MATH. SCAND. 36 (1975), 137–158

SEMIGROUPS OF CONVEX BIFUNCTIONS GENERATED BY LAGRANGE PROBLEMS IN THE CALCULUS OF VARIATIONS

R. TYRRELL ROCKAFELLAR*

To Werner Fenchel on his 70th birthday.

Abstract.

Duality theorems previously obtained for certain convex problems in the calculus of variations are applied to the study of the behavior of convex bifunctions under the operation of inf-multiplication. It is shown that each convex bifunction F from R^n to R^n generates a one-parameter semigroup of convex bifunctions from R^n to R^n having F as its infinitesimal generator. The polar semigroup is defined and its infinitesimal generator is also investigated. The results generalize the classical theory of one-parameter groups of linear transformations.

1. Introduction.

A *bifunction* from a space X to a space V is a mapping F which assigns to each $x \in X$ a function

$$Fx: V \to [-\infty, +\infty] .$$

The value of the function Fx at the point $v \in V$ is denoted by $(Fx)(v)$. There is thus a one-to-one correspondence between bifunctions from X to V and extended-real-valued functions on $X \times V$ (the "graph functions" of the bifunctions). The bifunction F is said to be *convex* if its graph function is convex, i.e. if X and V are real linear spaces and $(Fx)(v)$ depends convexly on (x,v).

Convex bifunctions were introduced in [2] in order to bring out a far-reaching analogy between many results on optimization, such as duality theorems and minimax theorems, and certain classical formulas of linear algebra. By means of Fenchel's theory of conjugate convex functions, a "convex algebra" parallel to linear algebra was developed.

* Research supported in part by grant AFOSR-72-2269.

Received October 31, 1974.

In this context, a linear transformation $A: \mathbb{R}^n \to \mathbb{R}^n$ is identified with its *indicator bifunction*, namely the convex bifunction ψ_A from $\mathbb{R}^n$ to $\mathbb{R}^n$ defined by

$$(1.1) \qquad (\psi_A x)(v) = \begin{cases} 0 & \text{if } v = Ax \\ +\infty & \text{if } v \neq Ax \,. \end{cases}$$

The purpose of this note is to pursue a further analogy. It is well-known that each linear transformation $A: \mathbb{R}^n \to \mathbb{R}^n$ generates a one-parameter group of linear transformations $B^{(\tau)}: \mathbb{R}^n \to \mathbb{R}^n$ having A as its infinitesimal generator, namely

$$(1.2) \qquad B^{(\tau)} = e^{\tau A} \quad \text{for } -\infty < \tau < +\infty \,.$$

We shall show that, similarly, each convex bifunction from $\mathbb{R}^n$ to $\mathbb{R}^n$ belonging to a certain "regular" class generates a one-parameter semigroup of convex bifunctions also belonging to this class. Moreover, such semigroups correspond to certain convex problems of Lagrange in the calculus of variations, and they reflect the many duality properties that have been demonstrated for such problems [3], [4], [5].

The semigroup operation in question is that of *inf-multiplication* of bifunctions, which is defined as follows. If F_1 and F_2 are bifunctions from $\mathbb{R}^n$ to $\mathbb{R}^n$, then F_1F_2 is the bifunction from $\mathbb{R}^n$ to $\mathbb{R}^n$ whose values are given by

$$(1.3) \qquad (F_1F_2x)(v) = \inf_{u \in \mathbb{R}^n}\{(F_2x)(u) + (F_1u)(v)\} \,,$$

where in the sum one uses the convention:

$$(1.4) \qquad -\infty + \infty = \infty + (-\infty) = +\infty \,.$$

This operation is associative and convexity-preserving [2, p. 406]. Moreover, it generalizes the operation of multiplication of linear transformations, in the sense that

$$(1.5) \qquad \psi_A\psi_B = \psi_{AB} \,.$$

The bifunction ψ_I (where I is the identity transformation) serves as the identity for inf-multiplication.

By a one-parameter semigroup of bifunctions from $\mathbb{R}^n$ to $\mathbb{R}^n$, we mean a parameterized family $E^{(\tau)}$, $0 < \tau < +\infty$, with the property that

$$(1.6) \qquad E^{(\tau)}E^{(\sigma)} = E^{(\tau+\sigma)} \quad \text{for all } \tau > 0, \sigma > 0 \,.$$

(The property can always be extended to $\tau = 0$ and $\sigma = 0$ by taking $E^{(0)} = \psi_I$.) Every one-parameter group of linear transformations as in (1.2) yields such a semigroup with

$$E^{(\tau)} = \psi_{B^{(\tau)}}; \tag{1.7}$$

in fact property (1.6) holds in this case for *all* real τ and σ by virtue of (1.5) and the identity

$$B^{(\tau)}B^{(\sigma)} = B^{(\tau+\sigma)} \quad \text{for all } \tau \in \mathsf{R},\ \sigma \in \mathsf{R}\,. \tag{1.8}$$

The close relationship between one-parameter semigroups of bifunctions and problems in the calculus of variations is easily perceived. Given any extended-real-valued function L on $\mathsf{R}^n \times \mathsf{R}^n$, or equivalently the bifunction F from R^n to R^n having

$$Fx = L(x, \cdot)\,, \tag{1.10}$$

let us define the bifunction $E_F^{(\tau)}$ for $0 < \tau < +\infty$ by

$$(E_F^{(\tau)}a)(b) = \inf \{\textstyle\int_0^\tau L(x(t), \dot{x}(t))\,dt \mid x(0) = a,\ x(\tau) = b\}\,, \tag{1.11}$$

where the infimum is over all absolutely continuous functions $x: [0, \tau] \to \mathsf{R}^n$ satisfying the given terminal conditions. Here the integral is (in the possible absence of measurability) to be interpreted as the *upper integral*, i.e. the infimum of $\int_0^\tau \alpha(t)\,dt$ over all summable functions $\alpha: [0, \tau] \to [-\infty, +\infty]$ satisfying $\alpha(t) \geqq L(x(t), \dot{x}(t))$ for almost every $t \in [0, \tau]$ at which the derivative $\dot{x}(t)$ exists. (If there are no such functions α, the integral is $+\infty$ by convention.)

Note that in the case of $F = \psi_A$, where A is a linear transformation, $E_F^{(\tau)}$ is the bifunction in (1.7). Indeed, one has

$$\int_0^\tau [\psi_A x(t)](\dot{x}(t))\,dt < +\infty$$

if and only if $\dot{x}(t) = Ax(t)$ almost everywhere, i.e.

$$x(t) = e^{tA}x(0), \quad 0 \leqq t \leqq \tau\,,$$

in which case the integral vanishes; thus in this case $(E_F^{(\tau)}a)(b)$ is 0 if $b = e^{\tau A}a$ and $+\infty$ if $b \neq e^{\tau A}a$.

In this sense, the general definition of the family $E_F^{(\tau)}$ extends the notion of the exponential of a linear transformation to that of the exponential of a bifunction. The following result confirms the analogy and sets the stage for our main efforts.

THEOREM 1. *The family $E_F^{(\tau)}$ defined by* (1.1) *is a one-parameter semigroup of bifunctions from R^n to R^n (called the semigroup generated by the bifunction F in* (1.10)*). If F is convex, then so is $E_F^{(\tau)}$ for all $\tau > 0$.*

PROOF. We must prove that

$$(1.12)\qquad (E_F^{(\tau+\sigma)}a)(b) = \inf_{c\in\mathbf{R}^n}\{(E_F^{(\sigma)}a)(c)+(E_F^{(\tau)}c)(b)\}\,.$$

First we fix any $c \in \mathbf{R}^n$ and demonstrate that

$$(1.13)\qquad (E_F^{(\tau+\sigma)}a)(b) \leqq (E_F^{(\sigma)}a)(c)+(E_F^{(\tau)}c)(b)\,.$$

If $(E_F^{(\sigma)}a)(c) = +\infty$ or $(E_F^{(\tau)}c)(b) = +\infty$, then (1.13) holds trivially (in view of convention (1.4)). Suppose therefore that there are numbers μ and ν with

$$(1.14)\qquad (E_F^{(\sigma)}a)(c) < \mu \quad\text{and}\quad (E_F^{(\tau)}c)(b) < \nu\,.$$

To establish (1.13), it will be enough to check that

$$(1.15)\qquad (E_F^{(\tau+\sigma)}a)(b) < \mu+\nu\,.$$

By virtue of (1.14), there exist functions $y:[0,\sigma] \to \mathbf{R}^n$ and $\beta:[0,\sigma] \to [-\infty, +\infty]$ such that y is absolutely continuous with $y(0)=a$ and $y(\sigma)=c$, while β is summable with $\int_0^\sigma \beta(t)\,dt < \mu$ and $L(y(t),\dot{y}(t)) \leqq \beta(t)$ almost everywhere that $\dot{y}(t)$ exists. At the same time, there exist functions $z:[0,\tau] \to \mathbf{R}^n$ and $\gamma:[0,\tau] \to [-\infty, +\infty]$, such that z is absolutely continuous with $z(0)=c$ and $z(\tau)=b$, while γ is summable with $\int_0^\tau \gamma(t)\,dt < \nu$ and $L(z(t),\dot{z}(t)) \leqq \gamma(t)$ almost everywhere that $\dot{z}(t)$ exists. Define

$$x(t) = \begin{cases} y(t) & \text{for } t \in [0,\sigma] \\ z(t-\sigma) & \text{for } t \in [\sigma,\tau+\sigma]\,, \end{cases}$$

$$\alpha(t) = \begin{cases} \beta(t) & \text{for } t \in [0,\sigma) \\ \gamma(t-\sigma) & \text{for } t \in (\sigma,\tau+\sigma]\,. \end{cases}$$

Then $x:[0,\tau+\sigma] \to \mathbf{R}^n$ is absolutely continuous with $x(0)=a$ and $x(\tau+\sigma) = b$, while $\alpha:[0,\tau+\sigma] \to [-\infty, +\infty]$ is summable with $L(x(t),\dot{x}(t)) \leqq \alpha(t)$ almost everywhere that $\dot{x}(t)$ exists. It follows that

$$\begin{aligned}(E_F^{(\tau+\sigma)}a)(b) &\leqq \int_0^{\tau+\sigma} L(x(t),\dot{x}(t))\,dt \leqq \int_0^{\tau+\sigma} \alpha(t)\,dt \\ &= \int_0^\sigma \beta(t)\,dt + \int_0^\tau \gamma(t)\,dt < \mu+\nu\,,\end{aligned}$$

which yields the desired inequality (1.15). Thus (1.13) is valid for all a, b, c, and in consequence the inequality $\leqq$ holds in (1.12).

We now argue towards the inequality $\geqq$ in (1.12). This is trivial if $(E_F^{(\tau+\sigma)}a)(b) = +\infty$, so it may be supposed that there is a number μ with

$$(1.16)\qquad (E_F^{(\tau+\sigma)}a)(b) < \mu\,.$$

The task is to demonstrate the existence of $c \in \mathbf{R}^n$ with

$$(1.17)\qquad (E_F^{(\sigma)}a)(c)+(E_F^{(\tau)}c)(b) < \mu\,.$$

The definitions imply from (1.16) that there exist functions $x:[0,\tau+\sigma]\to \mathsf{R}^n$ and $\alpha:[0,\tau+\sigma]\to[-\infty,+\infty]$ such that x is absolutely continuous with $x(0)=a$ and $x(\tau+\sigma)=b$, while α is summable with $\int_0^{\tau+\sigma}\alpha(t)\,dt<\mu$ and $L(x(t),\dot{x}(t))<\alpha(t)$ almost everywhere that $\dot{x}(t)$ exists. Let $c=x(\sigma)$. Then in particular we have

$$(E_F^{(\sigma)}a)(c) \leqq \int_0^\sigma L(x(t),\dot{x}(t))\,dt \leqq \int_0^\sigma \alpha(t)\,dt\,.$$

On the other hand, setting $y(t)=x(t-\sigma)$ for $t\in[0,\tau]$ we have y absolutely continuous with $y(0)=c$ and $y(\tau)=b$, so that

$$(E_F^{(\tau)}c)(b) \leqq \int_0^\tau L(y(t),\dot{y}(t))\,dt \leqq \int_0^\tau \alpha(t-\sigma)\,dt\,.$$

It follows that

$$(E_F^{(\sigma)}a)(c)+(E_F^{(\tau)}c)(b) \leqq \int_0^\sigma \alpha(t)\,dt+\int_\sigma^{\tau+\sigma}\alpha(t)\,dt = \int_0^{\tau+\sigma}\alpha(t)\,dt < \mu\,.$$

Thus (1.17) holds and the identity (1.12) has been established.

It remains only to prove the convexity assertion in Theorem 1. This amounts to showing that if

$$(E_F^{(\tau)}a_i)(b_i) < \mu_i \quad \text{for } i=1,2\,, \tag{1.18}$$

and

$$(a,b) = (1-\lambda)(a_1,b_1)+\lambda(a_2,b_2), \quad 0<\lambda<1\,, \tag{1.19}$$

then

$$(E_F^{(\tau)}a)(b) \leqq (1-\lambda)\mu_1+\lambda\mu_2\,. \tag{1.20}$$

By (1.18), there exist for $i=1,2$, certain functions $x_i:[0,\tau]\to\mathsf{R}^n$ and $\alpha_i:[0,\tau]\to[-\infty,+\infty]$ such that x_i is absolutely continuous with $x_i(0)=a_i$ and $x_i(\tau)=b_i$, while α_i is summable with $\int_0^\tau\alpha_i(t)\,dt<\mu_i$ and

$$L(x_i(t),\dot{x}_i(t))\leqq\alpha_i(t) \quad \text{almost everywhere that } \dot{x}_i(t) \text{ exists}\,. \tag{1.21}$$

Let

$$x(t) = (1-\lambda)x_1(t)+\lambda x_2(t)\,,$$

$$\alpha(t) = (1-\lambda)\alpha_1(t)+\lambda\alpha_2(t)\,,$$

where the convention (1.4) is used in the second formula. Then x is absolutely continuous with $x(0)=a$ and $x(\tau)=b$. Also, α is summable (the convention in the sum cannot affect this, since α_1 and α_2 can be infinite only on a set of measure zero). We have

$$\dot{x}(t) = (1-\lambda)\dot{x}_1(t)+\lambda\dot{x}_2(t)$$

for almost every $t \in [0, \tau]$ at which $\dot{x}(t)$ exists, and hence the assumed convexity of L implies by (1.21) (still using (1.4)) that

$$L(x(t), \dot{x}(t)) \leqq (1-\lambda)L(x_1(t), \dot{x}_1(t)) + \lambda L(x_2(t), \dot{x}_2(t)) \leqq \alpha(t)$$

for almost every t at which $\dot{x}(t)$ exists. Therefore

$$\begin{aligned}(E_F^{(\tau)}a)(b) &\leqq \int_0^\tau L(x(t), \dot{x}(t))\,dt \leqq \int_0^\tau \alpha(t)\,dt \\ &= (1-\lambda)\int_0^\tau \alpha_1(t)\,dt + \lambda \int_0^\tau \alpha_2(t)\,dt < (1-\lambda)\mu_1 + \lambda\mu_2 .\end{aligned}$$

This gives the relation (1.20), and the proof of Theorem 1 is finished.

To what extent is the "infinitesimal generator" F of the semigroup $E_F^{(\tau)}$, $0 < \tau < +\infty$, uniquely determined by the semigroup? If it is uniquely determined, can it be recovered from the semigroup by some kind of differentiation? These are intriguing questions, and except for the special classical cases, they are completely open.

We shall not tackle such questions here, but concentrate rather on duality properties in the convex case related to the theory of conjugate convex functions. The main goal is to describe a "regular" class of convex bifunctions F which is preserved not only by inf-multiplication but by passage to the semigroup $E_F^{(\tau)}$, and which is also closed under certain duality operations defined below. The chief results are given in section 3.

Some indications of the relationship between one-parameter semigroups of bifunctions and the dynamic evolution of certain economic models and their duals is furnished in [6].

2. Duality.

The *polar* of a bifunction F from R^n to R^n is the bifunction $F^\#$ from R^n to R^n defined by

$$(2.1) \qquad (F^\# c)(d) = \sup_{a,b}\{b \cdot d - a \cdot c - (Fa)(b)\} .$$

The graph function of $F^\#$ is thus the conjugate of the graph function of F, except for a change of sign in the first argument. It follows from Fenchel's theorem on conjugate functions (see [2, Theorem 12.2]) that $F^\#$ is always a *closed convex* bifunction (i.e. its graph function is either identically $-\infty$ or it is convex, lower semicontinuous and nowhere $-\infty$), while $F^{\#\#}$ is the greatest closed convex bifunction $\leqq F$. If $F = \psi_A$ for a linear transformation A, then $F^\# = \psi_{A^\#}$, where $A^\# = (A^*)^{-1}$.

The remarkable property of the polarity operation is that it is "essentially" an automorphism on the semigroup of *convex* bifunctions. To state the precise result, we introduce the following notation. If f is any extended-real-valued function on R^m, we set

$$\mathrm{dom} f = \{x \mid f(x) < +\infty\} . \tag{2.2}$$

If F is a bifunction from R^n to R^n, we set

$$\begin{aligned} \mathrm{dom} F &= \{a \in \mathsf{R}^n \mid \mathrm{dom} Fa \neq \emptyset\} \\ &= \{a \in \mathsf{R}^n \mid \exists b \in \mathsf{R}^n \text{ with } (Fa)(b) < +\infty\} , \end{aligned} \tag{2.3}$$

$$\begin{aligned} \mathrm{rge} F &= \textstyle\bigcup_{a \in \mathsf{R}^n} \mathrm{dom} Fa \\ &= \{b \in \mathsf{R}^n \mid \exists a \in \mathsf{R}^n \text{ with } (Fa)(b) < +\infty\} . \end{aligned} \tag{2.4}$$

These sets are convex if f and F are convex. We denote by $\mathrm{ri}\, C$ the *relative interior* of a convex set [2, § 6].

THEOREM 2 [2]. *If F_1 and F_2 are convex bifunctions from R^n to R^n such that*

$$\mathrm{ri}\,\mathrm{dom} F_1 \cap \mathrm{ri}\,\mathrm{rge} F_2 \neq \emptyset , \tag{2.5}$$

then

$$(F_1 F_2)^\# = F_1^\# F_2^\# . \tag{2.6}$$

PROOF. This is shown on p. 406 of [2] in the case where the graph functions of F_1 and F_2 nowhere have the value $-\infty$. (In the notation of [2], the polar $F^\#$ is $F_*{}^*$.) The general case is only a slight extension. Suppose, for instance, that the function $f_1(a,b) = (F_1 a)(b)$ has the value $-\infty$ somewhere. Then by definition (2.1) we have

$$(F_1^\# c)(d) = +\infty \quad \text{for all } c,d , \tag{2.7}$$

so that

$$(F_1^\# F_2^\# c)(d) = \inf_{w \in \mathsf{R}_n} \{(F_2^\# c)(w) + (F_1^\# w)(d)\} \equiv +\infty . \tag{2.8}$$

On the other hand, the fact that f_1 is convex and takes on $-\infty$ implies

$$f_1(a,b) = -\infty \quad \text{for all } (a,b) \in \mathrm{ri}\,\mathrm{dom} f_1 \tag{2.9}$$

[2, Theorem 7.2]. But since $\mathrm{dom} F_1$ is the projection of $\mathrm{dom} f_1$ in the first argument, we have

$$\mathrm{ri}\,\mathrm{dom} f_1 = \{(a,b) \mid a \in \mathrm{ri}\,\mathrm{dom} F_1,\ b \in \mathrm{ri}\,\mathrm{dom}(F_1 a)\} \tag{2.10}$$

[2, Theorem 6.8]. Thus, taking w to be any element of the non-empty intersection in (2.5), there exists $\bar{b}$ with $(F_1 w)(\bar{b}) = -\infty$, while at the same time there exists $\bar{a}$ with $(F_2 \bar{a})(w) < +\infty$. Then

$$(F_1 F_2 \bar{a})(b) \leqq (F_2 \bar{a})(w) + (F_1 w)(\bar{b}) = -\infty ,$$

implying from the formula for the polar $(F_1F_2)^\#$ that

$$((F_1F_2)^\# c)(d) = +\infty \quad \text{for all } c,d\,.$$

Comparing this with (2.7), we see that, again in this degenerate case, (2.6) is valid. The argument is similar if it is the graph function of F_2, rather than that of F_1, which takes on the value $-\infty$.

The interesting consequence of Theorem 2 for the study of one-parameter semigroups is the following.

COROLLARY. *If the family* $E^{(\tau)}$, $0 < \tau < +\infty$, *is a one-parameter semigroup of convex bifunctions from* R^n *to* R^n *such that* $\operatorname{dom} E^{(\tau)}$ *is nonempty and open for all* τ, *then the polar family* $E^{(\tau)\#}$ *is also a one-parameter semigroup.*

PROOF. Let $\tau > 0$ and $\sigma > 0$. Since $E^{(\tau+\sigma)} = E^{(\tau)}E^{(\sigma)}$, we have

$$\text{(2.11)} \qquad \begin{aligned} \operatorname{dom} E^{(\tau+\sigma)} \neq \emptyset &\Leftrightarrow \operatorname{dom} E^{(\tau)} \cap \operatorname{rge} E^{(\sigma)} \neq \emptyset \\ &\Leftrightarrow \operatorname{ri}\operatorname{dom} E^{(\tau)} \cap \operatorname{ri}\operatorname{rge} E^{(\sigma)} \neq \emptyset\,, \end{aligned}$$

the second equivalence holding by virtue of the openness of $\operatorname{dom} E^{(\tau)}$. Thus the hypothesis of the theorem is satisfied for all $\tau > 0$ and $\sigma > 0$, yielding the identity

$$\text{(2.12)} \qquad E^{(\tau+\sigma)\#} = E^{(\tau)\#}E^{(\sigma)\#}\,,$$

which says that the polar family forms a semigroup.

This corollary raises a question in the case where it is applicable to a semigroup of the form $E_F{}^{(\tau)}$. Is the polar semigroup $E_F{}^{(\tau)\#}$ of the form $E_G{}^{(\tau)}$ for some other convex bifunction G? This is true if F is the indicator ψ_A of a linear transformation A, since then $E_F{}^{(\tau)}$ is the indicator of $B^{(\tau)} = e^{\tau A}$, and hence $E_F{}^{(\tau)\#}$ is the indicator of

$$((e^{\tau A})^*)^{-1} = e^{-\tau A^*}\,.$$

The polar semigroup is thus generated by the indicator ψ_{-A^*}, where A^* is the adjoint of A.

We now prove a generalization in terms of the bifunction

$$\text{(2.13)} \qquad (Gp)(w) = \sup_{x,v}\{x\cdot w + p\cdot v - (Fx)(v)\}\,.$$

(If $F = \psi_A$, then $G = \psi_{-A^*}$.) Observe that in the notation of (1.10) we have

$$\text{(2.14)} \qquad Gp = M(p,\cdot)\,,$$

where

$$\text{(2.15)} \qquad M(p,w) = L^*(w,p)$$

(L^* = conjugate of L). The function M was termed in [3], [4], the *Lagrangian dual to L.*

THEOREM 3. *Let F be a convex bifunction from* R^n *to* R^n *such that* $\operatorname{dom} F = \mathsf{R}^n$. *Then*

$$\operatorname{dom} E_F^{(\tau)} \uparrow \mathsf{R}^n \quad \text{as } \tau \downarrow 0 , \tag{2.16}$$

and there exists $T \in (0, +\infty]$ *such that* $\operatorname{dom} E_F^{(\tau)}$ *is nonempty and open if* $0 < \tau < T$ *but empty if* $T \leqq \tau < +\infty$. *Moreover, one has*

$$E_F^{(\tau)\#} = E_G^{(\tau)} \quad \text{for } 0 < \tau < T , \tag{2.17}$$

where G is the bifunction defined in (2.13).

The proof of Theorem 3 will be based on duality theorems in [4] and the following result.

LEMMA. *Let F be a convex bifunction from* R^n *to* R^n *with* $\operatorname{dom} F = \mathsf{R}^n$, *and let L be the graph function of F, as in* (1.10). *Let* $x_0 : [\tau_0, \tau_1] \to \mathsf{R}^n$ *be an absolutely continuous function such that*

$$\int_{\tau_0}^{\tau_1} L(x_0(t), \dot{x}_0(t))\, dt < +\infty . \tag{2.18}$$

Let $r \in (0, +\infty)$ *be such that* $r > |x_0(t)|$ *for all* $t \in [\tau_0, \tau_1]$ ($|\cdot|$ = *Euclidean norm*), *let I be any bounded open real interval containing* $[\tau_0, \tau_1]$, *and let*

$$D = \{(t,x) \in I \times \mathsf{R}^n \mid |x| < r\} . \tag{2.19}$$

Then there is a function $\varphi : D \to \mathsf{R}^n$ *with the following properties.*

(a) $L(x, \varphi(t,x)) \leqq \alpha(t)$ *for all* $(t,x) \in D$, *where* $\alpha : I \to [-\infty, +\infty]$ *is a certain summable function.*

(b) $\dot{x}_0(t) = \varphi(t, x_0(t))$ *for almost every* $t \in [\tau_0, \tau_1]$.

(c) $\varphi(t,x)$ *is summable as a function of* $t \in I$ *for fixed* $x \in \mathsf{R}^n$ *and Lipschitz continuous as a function of* $x \in \mathsf{R}^n$ *for fixed* $t \in I$. *In fact, there is a summable function* $k : I \to [0, +\infty)$ *such that*

$$|\varphi(t,x') - \varphi(t,x)| \leqq k(t)|x' - x| . \tag{2.20}$$

(d) *If* $x \neq x_0(t)$, *then*

$$(x, \varphi(t,x)) \in \operatorname{ri} \operatorname{dom} L . \tag{2.21}$$

PROOF. Let S be an n-dimensional simplex containing every x with $|x| < 3r$. Let $a_0, a_1, \ldots, a_n$ be the vertices of S. Each a_i belongs to $\operatorname{dom} F$

by hypothesis, and hence the convex set $\operatorname{dom} Fa_i$ is nonempty; choose an element v_i of $\operatorname{ri}\operatorname{dom} Fa_i$ for $i=0,1,\ldots,n$. Then

$$(a_i,v_i) \in \operatorname{ri}\operatorname{dom} L \quad \text{for } i=0,1,\ldots,n\,, \tag{2.22}$$

because in general the convexity of F implies

$$\operatorname{ri}\operatorname{dom} L = \{(x,v) \mid x \in \operatorname{ri}\operatorname{dom} F,\ v \in \operatorname{ri}\operatorname{dom} Fx\} \tag{2.23}$$

[2, Theorem 6.8]. Let Q be the unique linear transformation from $\mathbf{R}^n$ to $\mathbf{R}^n$ such that

$$Q(a_i-a_0) = v_i - v_0 \quad \text{for } i=1,\ldots,n\,,$$

and let $q=v_0-Qa_0$. Then

$$v_i = Qa_i+q \quad \text{for } i=0,1,\ldots,n\,. \tag{2.24}$$

If $u \in S$, then u can be expressed as a convex combination of the vertices of S:

$$u = \textstyle\sum_{i=0}^n \lambda_i a_i \quad \text{where } \lambda_i \geqq 0,\ \sum_{i=0}^n \lambda_i = 1\,.$$

We then have

$$(u,Qu+q) = \textstyle\sum_{i=0}^n \lambda_i(a_i,v_i)$$

by (2.24), and hence via (2.22) and the choice of S:

$$(u,Qu+q) \in \operatorname{ri}\operatorname{dom} L \quad \text{whenever } |u|<3r\,, \tag{2.25}$$

$$L(u,Qu+q) \leqq \max_{i=0,1,\ldots,n} L(a_i,v_i) \quad \text{whenever } |u|<3r\,. \tag{2.26}$$

Now let I_0 be the set of $t \in [\tau_0,\tau_1]$ such that $\dot{x}_0(t)$ exists and $(x_0(t),\dot{x}_0(t)) \in \operatorname{dom} L$; the complement of I_0 in $[\tau_0,\tau_1]$ is of measure zero. For $t \in I_0$ and $|x|<r$, define

$$\varphi(t,x) = (1-\lambda)\dot{x}_0(t)+\lambda(Qu+q)\,, \tag{2.27}$$

where the elements $\lambda \in [0,1)$ and $u \in \mathbf{R}^n$ are determined by the relations

$$x = (1-\lambda)x_0(t)+\lambda u, \quad |u-x_0(t)| = 2r\,, \tag{2.28}$$

or in other words

$$\lambda = |x-x_0(t)|/2r, \quad \lambda u = x-[1-(|x-x_0(t)|/2r)]x_0(t)\,. \tag{2.29}$$

For $t \in I_0$, we then have

$$\varphi(t,x) = \dot{x}_0(t)+Q(x-x_0(t))+|x-x_0(t)|c(t)\,, \tag{2.30}$$

where

$$c(t) = (|x-x_0(t)|/2r)[q-\dot{x}_0(t)+Qx_0(t)]\,. \tag{2.31}$$

Note that $c(t)$ is summable over $t \in I_0$. We complete the definition of φ by setting

(2.32) $$\varphi(t,x) = Qx+q \quad \text{if } t \notin I_0 .$$

Properties (b) and (c) of the lemma are then evident. Expressions (2.27) and (2.28), with $0 \leqq \lambda < 1$, yield via (2.25) and the relation $(x_0(t), \dot{x}_0(t)) \in \operatorname{dom} L$ for $t \in I_0$ the fact that property (d) holds. The same expressions also give us from (2.26) that

$$\begin{aligned} L(x,\varphi(t,x)) &\leqq (1-\lambda)L(x_0(t),\dot{x}_0(t)) + \lambda L(u, Qu+a) \\ &\leqq \max\{L(x_0(t),\dot{x}_0(t)), L(a_0, v_0), \ldots, L(a_n, v_n)\} \end{aligned}$$

if $t \in I_0$ and $|x| < r$, while

$$\begin{aligned} L(x,\varphi(t,x)) &= L(x, Qx+q) \\ &\leqq \max\{L(a_0,v_0),\ldots,L(a_n,v_n)\} \end{aligned}$$

if $t \in I \setminus I_0$ and $|x| < r$. Since (2.17) is assumed, it is clear from these inequalities that property (a) is fulfilled.

Proof of Theorem 3. First consider any $\tau > 0$ and $a_0 \in \operatorname{dom} E_F^{(\tau)}$. There exists by definition an absolutely continuous function $x_0 : [0,\tau] \to \mathsf{R}^n$ such that

$$\int_0^\tau L(x_0(t)\dot{x}_0(t))\,dt < +\infty \quad \text{and} \quad x_0(0) = a_0 .$$

Corresponding to x_0, we can construct a function $\varphi : D \to \mathsf{R}^n$ with the properties described in the lemma above. Then, according to the theory of differential equations (cf. [1, p. 59]), there exists $\varepsilon > 0$ such that the equation

(2.33) $$\dot{x}(t) = \varphi(t,x(t)) \quad \text{a.e.,} \quad x(0) = a ,$$

has a solution x (an absolutely continuous function) over the interval $[0,\tau+\varepsilon]$ whenever $|a-a_0| < \varepsilon$. Property (a) of the lemma tells us that

$$\int_0^\sigma L(x(t),\dot{x}(t))\,dt < +\infty \quad \text{for } 0 < \sigma \leqq \tau+\varepsilon ,$$

and therefore $a \in \operatorname{dom} E_F^{(\sigma)}$ if $|a-a_0| < \varepsilon$ and $\sigma \in (0,\tau+\varepsilon]$. This verifies that the set $\operatorname{dom} E_F^{(\tau)}$ is open, nonincreasing in τ, and nonempty for an open interval $(0,T)$ of τ values.

To demonstrate the limit assertion (2.16), we borrow a fact established in the first part of the proof of the lemma: given any $r > 0$, one may construct a linear transformation $Q : \mathsf{R}^n \to \mathsf{R}^n$ and a vector $q \in \mathsf{R}^n$ such that (2.26) holds. One may then find $\tau > 0$ such that, for $|a| \leqq r$, the solution to the equation

$$\dot{x} = Qx+q, \quad x(0) = a ,$$

satisfies $|x(t)| < 3r$ for $t \in [0,\tau]$. Then by (2.16) the expression $L(x(t),\dot{x}(t))$ is bounded above as a function of $t \in [0,\tau]$, so that

$$\int_0^\tau L(x(t),\dot{x}(t))\,dt \;<\; +\infty\,,$$

and consequently $a \in \operatorname{dom} E_F^{(\tau)}$. Thus, for any $r > 0$ we have

$$\operatorname{dom} E_F^{(\tau)} \supset \{a \in \mathsf{R}^n \mid |a| \leqq r\}$$

for all τ sufficiently small, and (2.16) is valid.

We turn now to the proof of (2.17). If L is lower semicontinuous and nowhere has the value $-\infty$, the relation amounts to an earlier result [4, Corollary 2 on p. 8]; only a change of notation and terminology is involved.

Suppose next that L is not lower semicontinuous, although L still does not take on $-\infty$. Let $\bar{L}$ denote the greatest lower semicontinuous function on $\mathsf{R}^n \times \mathsf{R}^n$ majorized by L. Then $\bar{L}$ is convex, does not take on $-\infty$, and

$$\operatorname{ri}\operatorname{dom}\bar{L} = \operatorname{ri}\operatorname{dom}L\,, \tag{2.34}$$

with

$$\bar{L}(x,v) = L(x,v) \quad \text{if } (x,v) \in \operatorname{ri}\operatorname{dom}\bar{L} \tag{2.35}$$

[2, Theorem 7.4]. Moreover, $\bar{L}$ has the same conjugate as L, i.e. yields the same dual Lagrangian M [2, Theorem 12.2]. Let $\bar{F}$ be the convex bifunction from R^n to R^n whose graph function is $\bar{L}$. Then $\operatorname{dom}\bar{F} = \mathsf{R}^n$, because $\bar{L} \leqq L$. Since L is lower semicontinuous and does not take on $-\infty$, the case of formula (2.17) already established is applicable to $\bar{F}$. Inasmuch as $\bar{L}$ and L yield the same dual Lagrangian M, the bifunction G involved is the same for $\bar{F}$ as for F. Thus we have

$$E_{\bar{F}}^{(\tau)\#} = E_G^{(\tau)} \quad \text{for } 0 < \tau < \bar{T}\,, \tag{2.36}$$

where $(0,\bar{T})$ is the interval of τ values for which $\operatorname{dom} E_{\bar{F}}^{(\tau)}$ is nonempty. We demonstrate next that for $0 < \tau < \bar{T}$ one has $\operatorname{dom} E_F^{(\tau)} \neq \emptyset$ and

$$E_F^{(\tau)\#} = E_{\bar{F}}^{(\tau)\#}\,, \tag{2.37}$$

so that, in view of (2.36), (2.17) is indeed true under the present assumptions on L. Clearly

$$(E_{\bar{F}}^{(\tau)}a)(b) \leqq (E_F^{(\tau)}a)(b) \quad \text{for all } a,b\,, \tag{2.38}$$

because $\bar{L} \leqq L$. It suffices therefore to demonstrate that if

$$(E_{\bar{F}}^{(\tau)}a_0)(b_0) < \mu < +\infty \tag{2.39}$$

and $\delta > 0$, there exist a and b with

(2.40) $$(E_F^{(\tau)}a)(b) < \mu \quad \text{and} \quad |(a,b)-(a_0,b_0)| < \delta .$$

According to (2.39), there is an absolutely continuous function $x_0:[0,\tau] \to \mathbb{R}^n$ such that

(2.41) $$x_0(0) = a_0, \quad x_0(\tau) = b_0, \quad \int_0^\tau \bar{L}(x_0(t),\dot{x}_0(t))dt < \mu .$$

We construct a corresponding function of the form $\varphi:D \to \mathbb{R}^n$ having the properties described in the preceding lemma with respect to $\bar{L}$. Then for all points a in some neighborhood of a_0, the equation (2.33) has a solution over $[0,\tau]$. Fix a particular such point $a_1 \neq a_0$ and corresponding solution x_1. The Lipschitz property (2.20) in the lemma guarantees that no other solution to equation (2.33) coincides at any point with x_1, [1, p. 51], and therefore, since x_0 is a particular solution by (b) of the lemma, we have

$$x_1(t) \neq x_0(t) \quad \text{for all } t \in [0,\tau] .$$

Hence

(2.42) $$(x_1(t),\dot{x}_1(t)) \in \operatorname{ri}\operatorname{dom}\bar{L} \text{ for almost every } t \in [0,\tau]$$

by (d) of the lemma. Also,

(2.43) $$\int_0^\tau \bar{L}(x_1(t),\dot{x}_1(t))dt < +\infty$$

by (a) of the lemma. Consider a function of the form

(2.44) $$x(t) = (1-\lambda)x_0(t)+\lambda x_1(t), \quad \text{where } 0<\lambda<1 .$$

We have

(2.45) $$(x(t),\dot{x}(t)) = (1-\lambda)(x_0(t),\dot{x}_0(t))+\lambda(x_1(t),\dot{x}_1(t)) \in \operatorname{ri}\operatorname{dom}\bar{L}$$

by (2.42), and consequently

(2.46) $$\bar{L}(x(t),\dot{x}(t)) = L(x(t),\dot{x}(t)) \quad \text{for almost every } t \in [0,\tau]$$

by (2.35). The convexity of $\bar{L}$ implies

(2.47) $$\bar{L}(x(t),\dot{x}(t)) \leqq (1-\lambda)\bar{L}(x_0(t),\dot{x}_0(t))+\lambda\bar{L}(x_1(t),\dot{x}_1(t)) .$$

Let

$$a = x(0) = (1-\lambda)a_0+\lambda a_1 \quad \text{and} \quad b = x(\tau) = (1-\lambda)b_0+\lambda b_1$$

(where $a_1 = x_1(0)$ and $b_1 = x_1(\tau)$). Then

$$(E_F^{(\tau)}a)(b) \leqq \int_0^\tau L(x(t),\dot{x}(t))dt ,$$

so that by (2.46) and (2.47) we have

(2.48) $$(E_F^{(\tau)}a)(b) \leqq (1-\lambda)\int_0^\tau \bar{L}(x_0(t),\dot{x}_0(t))dt+\lambda\int_0^\tau \bar{L}(x_1(t),\dot{x}_1(t))dt .$$

Furthermore,

$$|(a,b)-(a_0,b_0)| = \lambda|(a_1,b_1)-(a_0,b_0)| . \tag{2.49}$$

It is clear from (2.39), (2.43), (2.48) and (2.49) that the desired inequalities (2.40) will be satisfied if λ is chosen sufficiently small in (2.44).

We are left with the case of (2.17) where L takes on the value $-\infty$ somewhere, and hence [2, Theorem 7.2]:

$$L(x,v) = -\infty \quad \text{for all } (x,v) \in \operatorname{ri}\operatorname{dom} L . \tag{2.50}$$

Trivially from the definition (2.13), we have

$$(Gp)(w) = +\infty \quad \text{for all } p,w ,$$

so that

$$(E_G^{(\tau)}c)(d) = +\infty \quad \text{for all } c,d .$$

To establish (2.17) in this case, therefore, it will be enough to show that for every $\tau \in (0,T)$, there exist a and b with $(E_F^{(\tau)}a)(b) = -\infty$. Given any $\tau \in (0,T)$, there does exist an absolutely continuous function $x_0:[0,\tau] \to \mathsf{R}^n$ with

$$\int_0^\tau L(x_0(t),\dot{x}_0(t))\,dt < +\infty .$$

Once more we construct a corresponding function $\varphi: D \to \mathsf{R}^n$ with the properties in the lemma. For all points a sufficiently near to a_0, the equation (2.33) has a solution x over $[0,\tau]$. Since x_0 solves the equation for the initial point a_0 (by (b) of the lemma), the Lipschitz property in (c) of the lemma ensures that if $a \neq a_0$ we have

$$x(t) \neq x_0(t) \quad \text{for all } t \in [0,\tau] .$$

But then

$$(x(t),\dot{x}(t)) \in \operatorname{ri}\operatorname{dom} L \quad \text{for almost every } t \in [0,\tau]$$

by (d) of the lemma, implying by way of (2.50) that

$$L(x(t),\dot{x}(t)) = -\infty \quad \text{for almost every } t \in [0,\tau] .$$

Therefore, the points $a = x(0)$, $b = x(\tau)$, satisfy

$$(E_F^{(\tau)}a)(b) \leqq \int_0^\tau L(x(t),\dot{x}(t))\,dt = -\infty .$$

This completes the proof of Theorem 3.

3. The case of regular convex bifunctions.

In order to crystalize a more complete duality in the context of Theorem 3, we need conditions ensuring that the "escape time" T associated with the bifunction F is $+\infty$, and that the relationship between $E_F^{(\tau)}$ and $E_G^{(\tau)}$ is reciprocal.

Let $L: \mathsf{R}^n \times \mathsf{R}^n \to (-\infty, +\infty]$ be a lower semicontinuous convex function with $\operatorname{dom} L \neq \emptyset$, and let $\hat{L}$ be the *recession function* of L, i.e.

$$\hat{L}(y,z) = \lim_{\lambda \to +\infty} [L(x_0+\lambda y, v_0+\lambda z) - L(x_0,v_0)]/\lambda , \tag{3.1}$$

where $(x_0,v_0) \in \operatorname{dom} L$ (the formula gives the same values independent of which (x_0,v_0) in $\operatorname{dom} L$ is selected [2, p. 66]). We shall associate with L the following sets: first the nonempty closed convex cone

$$K_1(L) = \operatorname{cl} \operatorname{dom} \hat{L} = \operatorname{cl}\{(y,z) \mid \hat{L}(y,z) < +\infty\} , \tag{3.2}$$

and second the recession cone of $\operatorname{cl} \operatorname{dom} L$, i.e.

$$K_2(L) = \{(y,z) \mid (x,v)+\lambda(y,z) \in \operatorname{cl} \operatorname{dom} L \text{ for all } (x,v) \in \operatorname{dom} L \text{ and } \lambda \geqq 0\} . \tag{3.3}$$

Some background facts about $K_2(L)$ [2, p. 63] are that it is a nonempty closed convex cone, and

(3.4) $(y,z) \in K_2(L)$ if there exists $(x,v) \in \operatorname{cl} \operatorname{dom} L$ such that $(x,v)+\lambda(y,z) \in \operatorname{cl} \operatorname{dom} L$ for all $\lambda \geqq 0$.

Furthermore,

$$(x,v)+\lambda(y,z) \in \operatorname{ri} \operatorname{dom} L \text{ for all } \lambda \geqq 0 \text{ if } (x,v) \in \operatorname{ri} \operatorname{dom} L \text{ and } (y,z) \in K_2(L) . \tag{3.5}$$

We shall say that F is a *regular convex bifunction* if its graph function L is, as above, a lower semicontinuous convex function on $\mathsf{R}^n \times \mathsf{R}^n$ which nowhere has the value $-\infty$ yet is not identically $+\infty$ (implying $F^{\#\#} = F$), and if in addition the following two conditions are satisfied:

$$(0,z) \in K_1(L) \quad \textit{implies} \quad z = 0 , \tag{3.6}$$

and

(3.7) *for every* $y \in \mathsf{R}^n$, *there exists* $z \in \mathsf{R}^n$ *with* $(y,z) \in K_2(L)$.

For example, if $F = \psi_A$, where A is a linear transformation from R^n to R^n, then F is regular convex, because $K_1 = K_2 =$ graph of A. For another example, suppose F is of the form

$$(Fx)(v) = f(x) + g(v-Ax) , \tag{3.8}$$

where f and g are convex functions on R^n and A is a linear transformation. (This case corresponds to problems of optimal control as studied in [3] and [5].) If f is finite, while g is cofinite (i.e. the conjugate of a finite convex function), then F is regular convex. (Property (3.7) holds because $(x, Ax) \in K_2$ for all x, while property (3.6) follows from the formula

$$\hat{L}(y,z) = \hat{f}(y) + \hat{g}(z - Ay),$$

where $\hat{f}$ and $\hat{g}$ are the recession functions of f and g; since g is cofinite, we have $\hat{g}(u) = +\infty$ for $u \neq 0$ [2, p. 116].)

Some initial results about regular convex bifunctions are collected in the next theorem.

THEOREM 4. *If F is a regular convex bifunction from* R^n *to* R^n, *then* $\operatorname{dom} F = \mathsf{R}^n$. *Furthermore, the associated bifunction G in* (2.13) *is likewise regular convex and satisfies the reciprocal formula*

$$(3.9) \qquad (Fx)(v) = \sup_{p,w}\{x \cdot w + p \cdot v - (Gp)(w)\}.$$

The class of regular convex bifunctions is closed under inf-*multiplication.*

PROOF. The fact that $\operatorname{dom} F = \mathsf{R}^n$ is clear from property (3.7) and the definition of $K_2(L)$. The validity of (3.9) is a consequence merely of the graph function L of F being a proper convex function which is lower semicontinuous; then by Fenchel's theorem (cf. [2, § 12]) the conjugate L^* (or equivalently the dual Lagrangian M in (2.15), the graph function of G) is likewise a proper convex function which is lower semicontinuous, and $L^{**} = L$. We invoke next the fact that the polar of the cone $K_1(L)$ is, according to [2, Theorem 13.3 and Corollary 14.2.1], the recession cone of the closure of $\operatorname{dom} L^*$; thus

$$(3.10) \qquad K_1(L)^\circ = \{(r,q) \mid (q,r) \in K_2(M)\},$$

$$(3.11) \qquad K_2(M)^\circ = \{(z,y) \mid (y,z) \in K_1(L)\}.$$

It follows that property (3.6) is equivalent to:

(3.12) for every $q \in \mathsf{R}^n$ there exists $r \in \mathsf{R}^n$ with $(q,r) \in K_2(M)$.

By symmetry, property (3.7) is likewise equivalent to:

$$(3.13) \qquad (0,r) \in K_2(M) \quad \text{only for } r = 0.$$

The regularity of F thus implies that of G.

Consider now two regular convex bifunctions F_1 and F_2 from R^n to R^n, and let $F_0 = F_1 F_2$. We shall demonstrate that F_0 is regular convex. For $i = 0, 1, 2$ let

$$(3.14) \qquad (H_i v)(x) = (F_i x)(v),$$

so that $H_0 = H_2 H_1$. The graph functions of F_1 and F_2 (or equivalently, of H_1 and H_2) are known to be closed convex functions which are not identically infinite, and we want to verify, among other things, that the same is true of the graph function of F_0 (or equivalently, of H_0). Since $\operatorname{dom} F_1 = \mathsf{R}^n = \operatorname{dom} F_2$ by what has already been established, it is immediate from the formula for inf-multiplication that $\operatorname{dom} F_0 = \mathsf{R}^n$. Thus the graph functions of F_0 (and H_0) are not identically $+\infty$, and it will suffice to show that H_0 is the polar of a convex bifunction whose graph function is not identically $+\infty$. In fact, we shall show that

$$(3.15) \qquad H_0 = (H_2^{\#} H_1^{\#})^{\#} ,$$

making use in this of Theorem 2 as well as the relations

$$(3.16) \qquad H_1^{\#\#} = H_1 \quad \text{and} \quad H_2^{\#\#} = H_2 ,$$

which are true by the cited properties of the graph functions of H_1 and H_2. Let G_i be the bifunction corresponding to F_i by formula (2.13). Then

$$(3.17) \qquad (H_i^{\#} w)(p) = \sup_{x,v} \{v \cdot p - x \cdot w - (F_i x)(v)\} = (G_i(-w))(p) .$$

We know from the preceding arguments that G_1 and G_2 are regular convex, since F_1 and F_2 are. Therefore $H_1^{\#}$ and $H_2^{\#}$ are regular convex by (3.17) and in particular $\operatorname{dom} H_1^{\#} = \mathsf{R}^n = \operatorname{dom} H_2^{\#}$. The latter implies that $\operatorname{dom} H_2^{\#} H_1^{\#} = \mathsf{R}^n$, and the graph function of $H_2^{\#} H_1^{\#}$ is thus not identically $+\infty$. Furthermore, Theorem 2 and (2.13) yield

$$(3.18) \qquad (H_2^{\#} H_1^{\#})^{\#} = H_2^{\#\#} H_1^{\#\#} = H_2 H_1 = H_0 .$$

The mentioned properties of the graph function of H_0 (and of F_0) are thereby shown to be correct.

Next we prove that F_0 again possesses property (3.7). Let L_i be the graph function of F_i, $i = 0, 1, 2$. Given any $y \in \mathsf{R}^n$, there exists by the regularity of F_2 some $s \in \mathsf{R}^n$ with $(y, s) \in K_2(L_2)$. For this s, there also exists some $z \in \mathsf{R}^n$ with $(s, z) \in K_2(L_1)$. We claim that then $(y, z) \in K_2(L_0)$. To see this, let (x_0, v_0) be an arbitrary element of $\operatorname{ri} \operatorname{dom} L_0$. We have by definition

$$L_0(x, v) = \inf_r \{L_2(x, r) + L_1(r, v)\} ,$$

and hence

$$(3.19) \qquad \operatorname{dom} L_0 = \{(x, v) \mid \exists r \text{ with } (x, r) \in \operatorname{dom} L_2,\ (r, v) \in \operatorname{dom} L_1\} .$$

Moreover, as will be proved in a moment,

$$(3.20) \qquad \operatorname{ri} \operatorname{dom} L_0 = \{(x, v) \mid \exists r \text{ with } (x, r) \in \operatorname{ri} \operatorname{dom} L_2,\ (r, v) \in \operatorname{ri} \operatorname{dom} L_1\} .$$

Skipping temporarily the verification of (3.20), we observe that this formula gives the existence of r_0 with $(x_0, r_0) \in \operatorname{ri}\operatorname{dom} L_2$ and $(r_0, v_0) \in \operatorname{ri}\operatorname{dom} L_1$. Since $(y, s) \in K_2(L_2)$, we then have by (3.5) that

$$(3.21) \qquad (x_0 + \lambda y, r_0 + \lambda s) \in \operatorname{dom} L_2 \quad \text{for all } \lambda \geqq 0 ,$$

while, since $(s, z) \in K_2(L_1)$,

$$(3.22) \qquad (r_0 + \lambda s, v_0 + \lambda z) \in \operatorname{dom} L_1 \quad \text{for all } \lambda \geqq 0 .$$

But (3.21) and (3.22) imply by way of (3.19) that

$$(x_0 + \lambda y, v_0 + \lambda z) \in \operatorname{dom} L_0 \quad \text{for all } \lambda \geqq 0 ,$$

or in other words by (3.5), $(y, z) \in K_2(L_0)$; thus F_0 satisfies (3.7).

Returning now to the omitted proof of (3.20), we represent $\operatorname{dom} L_0$ through (3.19) as the set $A(C \cap D)$, where

$$C = \{(x, r, r', v) \mid (x, r) \in \operatorname{dom} L_2,\ (r', v) \in \operatorname{dom} L_1\} ,$$

$$D = \{(x, r, r', v) \mid r' = r\} ,$$

$$A : (x, r, r', v) \to (x, v) .$$

Here C is convex, D is affine, and A is a linear transformation. The desired relation (3.20) is equivalent to the formula

$$\operatorname{ri}(A(C \cap D)) = A((\operatorname{ri} C) \cap D) ,$$

which is valid by [2, Corollary 6.5.1 and Theorem 6.6] if $(\operatorname{ri} C) \cap D \neq \emptyset$, i.e. if the right side of (3.20) is nonempty. But

$$\operatorname{dom} L_2 = \{(x, r) \mid x \in \operatorname{dom} F_2 \text{ and } r \in \operatorname{dom} F_2 x\} ,$$

and hence [2, Theorem 6.8]:

$$\operatorname{ri}\operatorname{dom} L_2 = \{(x, r) \mid x \in \operatorname{ri}\operatorname{dom} F_2 \text{ and } r \in \operatorname{ri}\operatorname{dom} F_2 x\} .$$

Similarly,

$$\operatorname{ri}\operatorname{dom} L_1 = \{(r, v) \mid r \in \operatorname{ri}\operatorname{dom} F_1 \text{ and } v \in \operatorname{ri}\operatorname{dom} F_1 r\} .$$

Since $\operatorname{dom} F_1 = \mathrm{R}^n = \operatorname{dom} F_2$, these formulas indicate that the right side of (3.20) is nonempty, as needed.

The final goal is to establish that F_0 again possesses property (3.6). By the symmetry displayed in the first part of the proof, this amounts to showing that G_0 (or equivalently $H_0^\#$, because of (3.17)) possess (3.7). We know that $H_1^\#$ and $H_2^\#$ are regular convex bifunctions and thus possess (3.7). Moreover, (3.7) has been shown to be preserved under inf-multiplication. Therefore $H_2^{\#}H_1^{\#}$ possesses (3.7). We also know from the argument for F_1F_2 that the product of two regular convex bifunctions is its own bipolar; in particular

$$(H_2^{\#}H_1^{\#})^{\#\#} = H_2^{\#}H_1^{\#} .$$

It follows from (3.15) that

$$H_0^{\#} = H_2^{\#}H_1^{\#} ,$$

and hence $H_0^{\#}$ does have property (3.7). The proof of Theorem 4 is now finished.

Our main theorem for one-parameter semigroups may now be presented.

THEOREM 5. *Let F be a regular convex bifunction from* R^n *to* R^n. *Then* $E_F^{(\tau)}$ *is a regular convex bifunction for all* $\tau > 0$, *and one has*

$$E_F^{(\tau)\#} = E_G^{(\tau)} \quad \text{for all } \tau > 0 , \tag{3.23}$$

where G is given by (2.13).

PROOF. First we show that $\operatorname{dom} E_F^{(\tau)} = \mathsf{R}^n$ for all $\tau > 0$. Let S be a simplex in R^n such that

$$|y| \leqq 1 \quad \text{implies} \quad y \in S , \tag{3.24}$$

and let $a_0, a_1, \ldots, a_n$ be the vertices of S. Since F is regular convex, property (3.7) holds, and hence there exist vectors v_i such that

$$(a_i, v_i) \in K_2(L) \quad \text{for } i = 0, 1, \ldots, n . \tag{3.25}$$

The vectors a_i are affinely independent, so we have

$$v_i = Qa_i + q \quad \text{for } i = 0, 1, \ldots, n \tag{3.26}$$

for a uniquely determined linear transformation $Q\colon \mathsf{R}^n \to \mathsf{R}^n$ and vector $q \in \mathsf{R}^n$. If y satisfies $|y| = 1$, there is a barycentric representation

$$y = \textstyle\sum_{i=0}^n \lambda_i a_i \quad \text{where } \lambda_i \geqq 0,\ \sum_{i=0}^n \lambda_i = 1 ,$$

and correspondingly one has

$$(y, Qy + q) = \textstyle\sum_{i=0}^n \lambda_i (a_i, v_i)$$

by (3.26). The convexity of $K_2(L)$ in (3.25) then yields

$$(y, Qy + q) \in K_2(L) .$$

The latter relation therefore holds for all y with $|y|=1$. But $K_2(L)$ is actually a cone containing the origin. Hence in fact $(\lambda y, \lambda(Qy+q)) \in K_2(L)$ whenever $|y|=1$ and $\lambda \geqq 0$, or to say the same thing another way,

$$(3.27) \qquad (y, Qy+|y|q) \in K_2(L) \quad \text{for all } y \in \mathsf{R}^n .$$

We now select any v_0 such that $v_0 \in \operatorname{ri}\operatorname{dom} F0$. This is possible because $\operatorname{dom} F = \mathsf{R}^n$ by Theorem 4, and hence $\operatorname{dom} Fx \neq \emptyset$ for all $x \in \mathsf{R}^n$. We have by [2, Theorem 6.8] that

$$\operatorname{ri}\operatorname{dom} L = \{(x,v) \mid x \in \operatorname{ri}\operatorname{dom} F,\ v \in \operatorname{ri}\operatorname{dom} Fx\},$$

so our choice of v_0 ensures

$$(3.28) \qquad (0, v_0) \in \operatorname{ri}\operatorname{dom} L .$$

Recalling (3.5), we see from (3.27) and (3.28) that

$$(3.29) \qquad (x, Qx+|x|q+v_0) \in \operatorname{ri}\operatorname{dom} L \quad \text{for all } x \in \mathsf{R}^n .$$

For every $a \in \mathsf{R}^n$, the equation

$$(3.30) \qquad \dot{x}(t) = Qx(t)+|x(t)|q+v_0, \quad x(0) = a ,$$

has a unique solution x over $[0, +\infty)$ such that $\dot{x}(t)$ is actually continuous. Then

$$(3.31) \qquad (x(t), \dot{x}(t)) \in \operatorname{ri}\operatorname{dom} L \quad \text{for all } t$$

by (3.29), and since L is finite and continuous relative to $\operatorname{ri}\operatorname{dom} L$ [2, Theorem 10.1], it follows that $L(x(t), \dot{x}(t))$ is finite and continuous as a function of t. Hence

$$(3.32) \qquad (E_F^{(\tau)}a)(x(\tau)) \leqq \int_0^\tau L(x(t), \dot{x}(t))dt < +\infty \quad \text{for all } \tau > 0 ,$$

implying $a \in \operatorname{dom} E_F^{(\tau)}$ for all $\tau > 0$. Thus $\operatorname{dom} E_F^{(\tau)} = \mathsf{R}^n$ for all $\tau > 0$.

Theorem 3 now gives us (3.23). But the same argument can be applied to G in place of F, since, by Theorem 4, G is regular convex and (3.9) holds. Therefore $\operatorname{dom} E_G^{(\tau)} = \mathsf{R}^n$ for all $\tau > 0$, and

$$(3.33) \qquad E_G^{(\tau)\#} = E_F^{(\tau)} \quad \text{for all } \tau > 0 .$$

The relations (3.23) and (3.33), along with $\operatorname{dom} E_F^{(\tau)} = \mathsf{R}^n = \operatorname{dom} E_G^{(\tau)}$, imply of course that the graph functions of $E_F^{(\tau)}$ and $E_G^{(\tau)}$ are lower semicontinuous convex functions which nowhere take on the value $-\infty$, and which are not identically $+\infty$.

Our next step is to demonstrate that property (3.7) holds for the graph functions

$$(3.34) \qquad L_\tau(a,b) = (E_F^{(\tau)}a)(b) .$$

Fix any $\tau > 0$ and $a \in \mathsf{R}^n$. Let $b = x(\tau)$, where, as above, x is the unique solution to (3.30). Then

$$(3.35) \qquad (a,b) \in \operatorname{dom} L_\tau$$

by (3.32), and (3.31) holds. Let $a' \in \mathsf{R}^n$; we must demonstrate the existence of $b' \in \mathsf{R}^n$ such that $(a',b') \in K_2(L_\tau)$, and for this it suffices by characterization (3.4) to display an element b' such that

$$(3.36) \qquad (a,b)+\lambda(a',b') \in \operatorname{dom} L_\tau \quad \text{for all } \tau > 0 \,.$$

Let $y(t)$, $0 \leqq t < +\infty$, be the unique solution to the differential equation

$$(3.37) \qquad \dot{y}(t) = Qy(t) + |y(t)|q, \quad y(0) = a' \,,$$

so that by (3.27) we have

$$(3.38) \qquad (y(t),\dot{y}(t)) \in K_2(L) \quad \text{for all } t > 0 \,.$$

Note that $\dot{y}(t)$ is continuous in t. From (3.38) and (3.31) we obtain (in view of property (3.4) of $K_2(L)$) that

$$(3.39) \qquad (x(t)+\lambda y(t), \dot{x}(t)+\lambda\dot{y}(t)) \in \operatorname{ri}\operatorname{dom} L \quad \text{for all } t > 0,\ \lambda \geqq 0 \,.$$

Using again the fact that L is finite and continuous on $\operatorname{ri}\operatorname{dom} L$, we see that (3.39) implies

$$\int_0^\tau L(x(t)+\lambda y(t), \dot{x}(t)+\lambda\dot{y}(t))\,dt < +\infty \quad \text{for all } \lambda \geqq 0 \,,$$

and consequently

$$(3.40) \qquad L_\tau(x(0)+\lambda y(0), x(\tau)+\lambda y(t)) < +\infty \quad \text{for all } \lambda \geqq 0 \,.$$

But this says that (3.36) is fulfilled for $b' = y(\tau)$.

The last step is to prove that L_τ possesses property (3.5). As shown at the beginning of the proof of Theorem 4, this is equivalent to showing that the function

$$(3.41) \qquad M_\tau(d,c) = \sup_{a,b}\{a\cdot c + b\cdot d - L_\tau(a,b)\} \,,$$

which is the graph function of the bifunction that corresponds to $E_F^{(\tau)}$ in the same manner that G corresponds to F in (2.13), has property (3.7). However, since (3.23) holds, it is also true that

$$(3.42) \qquad (E_G^{(\tau)}c)(d) = \sup_{(a,b)}\{b\cdot d - a\cdot c - L_\tau(a,b)\} = M_\tau(d,-c) \,.$$

Define the bifunction H by

$$(3.43) \qquad (Hp)(w) = (Gp)(-w) \,,$$

so that

$$(3.44) \qquad (E_H^{(\tau)}d)(c) \equiv (E_G^{(\tau)}c)(d) \,.$$

Since G is regular convex by Theorem 4, so is H, and hence by our preceding argument the graph function of $E_H^{(\tau)}$ satisfies property (3.7). In view of (3.42) and (3.44), we may conclude as desired that M_τ has property (3.7).

REFERENCES

1. E. A. Coddington and N. Levinson, *Theory of Ordinary Differential Equations*, McGraw–Hill, New York, London, 1955.
2. R. T. Rockafellar, *Convex Analysis*, (Princeton Mathematical Series 28), Princeton University Press, Princeton, 1970.
3. R. T. Rockafellar, *Conjugate convex functions in optimal control and the calculus of variations*, J. Math. Anal. Appl. 32 (1970), 174–222.
4. R. T. Rockafellar, *Existence and duality theorems for convex problems of Bolza*, Trans. Amer. Math. Soc. 159 (1971), 1–40.
5. R. T. Rockafellar, *State constraints in convex problems of Bolza*, SIAM J. Control 10 (1972), 691–715.
6. R. T. Rockafellar, *Convex algebra and duality in dynamic models of production*, in *Mathematical Models of Economics* (J. Łos and M. Łos, eds.), 351–378, North-Holland Publ. Co., Amsterdam, New York, 1974.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF WASHINGTON
SEATTLE, WASHINGTON 98195, USA