, and so $g_{ij/\alpha}(X,G)(0)$ is arbitrary for $0 < \operatorname{ord}(\alpha) \leqq k$ for some k large. Consequently, $P \neq 0$ as a polynomial implies we can find (M,G,X) so X is G-normalized at x_0 and so $P(X,G)(x_0) \neq 0$. This permits us to identify the polynomial P with the formula defined by P.

Let $P \in \mathscr{P}_{n,O(p,q),s}$ and $A = g_{i_1j_1/\alpha_1} \ldots g_{i_sj_s/\alpha_s} dx_{k_1} \ldots dx_{k_r}$ be a monomial of P. We have assumed $\operatorname{ord}(\alpha_i) > 0$. Let

$$\begin{aligned}\deg_k(A) &= \delta_{i_1,k} + \delta_{j_1,k} + \alpha_1(k) + \ldots + \delta_{i_s,k} + \delta_{j_s,k} + \alpha_s(k) \\ &\quad + \delta_{k_1,k} + \ldots + \delta_{k_r,k}\,.\end{aligned}$$

$$r^*(A) = \begin{cases} A & \text{if } \deg_m A = 0 \\ 0 & \text{if } \deg_m A > 0 \end{cases}$$

r^*A is defined for $m-1$ dimensional manifolds. $r^*A(G_0) = A(G_0 \otimes 1)$. Since $r^*P(G_0) = 0$ for every G_0, r^*P vanishes as a polynomial. This implies that $\deg_m A > 0$ for every monomial A of P. Although this seems to single out the last index, in the second section we will use invariance under $O(p,q)$ to prove $\deg_m A > 0$ implies that $\deg_k(A) > 0$ for $k = 1, \ldots, m$. We will use this fact in the second section to prove theorem 1.3.

We give another description of the map r^* using Weyl's theorem 2.11A [8] on the invariants of the group $O(p,q)$. We follow the approach of Atiyah, Bott, and Patodi [1]; let X be normal coordinates which are centred at $x_0 \in M$. In the coordinate system X, the ordinary derivatives of G at x_0 can be expressed in terms of the covariant derivatives of the curvature tensor. This enables us to express P as a polynomial $\bar{P}$ in the covariant derivatives of the curvature tensor; because of the Bianchi identities, $\bar{P}$ is *not* uniquely defined as a polynomial. We use the metric to identify $T(M)$ with $T^*(M)$. The components of the jth order covariant derivatives of the curvature tensor lie in $\otimes^{4+j}(TM)$. Let $V = T(\mathsf{R}^m)_0$ and let Q be a fixed bilinear form of type (p,q) on V. Let $W = \bigoplus_{j=0}^{k}(\otimes^{4+j}(V))$ for some large k. Let W_0 be the subspace of W which is generated by the components of the covariant derivatives of the curvature tensors of all germs of bilinear forms G on R^m such that $G(0) = Q$. It is clear that W_0 is an $O(p,q)$ invariant subspace of W; $\bar{P}$ is a polynomial map from W_0 to $\Lambda^s(V)$ which is equivariant under the action of $O(p,q)$.

If $m = p+q > 2$, then $O(p,q)$ is semi-simple. Consequently, any representation of $O(p,q)$ is completely reducible [5, 9]. Let W_1 be an $O(p,q)$ invariant subspace of W such that $W = W_0 \oplus W_1$. Let $\bar{P} = 0$ on W_1; this defines $\bar{P}$ as a polynomial map from $W \to \Lambda^s(V)$ which is equivariant under the action of $O(p,q)$. By Weyl's theorem, this implies that P can be expressed in terms of alternations and contractions of indices. We sum over all indices from 1 thru m to define $\bar{P}$; r^*P is defined by letting the corresponding sums range over indices from 1 thru $m-1$. This description relates r^* to classical invariance theory if $p+q>2$.

If $m=2$, then every representation of the group $O(1,1)$ is not completely reducible. However, it is well known that the standard representation of $O(1,1)$ and $O(2)$ on the tensor algebra is completely reducible and we can therefore apply the argument given above to this case as well. The fact that every invariant is expressible in terms of alternations and contractions of indices also follows from Epstein and Stredder's work [2, 7].

Let $(\bar{p},\bar{q})=(p,q-1)$ for $q>0$ and $(\bar{p},\bar{q})=(p-1,0)$ if $q=0$.

LEMMA 1.4. *$r^*: \mathscr{P}_{n,O(p,q),s} \to \mathscr{P}_{n,O(\bar{p},\bar{q}),s}$ is surjective.*

PROOF. Let $P \in \mathscr{P}_{n,O(\bar{p},\bar{q}),s}$. Express P in terms of contractions and alternations of indices ranging from 1 thru $m-1$; let $Q \in \mathscr{P}_{n,O(p,q),s}$ be defined by letting the corresponding sums range over indices from 1 thru m. Then $r^*Q=P$ and therefore r^* is surjective. It should be noted that in general the expansion of P in terms of contractions and alternations of indices is not in general unique. Therefore, r^* need not be injective for general values of n.

Let d be exterior differentiation and let δ be the adjoint of d with respect to the bilinear form G. These two operators induce maps:

$$d: \mathscr{P}_{n,O(p,q),s} \to \mathscr{P}_{n+1,O(p,q),s+1}$$

$$\delta: \mathscr{P}_{n,O(p,q),s} \to \mathscr{P}_{n+1,O(p,q),s-1}$$

which commute with r^*. Let $T_m=S^1\times\ldots\times S^1$ be the m dimensional torus. T_m always admits a metric of type (p,q) for any $p+q=m$. If $P \in \mathscr{P}_{n,O(p,q),0}$, let

$$P(G)(M) = \int_M P(G)(x)\,|d\,\mathrm{vol}(x)|\ .$$

Similarly, if $P \in \mathscr{P}_{n,O(p,q),m}$ and if M is oriented, let

$$P(G)(M) = \int_M P(G)(x)\ .$$

In the third section, we will use lemma 1.4 and theorems 1.2 and 1.3 to prove:

THEOREM 1.5. *Let $P \in \mathscr{P}_{n,O(p,q),s}$ then:*

(a) *Let $s<m$ and $dP=0$. Then $\exists Q \in \mathscr{P}_{n-1,O(p,q),s-1}$ such that $P=dQ+R$ where R is a Pontrjagin form. $R=0$ if $n \neq s$, $Q=0$ if $n=s$.*
(b) *Let $s=m$ and suppose that $P(G)(T_m)=0$ for every G of type (p,q) on T_m. Then $\exists Q \in \mathscr{P}_{n-1,O(p,q),m-1}$ such that $P=dQ+R$ where R is a Pontrjagin form. $R=0$ if $n \neq m$; $Q=0$ if $n=m$.*
(c) *Let $s>0$ and $\delta P=0$. Then $\exists Q \in \mathscr{P}_{n-1,O(p,q),s+1}$ such that $P=\delta Q$.*
(d) *Let $s=0$ and suppose that $P(G)(T_m)=0$ for every G of type (p,q) on T_m. Then $\exists Q \in \mathscr{P}_{n-1,O(p,q),1}$ such that $P=\delta Q+cE_m$. If $n \neq m$, then $c=0$.*

This result implies the following:

COROLLARY 1.6. *Let P be an $O(p,q)$-invariant s-form valued polynomial in the derivatives of a bilinear form G of type (p,q).*

(a) *Suppose that $P(G)$ is always a closed s form. Then P induces a map from bilinear forms of type (p,q) to the s-cohomology of M. If the cohomology class of P is independent of G, then the cohomology class defined by $P(G)$ is a Pontrjagin class.*
(b) *Let $s=0$ and suppose that $P(G)(M)$ is independent of the metric. Then there exists a constant c such that $P(G)(M)=c\chi(M)$ (the Euler characteristic) for any compact manifold M.*

To prove this corollary, we decompose P into homogeneous parts by lemma 1.1 and prove corollary 1.6 for each part separately. We may therefore assume that $P \in \mathscr{P}_{n,O(p,q),s}$. If $dP=0$ and if $s<m$, we apply theorem 1.5 to express $P=R+dQ$. This implies that the cohomology class represented by P is a Pontrjagin class. If $s=m$, the assumption that the cohomology class represented by $P(G)$ is independent of G implies that $P(G)(T_m)$ is independent of G. If G is the flat metric on T_m of type (p,q), then $P(G)(T_m)=0$. Therefore, $P(G)(T_m)=0$ for any G on T_m. We apply theorem 1.5 to express $P=R+dQ$ which again implies the cohomology class represented by P is a Pontrjagin class. Finally, let $s=0$. Since $P(G)(T_m)$ is independent of the metric G on T_m, $P(G)(T_m)=0$ for every G of type (p,q) on T_m. This implies that $P=dQ+cE_m$. The Chern–Gauss–Bonnet theorem for indefinite metrics implies $P(G)(M)=cE_m(G)(M)=c\chi(M)$ for any compact manifold M. This completes the proof of corollary 1.6.

We proved theorem 1.5 and corollary 1.6 for positive definite metrics previously in [4]. Corollary 1.6 settles in the affirmative a conjecture which was proposed by I. M. Singer [6].

Section 2.

In this section, we apply the methods of [3] to prove theorems 1.2 and 1.3. Let $\{e_1,\ldots,e_m\}$ be the standard basis for R^m and let Q be the bilinear form of type (p,q) defined in section 1. Let $i \neq j$, we define $F_{abij} \in SO(p,q)$ as follows: if $Q(e_i,e_i)=Q(e_j,e_j)$ let $a^2+b^2=1$. Let

$$F_{abij}(e_i) = ae_i+be_j,\ F_{abij}(e_j) = -be_i+ae_j$$
$$F_{abij}(e_k) = e_k \text{ otherwise}$$

If $Q(e_i,e_i)= -Q(e_j,e_j)$, let $a^2-b^2=1$. Let

$$F_{abij}(e_i) = ae_i+be_j,\ F_{abij}(e_j) = be_i+ae_j$$
$$F_{abij}(e_k) = e_k \text{ otherwise .}$$

The F_{abij} generate $SO(p,q)$.

For the remainder of this section, x_0 will be a fixed point of M. We only consider coordinates which are G-normalized at x_0. Our polynomials involve the $g_{ij/\alpha}$ variables for $\operatorname{ord}(\alpha)>0$. $\varphi \in O(p,q)$. Let $Y=\varphi^*X$ be defined by:

$$x_j = \varphi_{ij}y_i,\ \partial/\partial y_i = \varphi_{ij}\partial/\partial x_j \,.$$

We adopt the convention of summing over repeated indices. Let $\varphi^*: \mathscr{P} \to \mathscr{P}$ be the algebra isomorphism defined by the identity:

$$\varphi^*P(X,G)(x_0) = P(\varphi^*X,G)(x_0)\,.$$

We extend φ^* to be a morphism of s-form valued polynomials as well using this defining relation. If P is an s-form valued polynomial which is $O(p,q)$ invariant, $\varphi^*P=P$ for every $\varphi \in O(p,q)$.

LEMMA 2.1. *Let P be an s-form valued polynomial and $i \neq j$ fixed indices. Suppose ${F^*}_{abij}P=P$ for all admissible a,b. Then:*

(a) *Let A be a monomial of P and let A' be the monomial which is obtained from A by interchanging the i and j indices. Let*

$$t = \deg_i A + \deg_j A\,.$$

Then t is even and $c(A,P)= \pm c(A',P)$, so A' is also a monomial of P.

(b) *Let $g_{ij/\alpha}$ divide some monomial of P. Then $g_{jj/\beta}$ divides some monomial of P for some multi-index β.*

(c) *Let g_{uv} divide some monomial of P with u and v distinct from i and j. Let $\alpha=(a_1,\ldots,a_i,\ldots,a_j,\ldots,a_m)$ where for notational convenience we have assumed $i<j$. Let $\beta=(a_1,\ldots,0,\ldots,a_i+a_j,\ldots,a_m)$. Then $g_{uv/\beta}$ divides some monomial of P.*

PROOF. We must first discuss the action of F^*_{abij} on $\mathscr{P}$. Suppose $Q(e_i,e_i) = -Q(e_j,e_j)$. Let $A = A_0 dx_I$ be a monomial. We compute $F^*_{abij}A_0$ by formally replacing every i index by $ai+bj$ and every j index by $bi+aj$. The indices of dx_I are contravariant rather than covariant so we replace every i index by $ai-bj$ and every j index by $-bi+aj$ in dx_I. We expand and apply the symmetries. If $Q(e_i,e_i) = Q(e_j,e_j)$, we do not need to distinguish between covariant and contravariant indices. We replace every i index by $ai+bj$ and every j index by $-bi+aj$ and expand similarly. Let $P = P_0 + \ldots + P_t + \ldots$ where every monomial A_t of P_t satisfies $t = \deg_i A_t + \deg_j A_t$. Since F^*_{abij} only affects the i and j indices, the P_t are invariant separately. We may therefore assume without loss of generality that $P = P_t$ for some t. If A is a monomial of P, let $F^*_{abij}A = a^r b^{t-r}$ summed over r. The monomials of Q_r are obtained from A by changing exactly $t-r$ indices $i \to j$ or $j \to i$ and by leaving the remaining r indices fixed.

We will use the following lemma in the proof of lemma 2.1:

LEMMA 2.2. *Let $p(a,b) = c_r a^r b^{t-r}$ be homogeneous of order t in the variables a and b.*

(a) *if $p(a,b) = 0$ for all $a^2+b^2 = 1$ then $p = 0$ as a polynomial.*
(b) *if $p(a,b) = 1$ for all $a^2+b^2 = 1$ then $p(a,b) = (a^2+b^2)^{t/2}$ as a polynomial.*
(c) *if $p(a,b) = 0$ for all $a^2-b^2 = 1$ then $p = 0$ as a polynomial.*
(d) *if $p(a,b) = 1$ for all $a^2-b^2 = 1$ then $p(a,b) = (a^2-b^2)^{t/2}$ as a polynomial.*

We prove lemma 2.2 as follows: let $t = a/b$ and $f(t) = c_s t^s$. Then $f(t) = 0$ for infinitely many choices of t under the assumptions of (a) or (c). Therefore all the $c_s = 0$. (b) and (d) follow directly from (a) and (c).

We prove lemma 2.1 (a) as follows: $t = \deg_i A + \deg_j A$ is independent of which monomial A of P we consider. Let $a = -1$, $F^*_{abij}A = (-1)^t A$. Since P is invariant, t must be even. Let B be a monomial of P. Decompose $F^*_{abij}B = ca^t A + db^t A + \text{other terms}$. If c is non-zero, then B transforms to A by not changing indices. This implies $B = A$ and $c = 1$. If d is non-zero, B transforms to A by changing all indices $i \to j$ or $j \to i$. This implies $B = A'$ and $d = \pm 1$. Therefore

$$F^*_{abij}P = a^t c(A,P)A \pm b^t c(A',P)A + \text{other terms}.$$

Since P is invariant, $c(A,P) = a^t c(A,P) \pm b^t c(A',P) + \ldots$. This implies $c(A,P) = \pm c(A',P)$ by lemma 2.2.

We prove lemma 2.1(b) as follows: suppose it false. Let $A=(g_{ij/\alpha})^k A_0$ for $k>0$ where $g_{ij/\alpha}$ does not divide A_0. Let $A_1=g_{jj/\alpha}(g_{ij/\alpha})^{k-1}A_0$. $F^*{}_{abij}A=ka^{t-1}bA_1+\ldots$. Let B be a monomial of P such that $F^*{}_{abij}B=ca^{t-1}bA_1+\ldots$ for $c\neq 0$. Then B transforms to A_1 by changing one index $i\to j$ or $j\to i$. Since $g_{jj/\beta}$ does not divide B for any β by assumption, this implies that we must have changed $i\to j$ and $B=A$. Therefore $F^*{}_{abij}P=kc(A,P)a^{t-1}bA_1+\ldots$. Since P is invariant, by lemma 2.2 $kc(A,P)=0$. This contradicts the assumption that A is a monomial of P and $k>0$, and proves lemma 2.1(b).

We prove lemma 2.1(c) as follows: let

$$\alpha_n = (a_1,\ldots,a_i-n,\ldots,a_j+n,\ldots,a_m) \quad \text{for } n=0,\ldots,a_i\,.$$

Choose n maximal so that g_{uv/α_n} divides some monomial of P. If $n=a_i$, the lemma is true so we assume that $n<a_i$. Let $A_1=g_{uv/\alpha_n}A_0$. Suppose $F^*{}_{abij}B=cba^{t-1}g_{uv/\alpha_{n+1}}A_0+\ldots$. Then B transforms to this monomial by changing $i\to j$ or $j\to i$. By hypothesis, $g_{uv/\alpha_{n+1}}$ does not divide B. Since u and v are distinct from i and j, the index which was changed cannot be one of these. This implies that the index which was changed was $i\to j$ and therefore $B=A_1$. If g_{uv/α_n} divides A_1 with multiplicity k, then $F^*{}_{abij}A_1=k(a_i-n)a^{t-1}bg_{uv/\alpha_{n+1}}A_0+\ldots$. This implies

$$F^*{}_{abij}P = c(A_1,P)\cdot k(a_i-n)a^{t-1}bg_{uv/\alpha_{n+1}}A_0+\ldots.$$

Since P is invariant and since $g_{uv/\alpha_{n+1}}$ divides no monomial of P by hypothesis, this implies $k(a_i-n)=0$ by lemma 2.2. Since $k>0$ and $a_i>n$ this is a contradiction. This completes the proof of lemma 2.1.

We can use lemma 2.1 to prove theorem 1.3. Suppose that $P\neq 0$, $P\in\mathscr{P}_{n,O(p,q),0}$, $r^*P=0$, and $n\leqq m$. Since $P\neq 0$, we can choose G so that $P(G)(x_0)\neq 0$. We further normalize our choice of coordinate systems X for the remainder of this section by assuming not only that X is G-normalized at x_0 but also that the first derivatives of G vanish at x_0 in the coordinate system X. Since this is true for normal coordinates, such coordinate systems surely exist. This class of coordinate systems is invariant under the action of $O(p,q)$. By restricting to such coordinate systems, we can assume that our polynomials involve the $g_{ij/\alpha}$ variables for $\operatorname{ord}(\alpha)\geqq 2$.

Since $P(G)(x_0)\neq 0$, there must be some monomial A of P of the form

$$A = g_{i_1j_1/\alpha_1}\cdots g_{i_rj_r/\alpha_r}\operatorname{ord}(\alpha_i)\geqq 2\,.$$

Let $T_i\in O(p,q)$ be defined by:

$$T_i(e_i) = -e_i,\ T_i(e_j) = e_j\,.$$

Then $T_i^*(A) = (-1)^{\deg_i A} A$. Since P is invariant under $O(p,q)$, this implies that $\deg_i A$ must be even for all i. Since $r^*P = 0$, $\deg_m A > 0$ for every monomial A of P. By lemma 2.1(a), this implies $\deg_k A > 0$ for every monomial A of P and $k = 1,\ldots,m$. Therefore $\deg_k A \geqq 2$ for $k = 1,\ldots,m$. We estimate:

$$2m \leqq \deg_1 A + \ldots + \deg_m A = 2r + \operatorname{ord}(\alpha_1) + \ldots + \operatorname{ord}(\alpha_r) = 2r + n .$$
$$2r \leqq \operatorname{ord}(\alpha_1) + \ldots + \operatorname{ord}(\alpha_r) = n .$$

Consequently $2m \leqq 2n$ and $n \geqq m$. We have assumed that $n \leqq m$. Consequently $P = 0$ if $n < m$ which proves theorem 1.3(a). If $n = m$, then all these inequalities must be equalities. Therefore:

$$m = n = 2r;\ \deg_i A = 2 \text{ for } i = 1,\ldots,m;$$
$$\operatorname{ord}(\alpha_i) = 2 \text{ for } i = 1,\ldots,r .$$

We apply lemma 2.1(a) and (b) to choose a monomial A of P of the form:

$$A = g_{11/\alpha_1} g_{i_2 j_2/\alpha_2} \cdots$$

Since $\deg_1 A = 2$, this implies that the index 1 appears nowhere else in the monomial A. We can therefore apply lemma 2.1(a) and (c) together with the fact that $\operatorname{ord}(\alpha_1) = 2$ to choose a monomial A of P of the form:

$$A = g_{11/22} g_{i_2 j_2/\alpha_2} \cdots .$$

Again, $\deg_1 A = \deg_2 A = 2$ and therefore these indices appear nowhere else in the monomial A. Let $P = g_{11/22} P_0 + P_1$ where $g_{11/22}$ does not divide P_1. Then $F^*_{abij} P_0 = P_0$ for $i,j > 2$. Consequently we can apply lemma 2.1(a) and (b) to choose a monomial A of P of the form

$$A = g_{11/22} g_{33/\alpha_2} \cdots$$

We continue inductively to show $A_0 = g_{11/22} \cdots g_{m-1,m-1/mm}$ is a monomial of P.

Let $V_m \subset \mathscr{P}_{m,O(p,q),0}$ be the kernel of r^*. If $V_m \neq 0$, m must be even. If $P \in V_m$, $P \neq 0$, $c(A_0,P) \neq 0$ since A_0 is a monomial of P. Therefore, $\dim(V_m) \leqq 1$. If m is even, $E_m \neq 0 \in V_m$. Therefore E_m is a basis for V_m which proves 1.3.

We will need the following lemma in the proof of theorem 1.2:

LEMMA 2.3. *Let $P \neq 0$ be an s-form valued polynomial invariant under the action of $O(p,q)$. Then there is a monomial A of P with:*

$$A = g_{i_1 j_1/\alpha_1} \cdots g_{i_r j_r/\alpha_r} dx_{k_1} \cdots dx_{k_s}$$

such that $\deg_k A = 0$ *for* $k > 2r$.

PROOF. Let A be any monomial of P which is represented as above. If $2r \geqq m$, the lemma is proved. Let $A=B\cdot C$ where $B=g_{i_1j_1/\alpha_1}\dots g_{i_kj_k/\alpha_k}$ is such that $\deg_i B=0$ for $i>2k$. Choose A so that $k=k(A)$ is maximal among all possible choices of A. First we assume that $k<r$. Let $P=BP_0+P_1$ where the monomials of P_1 are not divisible by B. $P_0\neq 0$ by hypothesis and P_0 involves derivatives of the metric. It is clear that P_0 is invariant under $F^*{}_{abij}$ for $i,j>2k$. Let $g_{ij/\alpha}$ divide some monomial of P_0. If $i,j>2k$, we can suppose $i=j=2k+1$ by applying lemma 2.1(a) and (b). Otherwise, by applying lemma 2.1(a) if needed we can suppose $i,j\leqq 2k+1$. Thus in any event, we can choose $g_{ij/\alpha}$ dividing some monomial of P_0 such that $i,j\leqq 2k+1$. Express $P_0=P_2+P_3$ where every monomial of P_2 is divisible by $g_{ij/\alpha}$ for some α and no monomial of P_3 is divisible by $g_{ij/\alpha}$ for any α. Since $i,j\leqq 2k+1$, $F^*{}_{abuv}P_2=P_2$ for $u,v\geqq 2k+2$. We apply lemma 2.1(c) repeatedly to the indices $u\geqq 2k+2$ to construct $g_{ij/\alpha}$ which divides some monomial of P_2 where $\alpha=(a_1,\dots,a_{2k+2},0,\dots,0)$. If $B'=Bg_{ij/\alpha}$, then $\deg_u B'=0$ for $u>2k+2$. Since there is some monomial of P of the form $A'=B'C'$, this contradicts the maximality of k in the choice of A. Therefore, $k=r$.

We have constructed a monomial A of P of the form:

$$A = g_{i_1j_1/\alpha_1}\dots g_{i_rj_r/\alpha_r}dx_{k_1}\dots dx_{k_s} = Bdx_{k_1}\dots dx_{k_s}\,.$$

Let $T_i\in O(p,q)$ be defined by:

$$T_i(e_i) = -e_i,\quad T_i(e_j) = e_j \quad \text{for } j\neq i\,.$$

Then $T^*{}_iA=(-1)^{\deg_i A}A$. Since P is invariant, $\deg_i A=\deg_i B+\delta_{i,k_1}+\dots+\delta_{i,k_s}$ is even. Since $\deg_i B=0$ for $i>2r$, $\deg_i A<2$ implies $\deg_i A=0$ for $i>2r$. This proves lemma 2.3.

We use this lemma to generalize theorem 1.3(a). Let $(\bar p,\bar q)=(p,q-1)$ for $q>0$ and let $(\bar p,\bar q)=(p-1,0)$ if $q=0$.

THEOREM 2.4. *$r^*\colon \mathscr{P}_{n,O(p,q),s}\to\mathscr{P}_{n,O(\bar p,\bar q),s}$ is bijective if $n<p+q=m$.*

PROOF. By lemma 1.4, r^* is always surjective. Suppose that r^* is not injective. Let $0\neq P\in\mathscr{P}_{n,O(p,q),s}$ with $r^*P=0$. By lemma 2.3, choose a monomial A of P of the form:

$$A = g_{i_1j_1/\alpha_1}\dots g_{i_rj_r/\alpha_r}dx_{k_1}\dots dx_{k_s}$$

such that $\deg_i A=0$ for $i>2r$. Since $r^*P=0$, $\deg_m A>0$. Therefore $2r\geqq m$. We have restricted to the polynomial algebra in the $g_{ij/\alpha}$ variables with $\operatorname{ord}(\alpha)>1$. Therefore, $n=\operatorname{ord}(\alpha_1)+\dots+\operatorname{ord}(\alpha_r)\geqq 2r\geqq m$. This contradicts the assumption that $n<m$ and proves theorem 2.4.

We use theorem 2.4 to reduce the proof of theorem 1.2 to the case that $q=0$. This case was proved previously in [1, 3]. Let $n \leqq s \leqq p+q=m$, then:

$$(r^*)^m: \mathscr{P}_{n, O(p, q+m), s} \to \mathscr{P}_{n, O(p, q), s}$$

$$(r^*)^m: \mathscr{P}_{n, O(q+m, p), s} \to \mathscr{P}_{n, O(m, 0), s} .$$

By theorem 2.4, these two maps are bijective. If G is a metric of type (p,q), then $-G$ is a metric of type (q,p). Let τ be the isomorphism

$$\tau: \mathscr{P}_{n, O(p, q), s} \leftrightarrow \mathscr{P}_{n, O(q, p), s}$$

be defined by $\tau(P)(G)=P(-G)$. This map induces a bijection of

$$\tau: \mathscr{P}_{n, O(p, q+m), s} \to \mathscr{P}_{n, O(q+m, p), s} .$$

Consequently:

$$\begin{aligned} \dim(\mathscr{P}_{n, O(p, q), s}) &= \dim(\mathscr{P}_{n, O(p, q+m), s}) = \dim(\mathscr{P}_{n, O(q+m, p), s}) \\ &= \dim(\mathscr{P}_{n, O(m, 0), s}) . \end{aligned}$$

Since $\mathscr{P}_{n, O(m, 0), s}=0$ for $n<s$, $\mathscr{P}_{n, O(p, q), s}=0$ for $n<s$. This proves theorem 1.2(a). If s is not divisible by 4, $\mathscr{P}_{s, O(m, 0), s}=0$ which implies that $\mathscr{P}_{s, O(p, q), s}=0$. Finally, let $s=4k$ and let $\pi(k)$ be the number of partitions of the integer k. Then $\dim(\mathscr{P}_{4k, O(p, q), 4k})=\dim(\mathscr{P}_{4k, O(m, 0), 4k})=\pi(k)$. Since the Pontrjagin classes span a subspace of dimension $\pi(k)$, $\mathscr{P}_{4k, O(p, q), 4k}$ is spanned by the Pontrjagin classes. This completes the proof of theorem 1.2.

We could also generalize the methods of [3] to prove theorem 1.2 directly without using H. Weyl's theorem or the group representation theorems which were used in lemma 1.4 to prove that r^* is surjective. In the next section we use theorem 2.4 to complete the proof of theorem 1.5.

Section 3.

Let $\nu=O(p,q)$ or $SO(p,q)$.

THEOREM 3.1. *Let $P \in \mathscr{P}_{n,\nu,s}$ with $n \neq s$.*

(a) *If $s<m$ and if $dP=0$, then $\exists Q \in \mathscr{P}_{n-1,\nu,s-1}$ such that $P=dQ$.*
(b) *If $s=m$ and if $P(G)(T_m)=0$ for every G of type (p,q) on T_m, then $\exists Q \in \mathscr{P}_{n-1,\nu,m-1}$ such that $P=dQ$.*

We will prove theorem 3.1 later in this section. We apply this result as follows: Let $P \in \mathscr{P}_{n, O(p, q), s}$ satisfy the assumptions of theorem 1.5(a) or theorem 1.5(b). If $n=s$, P is a Pontrjagin form by theorem 1.2. If $n \neq s$, we can apply theorem 3.1 to conclude that $P=dQ$ for some $Q \in \mathscr{P}_{n-1, O(p,q), s-1}$. This proves (a) and (b) of theorem 1.5.

We dualize theorem 3.1 to prove (c) and (d) of theorem 1.5 as follows:

THEOREM 3.2. *Let $P \in \mathscr{P}_{n,\nu,s}$ with $n \neq m-s$.*

(a) *If $s>0$ and if $\delta P=0$, then $\exists Q \in \mathscr{P}_{n-1,\nu,s+1}$ such that $P=\delta Q$.*

(b) *If $s=0$ and if $P(G)(T_m)=0$ for every G of type (p,q) on T_m, then $\exists Q \in \mathscr{P}_{n-1,\nu,1}$ such that $P=\delta Q$.*

PROOF. First we suppose that $\nu=SO(p,q)$. Let orn be a local orientation of M. Then $*P \in \mathscr{P}_{n,SO(p,q),m-s}$. Since $n \neq m-s$ and $*P$ satisfies the hypothesis of theorem 3.1, we can express $*P=dQ$. Therefore, $P= \pm *d*Q= \pm \delta Q$ for $Q \in \mathscr{P}_{n-1,SO(p,q),s+1}$. This proves theorem 3.2 if $\nu= SO(p,q)$. Next, let $P \in \mathscr{P}_{n,O(p,q),s} \subset \mathscr{P}_{n,SO(p,q),s}$ satisfy the hypothesis of theorem 3.2. Then $P=\delta Q$ for $Q \in \mathscr{P}_{n-1,SO(p,q),s+1}$ by what we just proved. Let orn be a local orientation of M and set

$$Q_0 = (Q(G,\mathrm{orn})+Q(G,-\mathrm{orn}))/2 \in \mathscr{P}_{n-1,O(p,q),s+1}\,.$$

Since $P(G,\mathrm{orn})=P(G,-\mathrm{orn})$, we still have $P=\delta Q_0$ which completes the proof of theorem 3.2.

We apply theorem 3.2 to prove theorem 1.5(c) and theorem 1.5(d) as follows: let $P \in \mathscr{P}_{n,O(p,q),s}$ satisfy the assumptions of theorem 1.5(c) or theorem 1.5(d). If $n \neq m-s$, then $P=\delta Q$ by theorem 3.2. We may therefore assume that $m-s=n$. Let $P_0=r^*P \in \mathscr{P}_{n,O(\bar{p},\bar{q}),s}$ for $\bar{p}+\bar{q}=m-1$. Since integration and δ commute with r^*, P_0 satisfys the same hypothesis as P does. Since $n=m-s \neq (m-1)-s$, we can apply theorem 3.2 to conclude $P_0=\delta Q_0$ for $Q_0 \in \mathscr{P}_{n-1,O(\bar{p},\bar{q}),s+1}$. By lemma 1.4, r^* is surjective. Let $Q \in \mathscr{P}_{n-1,O(p,q),s+1}$ be chosen so $r^*Q=Q_0$. Then $r^*(P-\delta Q)=0$. If $s \neq 0$, then $n<m$ and r^* is injective by theorem 2.4. This proves $P=\delta Q$ and proves theorem 1.5(c). If $s=0$, then $n=m$. By theorem 1.3, $r^*(P-\delta Q)=0$ implies $P-\delta Q=cE_m$. This proves theorem 1.5(d) and completes the proof of theorem 1.5.

The remainder of this paper is devoted to the proof of theorem 3.1. We adopt the following notational conventions: indices i, j, k range from 1 thru m; we sum over repeated indices unless otherwise indicated. If $\alpha=(\alpha(1),\ldots,\alpha(m))$ is a multi-index, let $\{i_1^{\alpha},\ldots,i_s^{\alpha}\}$ be the un-ordered collection of indices which corresponds to α which we defined in section 1. Let V be an m dimensional vector space and let $S_r(V) \subset \otimes^r V$ be the subspace of symmetric tensors. If $\{v_1,\ldots,v_r\}$ is an unordered collection of elements of V, let $v_1\circ\ldots\circ v_r \in S_r(V)$ be the symmetric tensor product of these vectors. Let V^* be the dual space of V; $S_r(V^*)=S_r(V)^*$. If $R\in S_r(V)\otimes\Lambda^s(V^*)$ and if $w\in S_r(V^*)$, define $R(w)\in\Lambda^s(V^*)$.

Let $(e_1,\ldots,e_m)$ be a basis for V and let $(e_1{}^*,\ldots,e_m{}^*)$ be the dual basis for V^*. If $\operatorname{ord}(\alpha)=r$, define

$$e_\alpha = e_{i_1}{}^\alpha\circ\ldots\circ e_{i_r}{}^\alpha \quad\text{and}\quad \binom{r}{\alpha} = r!/(\alpha(1)!\ldots\alpha(m!)) .$$

LEMMA 3.3. *Let* $R\in S_r(V)\otimes\Lambda^s(V^*)$ *and suppose* $R(e_1{}^*\circ\ldots\circ e_1{}^*)=0$ *for any basis* $(e_1,\ldots,e_m)$. *Then* $R=0$.

PROOF. Let $a=(a_1,\ldots,a_m)\in \mathsf{R}^m$ with $a_1\neq 0$. Let $(\bar e_1,\ldots,\bar e_m)$ be the basis:

$$\bar e_1 = e_1/a_1,\ \bar e_i = e_i - a_i e_1/a_1 \ \text{ for } i>1; \quad \bar e_1{}^* = a_1e_1{}^*+\ldots+a_me_m{}^* .$$

Let $a^\alpha=(a_1)^{\alpha(1)}\ldots(a_m)^{\alpha(m)}$, then:

$$0 = R(\bar e_1{}^*\circ\ldots\circ\bar e_1{}^*) = \textstyle\sum_{\operatorname{ord}(\alpha)=r} a^\alpha\binom{r}{\alpha}R(e_\alpha{}^*) .$$

Since this identity holds for all $a\in\mathsf{R}^m$ with $a_1\neq 0$, it implies $R(e_\alpha{}^*)=0$ for $\operatorname{ord}(\alpha)=r$. Since the $e_\alpha{}^*$ are a dual basis for $S_r(V)$, this implies $R=0$ and proves the lemma.

Let $(e_1,\ldots,e_m)$ be a basis for V and $R\in S_r(V)\otimes\Lambda^s(V^*)$. Expand:

$$R = R(i_1,\ldots,i_r;j_1,\ldots,j_s)(e_{i_1}\circ\ldots\circ e_{i_r})\otimes(e_{j_1}{}^*\wedge\ldots\wedge e_{j_s}{}^*) .$$

This expression is summed over all possible indices ranging from 1 thru m. We use $R(i_1,\ldots;j_1,\ldots)$ instead of the more classical $R_{i_1\ldots}{}^{j_1\ldots}$ for notational convenience. We may assume that $R(\ldots)$ is symmetric in the first r indices and anti-symmetric in the last s indices. This assumption permits us to replace symmetric and wedge product in this expansion by ordinary tensor product. We can also restrict the sum to range over $j_1<\ldots<j_s$ by multiplying everything by $s!$.

If $v\in V$ and $w\in V^*$, let (v,w) be the natural pairing. Let

$$\theta = v_1\otimes\ldots\otimes v_r\otimes w_1\otimes\ldots\otimes w_s\in(\otimes^r V)\otimes(\otimes^s V^*) .$$

Define:

$$T(\theta) = (v_1,w_1)v_2\otimes\ldots\otimes v_r\otimes w_2\otimes\ldots\otimes w_s\in(\otimes^{r-1}V)\otimes(\otimes^{s-1}V^*) .$$

The element $e_i\otimes e_i{}^*\in V\otimes V^*$ is invariantly defined. Let

$$S(\theta) = \textstyle\sum_i e_i\otimes v_1\otimes\ldots\otimes v_r\otimes e_i{}^*\otimes w_1\otimes\ldots\otimes w_s\in(\otimes^{r+1}V)\otimes(\otimes^{s+1}V^*) .$$

The two maps S and T are invariantly defined. Since $S_r(V)\otimes\Lambda^s(V^*)\subset(\otimes^r V)\otimes(\otimes^s V^*)$, S and T induce maps:

$$T: S_r(V)\otimes\Lambda^s(V^*)\to S_{r-1}(V)\otimes\Lambda^{s-1}(V^*)$$

$$S: S_r(V)\otimes\Lambda^s(V^*)\to S_{r+1}(V)\otimes\Lambda^{s+1}(V^*)\,.$$

In terms of a basis for V,

$$T(R) = R(i_1\ldots;j_1\ldots)(e_{i_1},e_{j_1}{}^*)(e_{i_2}\circ\ldots\circ e_{i_r})\otimes(e_{j_2}{}^*\wedge\ldots\wedge e_{j_s}{}^*)$$

$$S(R) = R(i_1\ldots;j_1\ldots)(e_k\circ e_{i_1}\circ\ldots\circ e_{i_r})\otimes(e_k{}^*\wedge e_{j_1}{}^*\wedge\ldots\wedge e_{j_s}{}^*)\,.$$

We will need the following technical lemma in the proof of theorem 3.1:

LEMMA 3.4. *Let $R\in S_r(V)\otimes\Lambda^s(V^*)$ for $r>0$. If $S(R)=0$, $S(T(R))=(m+1-s)R$.*

PROOF. Let $Q=ST(R)-(m+1-s)R$. By lemma 3.4, it suffices to prove that $Q(e_1{}^*\circ\ldots\circ e_1{}^*)=0$ or equivalently that $Q(1,\ldots,1;j_1,\ldots,j_s)=0$. For the remainder of the proof of this lemma, let $j_1<\ldots<j_s$. Suppose first $j_1>1$. Let $v=(e_1\circ\ldots\circ e_1)\in S_{r-1}(V)$ and let $w=e_{j_2}{}^*\wedge\ldots\wedge e_{j_s}{}^*\in\Lambda^{s-1}(V^*)$. Since $S(R)=0$,

$$0 = R(1,\ldots,1;j_1\ldots j_s)e_1{}^*\wedge e_{j_1}{}^*\wedge w\,.$$

For $1<j_1<\ldots<j_s$, $e_1{}^*\wedge e_{j_1}{}^*\wedge w\neq 0$. Therefore $R(1,\ldots,1;j_1,\ldots,j_s)=0$ for $j_1>1$. By definition, $S(T(R))(1,\ldots,1;j_1,\ldots,j_s)=0$ for $1<j_1<\ldots<j_s$. This proves $Q(1,\ldots,1;j_1,\ldots,j_s)=0$ for $j_1>1$. We assume therefore that $j_1=1$.

We must show $Q(1,\ldots,1;\ 1,j_2,\ldots,j_s)=0$. For notational convenience we may assume without loss of generality that $j_2=2,\ldots,j_s=s$. If $k>s$,

$$S(R) = s!\big(R(1,\ldots,1;\ 1,\ldots,s)(-1)^s+R(1,\ldots,1,k;\ 2,\ldots,s,k)\big)(v\circ e_1\circ e_k)$$
$$\otimes(e_1{}^*\wedge w\wedge e_k{}^*)$$
$$+\ \text{other terms not involving } (v\circ e_1\circ e_k)\otimes(e_1{}^*\wedge w\wedge e_k{}^*)\,.$$

Since $S(R)=0$,

$$R(1,\ldots,1;\ 1,\ldots,s) = (-1)^{s-1}R(1,\ldots,1,k;\ 2,\ldots,s,k)\,.$$

The only monomial of $T(R)$ which makes a contribution to $(v\circ e_1)\otimes(e_1{}^*w)$ under the action of S is $v\otimes w$. Let

$$T(R) = s!\big(R(1,\ldots,1;\ 1,\ldots,s)+\textstyle\sum_{k>s}R(1,\ldots,1,k;\ 2,\ldots,s,k)(-1)^{s-1}\big)(v\otimes w)$$

$$+ \text{ other terms not involving } v\otimes w$$

$$= s!(m+1-s)R(1,\dots,1;\,1,\dots,s)(v\otimes w)$$

$$+ \text{ other terms not involving } v\otimes w\,.$$

Therefore:

$$S(T(R)) = s!(m-s+1)R(1,\dots,1;\,1,\dots,s)(v\circ e_1)\otimes(e_1{}^*\wedge w)$$

$$+ \text{ other terms not involving } (v\circ e_1)\otimes(e_1\wedge w)\,.$$

This implies that

$$S(T(R))(1,\dots,1;\,1,\dots,s) = (m-s+1)R(1,\dots,1;\,1,\dots,s)$$

and completes the proof of lemma 3.4.

Let G be a metric of type (p,q) on an m dimensional manifold M and let $(e_1,\dots,e_m)$ be a local frame field for TM. Let ∇ be the Levi–Civita connection of G. If θ is a tensor field on M, let $\theta_{;k}$ denote covariant differentiation in the direction e_k. Let $\theta_{;k_1\dots k_r}$ denote multiple covariant differentiation. If h is a real-valued function on M, let $h_{;i_1,\dots,i_r}$ denote the covariant derivatives of h. In general, this will not be a symmetric tensor field. Let

$$h_\alpha = h_{:i_1{}^\alpha\dots i_r{}^\alpha} = 1/r!\,\left(\textstyle\sum_\tau h_{;i^\alpha{}_{\tau(1)}\dots i^\alpha{}_{\tau(r)}}\right)$$

where the sum ranges over the permutations of the integers 1 thru r. Let h_r be the invariantly defined tensor field:

$$h_r = h_{:i_1\dots i_r}(e_{i_1}{}^*\circ\dots\circ e_{i_r}{}^*) \in S_r(T^*M)\,.$$

Fix m, (p,q) and $\nu=O(p,q)$ or $SO(p,q)$ for the remainder of this section. Let $\mathscr{R}_{r,s}$ be the set of ν-invariant polynomials in the $\{g_{ij/\alpha}, g^{-1}\}$ variables which take values in $S_r(TM)\otimes\Lambda^s(T^*M)$. For example,

$$\mathscr{R}_{O,s} \cong \oplus_{n=0}^{\infty}\,\mathscr{P}_{n,O(p,q)s,}\,.$$

If G is a metric of type (p,q) and if $R_r\in\mathscr{R}_{r,s}$, then $R_r(G)\in\Gamma(S_r(TM)\otimes\Lambda^s(T^*M))$. If h is a real-valued function, $h_r\in\Gamma(S_r(T^*M))$. We define $R_r(G,h)\in\Gamma(\Lambda^s T^*M)$ in the obvious fashion. Let

$$\mathscr{R}_s = \oplus_{r=0}^{\infty}\,\mathscr{R}_{r,s}\,.$$

If $0\neq R\in\mathscr{R}_s$, let $R=R_0+\dots+R_r$ where $R_j\in\mathscr{R}_{j,s}$. Choose r so $R_r\neq 0$; let $\mathrm{ord}_h(R)=r$. We define $\mathrm{ord}_h(R)=-1$ if $R=0$. Let

$$R(G,h) = R_0(G,h)+\dots+R_r(G,h_r)\in\Lambda^s T^*M\,.$$

Let $dR(G,h)=d(R(G,h))$.

LEMMA 3.5. *Let* $R \in \mathscr{R}_s$. *Then*:
(a) $dR \in \mathscr{R}_{s+1}$
(b) $\operatorname{ord}_h(dR) \leqq \operatorname{ord}_h(R)+1$
(c) $\operatorname{ord}_h(dR-S(R)) \leqq \operatorname{ord}_h(R)$.

PROOF. We may assume without loss of generality that $R=R_r$ for some r. Then:

$$dR(G,h) = h_{:i_1\dots i_r;k}R(i_1\dots i_r;j_1\dots j_s)e_k{}^* \wedge e_{j_1}{}^* \wedge \dots \wedge e_{j_s}{}^*$$
$$+h_{:i_1\dots i_r}R(i_1\dots i_r;j_1\dots j_s)_k e_k{}^* \wedge e_{j_1}{}^* \wedge \dots \wedge e_{j_s}{}^*$$

If we commute indices in covariant differentiation, we introduce expressions in the curvature tensor. Therefore:

$$h_{:i_1\dots i_r;k} = h_{:i_1\dots i_rk}+\textstyle\sum_{j<r} Q_{j,i_1,\dots,i_r,k}(h_j) .$$

Therefore:

$$dR(G,h) = h_{:i_1\dots i_rk}R(i_1,\dots,i_r;j_1,\dots,j_s)e_k{}^* \wedge e_{j_1}{}^* \dots \wedge e_{j_s}{}^*+\textstyle\sum_{j\leqq r} Q_j'(h_j)$$

where $Q_j' \in \mathscr{R}_{j,s+1}$.
The tensor $S(R)$ was defined so that:

$$S(R)(h_{r+1}) = h_{:i_1\dots i_{r+1}}R(i_1\dots i_r;j_1\dots j_s)e^*_{i_{r+1}} \wedge e_{j_1}{}^* \wedge \dots \wedge e_{j_s}{}^* .$$

Therefore:

$$dR(G,h) = S(R)(h_{r+1})+\textstyle\sum_{j\leqq r} Q_j'(h_j) .$$

Since $S(R) \in \mathscr{R}_{r+1,s+1}$, this completes the proof of lemma 3.5.

We will use the following lemma in the proof of theorem 3.1:

LEMMA 3.6. *Let* $R \in \mathscr{R}_s$ *then*:
(a) *If* $dR=0$, $\exists Q \in \mathscr{R}_{s-1}$ *such that* $R-dQ \in \mathscr{R}_{0,s}$.
(b) *If* $s<m$, $R \in \mathscr{R}_{0,s}$ *and if* $dR=0$, *then* $R=0$.
(c) *If* $R \in \mathscr{R}_{0,m}$ *and if* $R(G,h)(T_m)=0$ *for every* (G,h) *on* T_m, *then* $R=0$.

PROOF. We prove (a) as follows: suppose $dR=0$. Choose Q so that $\operatorname{ord}_h(R-dQ)=r$ is minimal. If $r\leqq 0$, the lemma is proved so we may assume the contrary. Let $R'=R-dQ=R_0'+\dots+R_r'$. By lemma 3.5, $\operatorname{ord}_h(dR'-S(R_r'))<r+1$. Since $dR'=0$ and $S(R_r') \in \mathscr{R}_{r+1,s+1}$, this implies that $S(R_r')=0$. Let $Q'=T(R_r')/(m+1-s) \in \mathscr{R}_{r-1,s-1}$. By lemma 3.4, $S(Q')=R_r'$. Therefore $\operatorname{ord}_h(R'-S(Q'))<r$. Since $\operatorname{ord}_h(dQ'-S(Q'))<\operatorname{ord}_h(dQ')=r$ by lemma 3.5, this implies that $\operatorname{ord}_h(R-d(Q+Q'))=\operatorname{ord}_h(R'-dQ')<r$. This contradicts the choice of Q so that $\operatorname{ord}_h(R-dQ)$ is minimal and proves (a).

We prove (b) as follows. Let $R \in \mathscr{R}_{0,s}$ with $s<m$ and $dR=0$. Then $S(R)=0$ by lemma 3.5. However, since $R \in \mathscr{R}_{0,s}$,

$$S(R) = R(j_1,\ldots,j_s)e_k \otimes e_k{}^* \wedge e_{j_1}{}^* \wedge \ldots \wedge e_{j_s}{}^* .$$

Since $s<m$, we can choose k so that $e_k \otimes e_k{}^* \wedge e_{j_1}{}^* \wedge \ldots \wedge e_{j_s}{}^* \neq 0$ and therefore $R(j_1,\ldots,j_s)=0$. This implies $R=0$ and proves (b). Finally, let $R \in \mathscr{R}_{0,m}$ and suppose that $R(G,h)(T_m)=0$ for every (G,h) on T_m. Since $r=0$, $R(G) \in \Gamma(\Lambda^m(T^*M))$ and $R(G,h)=hR(G)$. Since

$$0 = \int_{T_m} hR(G)(x)$$

for every function h on T_m, this implies $R(G)(x)=0$ for all metrics of type (p,q) on T_m. As noted in the first section, this implies that $R=0$ and completes the proof of lemma 3.6.

We deduce the following

Corollary 3.7. *Let $R \in \mathscr{R}_s$ then:*

(a) *If $s<m$ and $dR=0$, then $\exists\, Q \in \mathscr{R}_{s-1}$ such that $R=dQ$.*
(b) *If $s=m$ and if $R(G,h)(T_m)=0$ for every (G,h) on T_m, then $\exists\, Q \in \mathscr{R}_{s-1}$ such that $R=dQ$.*

Proof. If $s<m$ and $dR=0$, we apply lemma 3.6(a) to construct Q such that $R-dQ \in \mathscr{R}_{0,s}$. Since $d(R-dQ)=0$, by lemma 3.6(b), $R-dQ=0$. Similarly, if $s=m$ and if $R(G,h)(T_m)=0$ for every (G,h) on T_m, then $dR=0$ automatically. By lemma 3.6(a), we can construct Q such that $R-dQ \in \mathscr{R}_{0,m}$. Since $(R-dQ)(G,h)(T_m)=0$ as well, we apply lemma 3.6(c) to conclude that $R-dQ=0$.

In [4] we gave a proof of theorem 1.5 under the assumption that $q=0$. Since our argument involved integration over $O(m)$, that proof does not generalize directly to this case. We use corollary 3.7 of this paper and adapt the argument given in [4] to complete the proof of theorem 3.1 for general q. Let $h(x)$ be a real valued function. Let $k=1$ if $s>n$ and $k=-1$ if $s<n$. If t is large, $H=(h(x)+t)^{2k}G$ will be a metric of type (p,q) on M. We can express the ordinary derivatives of h in terms of the covariant derivatives of h in any coordinate system. Therefore, we may express:

$$\begin{aligned} P(G,h) &= \big(P(h(x)+t)^{2k}G\big) \\ &= (h+t)^j P_0(G) + (h+t)^{j(\alpha)} h_\alpha P_\alpha(G) + (h+t)^{j(\alpha,\beta)} h_\alpha h_\beta P_{\alpha,\beta}(G) + \ldots . \end{aligned}$$

If $h=0$ and $t=1$, then $H=G$ so $P(G)=P_0(G)$. By lemma 1.1, $P(c^{2k}G)=c^{k(s-n)}P(G)$. Therefore $j=k(s-n)=|s-n|>0$ by hypothesis. Similarly, $j(\alpha)=j-1$, $j(\alpha,\beta)=j-2,\ldots$. Therefore:

$$P(G,h) = t^jP(G)+t^{j-1}(jhP(G)+h_\alpha P_\alpha(G))+O(t^{j-2}) .$$

We restrict to $0<\operatorname{ord}(\alpha)\leqq n$ in these expressions. Let

$$R(G,h) = jhP(G)+h_\alpha P_\alpha(G) .$$

If $dP(G)=0$ for any G, then $dR(G,h)=0$ for any (G,h); if $P(G)(T_m)=0$ for any G, then $R(G,h)(T_m)=0$ for any (G,h) on T_m. Therefore R satisfies the same hypothesis as P does.

Let

$$\begin{aligned} R(G,h) &= R_0(G,h)+\ldots+R_r(G,h)\\ R_r(G,h) &= h_{:i_1\ldots i_r}R_r(i_1\ldots i_r;j_1\ldots j_s)e_{j_1}{}^*\wedge\ldots\wedge e_{j_s}{}^* . \end{aligned}$$

If $v\in S_r(T^*M)_{x_0}$, we can choose a function h so that $h_r(x_0)=v$. This implies that the $R_r(i_1,\ldots,i_r;j_1,\ldots,j_s)$ are uniquely determined by the equation above. Define

$$\begin{aligned} R_r &= R(i_1,\ldots,i_r;j_1,\ldots,j_s)(e_{i_1}\circ\ldots\circ e_{i_r})\otimes(e_{j_1}{}^*\wedge\ldots\wedge e_{j_s}{}^*)\\ &\qquad \in \Gamma(S_r(TM)\otimes\Lambda^s(T^*M)) . \end{aligned}$$

Since P is invariant under $\nu=O(p,q)$ or $SO(p,q)$, R_r is ν-invariant. Therefore $R_r\in\mathscr{R}_{r,s}$ and $R\in\mathscr{R}_s$. Since R satisfies the hypothesis of corollary 3.7, $R=dQ$ for some $Q\in\mathscr{R}_{s-1}$. Let $Q=Q_0+\ldots+Q_{r-1}$ and let $h=1$. Then $dR=Q$ implies that $dQ_0=R_0=jP$. Since $j\neq 0$ we can divide by j to express $P=d(Q_0/j)$ where $Q_0\in\oplus_k\mathscr{P}_{k,\nu,s-1}$. By decomposing Q_0 into homogeneous parts by lemma 1.1, we may assume that in fact $Q_0\in\mathscr{P}_{n-1,\nu,s-1}$. This completes the proof of theorem 3.1.

In [4] we considered the invariants of isometric imbeddings of a Riemannian manifold of dimension m into a manifold of dimension $m+r$. The results of that paper do *not* generalize to the case of indefinite metrics since the restriction of a non-degenerate bilinear form to a subspace need not stay non-degenerate.

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