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INTERSECTION OF MOVING CONVEX SETS IN A NORMED SPACE

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To Werner Fenchel on his 70th birthday

1. Introduction.

Let I be an interval of R; let $t \mapsto A(t)$ or $t \mapsto A_t$ denote a *multifunction* (i.e. a set-valued mapping) from I into a metric space (E,d). When t is interpreted as the time the language of kinematics may be used and A is called a *moving set* in E.

Several recent papers concerning evolution processes or selection properties (cf. [3], [4], [8], [9]) have drawn the attention to the concept of the *variation* of such a multifunction in the sense of the *Hausdorff distance* between subsets of (E,d).

A classical preliminary in the study of Hausdorff distance consists in defining, for every two subsets C and D of (E,d), the non symmetric *écart*

$$(1) \qquad e(C,D) = \sup\nolimits_{x\in C}\inf\nolimits_{y\in D} d(x,y) = \sup\{d(x,D) : x \in C\}$$

which we propose to call *the metric excess of C over D*. The considered sets may be empty; let us agree that "sup" and "inf" above are understood in the sense of the ordered set $\overline{\mathsf{R}}_+ = [0,+\infty]$. The supremum of an empty collection of elements of this ordered set is 0 and the infimum is $+\infty$. Thus $e(\emptyset,D)=0$ for any D, and $e(C,\emptyset)=+\infty$ for any $C \neq \emptyset$. One easily proves that the écart e satisfies the triangle inequality; clearly $e(C,D)=0$ if and only if C is contained in the closure $\operatorname{cl} D$ of D.

The Hausdorff distance between C and D is then

$$h(C,D) = \max\{e(C,D), e(D,C)\},$$

possibly infinite; it is zero if and only if C and D have the same closure.

Let $[s,t]$ be a compact subinterval of I; for every finite subdivision of $[s,t]$, namely

$$S: \quad s = \tau_0 \leqq \tau_1 \leqq \ldots \leqq \tau_n = t,$$

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put

$$V(S) = \sum_{i=1}^{n} h(A(\tau_{i-1}), A(\tau_i)) \in \overline{\mathsf{R}}_+ .$$

In accordance with classical terminology, the supremum of $V(S)$ for S ranging over all the finite subdivisions of $[s,t]$ is called *the variation of the multifunction A over the interval* $[s,t]$; it is denoted $\operatorname{var}(A; s,t)$.

If, for every $[s,t] \subset I$, this variation is finite, let us say that A *is of finite variation in* I; in this case there exists a non decreasing function $v: I \to \mathsf{R}$ (unique up to an additive constant), called *the indefinite variation of A* in I, such that for every $[s,t] \subset I$

$$\operatorname{var}(A; s,t) = v(t) - v(s) .$$

Let us say that the multifunction A *is of continuous finite variation* in I if v is continuous in I.

The *absolute continuity* of A over a compact subinterval K of I is defined, by means of Hausdorff distance, in the conventional way, i.e. for every $\varepsilon > 0$, there exists $\eta > 0$, such that the implication

$$\sum_i |\tau_i - \sigma_i| < \eta \Rightarrow \sum_i h(A(\sigma_i), A(\tau_i)) < \varepsilon$$

holds for any finite family $]\sigma_i, \tau_i[$ of non overlapping subintervals of K. This is equivalent to A being of finite variation in K, with the numerical function v absolutely continuous. In this case v is almost everywhere differentiable in K; its derivative, denoted by $\dot{v}$, is a nonnegative element of $L^1(K)$ which, kinematically speaking, may be called *the speed function of the moving set A.*

The multifunction A is said to be *Lipschitzian* over I if there exists $\lambda \geqq 0$ such that, for every σ and τ in I,

$$h(A(\sigma), A(\tau)) \leqq \lambda |\sigma - \tau| .$$

This holds if and only if A is absolutely continuous on every compact subinterval of I, with $\dot{v} \leqq \lambda$ almost everywhere.

Having recalled this we now turn to the main purpose of the present paper: Suppose that the metric space E is actually a *normed real linear space*, that K is a compact interval of R and that $t \mapsto A(t)$ and $t \mapsto B(t)$ are two multifunctions from K into E with *convex* values. If these two multifunctions are of continuous finite variation, respectively are absolutely continuous, respectively are Lipschitzian, does the same hold for the multifunction $t \mapsto A(t) \cap B(t)$? It will be shown that the answer is yes under the addtional assumptions that $A(t)$ has, for every t, a non empty intersection with the interior of $B(t)$ and that $\operatorname{diam} A(t) \cap B(t) < +\infty$. In this connection, for x fixed in E, the numerical function $t \mapsto d(x, A(t) \cap B(t))$ is also studied.

The proofs rest on various inequalities which may be of more general interest. For instance they could be used in studying the *retraction* of the multifunction $t \mapsto A(t) \cap B(t)$, a concept similar to that of variation, in the definition of which h is replaced by e (cf. [13]).

All this was motivated by the author's theory of the evolution of *elastoplastic* mechanical systems [7], [9], [12].

2. Summary.

Throughout the paper, the distance function d on E is supposed finite-valued; but, as E is not necessarily bounded, the diameters of subsets or the expressions e and h associated with two subsets may take the value $+\infty$.

The suprema or infima of expressions involving d, e or h, will always be understood in the sense of the ordering of $\overline{\mathsf{R}}_+ = [0, +\infty]$: In this ordered set $\sup \varnothing = 0$ and $\inf \varnothing = +\infty$.

Sections 3 to 5 establish some technical inequalities, mainly concerning the case where E is a real normed linear space. In particular, $e(A, E \setminus B)$ is studied; clearly A meets the interior of B if and only if this is > 0.

The elementary facts about convex functions and their conjugates, a theory initiated by W. Fenchel [2], used in these sections, may today be considered as classical; the reader could refer to [5], [6] or [14]. (The latter treatise, though restricted to finite dimensional spaces, supplies much of the fundamental information.)

Sections 6 and 7 introduce what can be called the two *metric semi-continuities* of multifunctions from an arbitrary topological space T into a metric space E (here actually a normed space), i.e. continuity-like properties defined by means of the non-symmetric écart e. Other situations involving these concepts can be found in [1], [13], [15]. Considering two multifunctions $t \mapsto A_t$ and $t \mapsto B_t$ from T into the normed space E, with *convex* values and possessing such a semi-continuity at the point $s \in T$, it is proved that the assumption $A_s \cap \operatorname{int} B_s \neq \varnothing$, with $\operatorname{diam} A_s \cap B_s < \infty$, ensures the same semi-continuity for $t \mapsto A_t \cap B_t$. This yields also semi-continuity properties, in the classical sense, for the numerical functions $t \mapsto \operatorname{diam} A_t \cap B_t$ and $t \mapsto e(A_t, E \setminus B_t)$.

Section 8 makes use of all what precedes to answer the question formulated in the introduction. The final section 9 shows that, in what concerns the study of the numerical function $t \mapsto d(x, A_t \cap B_t)$, with x fixed in E, the boundedness assumption on $A_t \cap B_t$ may be dropped.

All this reproduces, with some improvements, most of the material formerly presented in two multigraph seminar reports [10]; short summaries appeared in [11] and [12] (to which a correction must be brought: In the formulation of what corresponds to paragraph 4 below, the assumption $\operatorname{int} B_2 \neq \emptyset$ had been forgotten).

3. Diameters.

Let A be a subset of a metric space (E,d); the *diameter* of A, i.e.

$$\operatorname{diam} A = \sup\{d(x,y) : x \in A, y \in A\}$$

takes its value in $\overline{\mathsf{R}}_+ = [0, +\infty]$. As already stipulated such a supremum is understood in the sense of the ordering of $\overline{\mathsf{R}}_+$, so that $\operatorname{diam} \emptyset = 0$.

If A' denotes another subset of E, one has, with the definition (1) of the *excess* e,

$$\operatorname{diam} A' \leqq \operatorname{diam} A + 2e(A',A) . \tag{2}$$

In fact, this inequality is trivial when A or A' is empty. Otherwise let x' and y' be elements of A' and let $\varepsilon > 0$; there exist x and y in A such that

$$d(x',x) \leqq d(x',A) + \varepsilon, \; d(y',y) \leqq d(y',A) + \varepsilon .$$

Using the distance inequality for the chain of points x', x, y, y', then taking suprema for x' and y' ranging over A' one finally obtains

$$\operatorname{diam} A' \leqq \operatorname{diam} A + 2e(A',A) + 2\varepsilon .$$

Since ε is arbitrary this proves (2).

In a similar way the following inequality can be established:

For every $a \in A$ and $b \in E$

$$d(a,b) \leqq \operatorname{diam} A + d(b,A) . \tag{3}$$

4. Balls contained in convex sets.

If (E,d) denotes an arbitrary metric space and B a subset of E, the open ball with center a and radius ϱ is contained in B if and only if $\varrho \leqq d(a, E \setminus B)$. One concludes that, if A denotes another subset of E, the following equivalence holds

$$A \cap \operatorname{int} B \neq \emptyset \Leftrightarrow e(A, E \setminus B) > 0 . \tag{4}$$

From now on let us suppose that E is a real normed linear space. Let F be its dual; in both spaces the norm will be denoted by $\|\cdot\|$.

If B_1 and B_2 are two nonempty *convex*, not necessarily closed, subsets of E, their respective *support functions* γ_1 and γ_2 are sublinear functions from F into $]-\infty, +\infty]$, taking the value zero at the origin. An expression of $e(B_1,B_2)$ in terms of γ_1 and γ_2 is easily found (cf. [13]) and, in particular, denoting by σ the *unit sphere* of F, one obtains

$$\forall y \in \sigma:\ \gamma_1(y) \leqq \gamma_2(y) + e(B_1,B_2)\ . \tag{5}$$

LEMMA. *Let B_1 and B_2 denote two convex subsets of the normed space E, with* $\operatorname{int} B_2 \neq \emptyset$ *and let $a \in E$; then*

$$d(a, E \setminus B_1) \leqq d(a, E \setminus B_2) + e(B_1,B_2)\ . \tag{6}$$

PROOF. In order to simplify the calculations, assume that $a=0$. Suppose B_1 nonempty, since otherwise the inequality is trivial.

Let ϱ be a real number such that

$$0 < \varrho \leqq d(0, E \setminus B_1)\,; \tag{7}$$

this means that the open ball with center 0 and radius ϱ is contained in B_1. As the support function of the ball is $y \mapsto \varrho\|y\|$, this inclusion implies

$$\forall y \in F:\ \varrho\|y\| \leqq \gamma_1(y)$$

or equivalently, as γ_1 is positively homogeneous,

$$\forall y \in \sigma:\ \varrho \leqq \gamma_1(y)\ . \tag{8}$$

Let us make use now of inequality (5) and suppose $e(B_1,B_2) < +\infty$ (otherwise the lemma is trivial); then (8) implies

$$\forall y \in \sigma:\ \varrho - e(B_1,B_2) \leqq \gamma_2(y)\ .$$

Again, let us interpret this as an inequality between two support functions, therefore equivalent to the relation of inclusion between the corresponding *closed* convex sets. Suppose first $\varrho - e(B_1,B_2) > 0$; the preceding inequality means that the closed ball centered at the origin with this radius is contained in $\operatorname{cl} B_2$. By an elementary property of convex sets in topological linear spaces, the hypothesis $\operatorname{int} B_2 \neq \emptyset$ ensures $\operatorname{cl} B_2 = \operatorname{cl}\operatorname{int} B_2$; thus the corresponding open ball is contained in B_2, i.e.

$$\varrho - e(B_1,B_2) \leqq d(0, E \setminus B_2)\ . \tag{9}$$

This, on the other hand, is trivial in the case $\varrho - e(B_1,B_2) \leqq 0$. The fact that (7) implies (9) proves the lemma.

PROPOSITION. *Let B_1 and B_2 denote two convex subsets of the normed space E, with* $\operatorname{int} B_2 \neq \emptyset$, *and let A_1 and A_2 denote arbitrary subsets of E. Then*

$$e(A_1, E \setminus B_1) \leqq e(A_2, E \setminus B_2) + e(A_1,A_2) + e(B_1,B_2)\ . \tag{10}$$

PROOF. Taking the suprema of both members of (6) for a ranging over A_1 yields

$$e(A_1, E \setminus B_1) \leqq e(A_1, E \setminus B_2) + e(B_1, B_2) .$$

Now, as e satisfies the triangle inequality, one has

$$e(A_1, E \setminus B_2) \leqq e(A_1, A_2) + e(A_2, E \setminus B_2) .$$

5. Main inequality.

LEMMA. *Let X be a real topological linear space and Y its topological dual. Let f and g denote two convex functions defined on X, with values in $]-\infty, +\infty]$; suppose there exists a point $a \in X$ at which both functions take finite values and that one of them is continuous at this point. Then, denoting by $*$ their conjugate functions (i.e. the polar functions in the terminology of [6]) defined on Y, the function $(f+g)^*$ equals the infimal convolute of f^* and g^*.*

If A and B are two convex subsets of X such that $A \cap \operatorname{int} B \neq \emptyset$, the support function of $A \cap B$ equals the infimal convolute of the support functions of A and B.

PROOF. By the definition of polar functions, the continuous affine function $x \mapsto \langle x,y \rangle - r$, with $y \in Y$ and $r \in \mathsf{R}$ is a minorant, for instance of f if and only if $r \geqq f^*(y)$. By the definition of the infimal convolute $f^* \triangledown g^*$

$$(f^* \triangledown g^*)(y) = \inf \{ f^*(u) + g^*(v) : u+v=y \}$$

the assumption

$$r > (f^* \triangledown g^*)(y)$$

implies the existence of s and t in R, and of u and v in Y, such that

$$r = s+t, \ y = u+v, \ s > f^*(u), \ t > g^*(v) .$$

Then the affine functions $\langle \cdot, u \rangle - s$ and $\langle \cdot, v \rangle - t$ are minorants of f and g, respectively; consequently $\langle \cdot, y \rangle - r$ is a minorant of $f+g$, thus

$$r \geqq (f+g)^*(y) . \tag{11}$$

This proves $f^* \triangledown g^* \geqq (f+g)^*$.

To establish the reverse inequality, we are going to prove that (11) implies

$$r \geqq (f^* \triangledown g^*)(y) .$$

This will be done by showing that every continuous affine minorant m

of $f+g$ equals the sum of continuous affine minorants of f and g. In fact, for such an m one has

$$\forall x \in X:\ f(x) \geqq m(x)-g(x)\ .$$

Therefore, in the topological linear space $X \times \mathsf{R}$ the two convex sets

$$\{(x,r) \in X \times \mathsf{R} :\ f(x) < r\}$$

and

$$\{(x,r) \in X \times \mathsf{R} :\ r < m(x)-g(x)\}$$

have an empty intersection. As f possesses a point of continuity the first of these sets possesses a nonempty interior; by a standard separation theorem it follows that there exists a closed hyperplane separating these two subsets of $X \times \mathsf{R}$. The fact that both sets meet the line $\{(x,r): x=a\}$, and again the continuity of f at the point a, ensures that this hyperplane is "nonvertical", i.e. it is the graph of an affine continuous functions $n: X \to \mathsf{R}$ such that

$$m-g \leqq n \leqq f\ .$$

Thus n and $m-n$ are continuous affine minorants of f and g, respectively; their sum equals m.

The last part of the lemma is a special case of what precedes: Take as f and g the respective indicator functions of A and B (i.e. the functions taking the value zero on the sets and $+\infty$ outside).

PROPOSITION. *Let A and B denote two convex subsets of the normed space E. If there exists an open ball with radius $\varrho > 0$ contained in B, whose center a belongs to A, one has*

(12) $$\forall x \in E:\ d(x, A \cap B) \leqq (1+\varrho^{-1}\|x-a\|)\big(d(x,A)+d(x,B)\big)\ .$$

PROOF. Let φ and γ denote the support functions of A and B, defined on the dual F of E. Let $x \in E$; standard arguments from the theory of conjugate convex functions yield

$$d(x,A) = \sup\{\langle x,u\rangle - \varphi(u) :\ u \in F,\ \|u\| \leqq 1\}$$

and a similar expression for $d(x,B)$, hence

$$d(x,A)+d(x,B) = \sup\{\langle x,u+v\rangle - \varphi(u) - \gamma(v) :\ \|u\| \leqq 1,\ \|v\| \leqq 1\}\ .$$

As the function θ defined on $F \times F$ by

$$\theta(u,v) = \langle x,u+v\rangle - \varphi(u) - \gamma(v)$$

is positively homogeneous, one has equivalently, for every $k > 0$,

$$k(d(x,A)+d(x,B)) = \sup\{\theta(u,v) : \|u\| \leqq k,\ \|v\| \leqq k\}. \tag{13}$$

Let us make calculation easier by supposing a translation performed in E such that $a=0$; in that case the hypothesis $a \in A$ implies

$$\forall u \in F:\ \varphi(u) \geqq 0 \tag{14}$$

and the hypothesis that the open ball with center $a=0$ and radius ϱ is contained in B implies

$$\forall v \in F:\ \gamma(v) \geqq \varrho\|v\|. \tag{15}$$

By the lemma, the support function of $A \cap B$ is the inf-convolute $\varphi \triangledown \gamma$, thus

$$\begin{aligned} d(x,A\cap B) &= \sup\{\langle x,w\rangle - (\varphi \triangledown \gamma)(w) : \|w\| \leqq 1\} \\ &= \sup\{\sup\{\langle x,w\rangle - \varphi(u) - \gamma(v) : u+v=w\} : \|w\| \leqq 1\} \\ &= \sup\{\theta(u,v) : \|u+v\| \leqq 1\}. \end{aligned}$$

Now, in view of (14) and (15) one has the implication

$$\|u+v\| \leqq 1 \Rightarrow \theta(u,v) \leqq \|x\| - \varrho\|v\|.$$

Therefore, when $\|v\| > \|x\|/\varrho$, the value of $\theta(u,v)$ is less than $\theta(0,0)=0$; hence,

$$d(x,A\cap B) = \sup\{\theta(u,v) : \|u+v\| \leqq 1,\ \|v\| \leqq \|x\|/\varrho\}.$$

But

$$\|u+v\| \leqq 1,\ \|v\| \leqq \|x\|/\varrho \Rightarrow \|u\| \leqq 1 + \|x\|/\varrho.$$

After putting $k = 1 + \|x\|/\varrho$ in (13), the comparison of the sets over which the suprema of $\theta(u,v)$ are taken yields

$$d(x,A\cap B) \leqq (1 + \|x\|/\varrho)(d(x,A) + d(x,B));$$

since $a=0$, this is (12).

Corollary. *Let A and B denote two convex subsets of the normed space E; let α and ϱ be real numbers such that*

$$0 < \alpha < \varrho < e(A, E \setminus B). \tag{16}$$

Then, for every $x \in E$ such that

$$d(x,A) + d(x,B) \leqq \alpha, \tag{17}$$

one has

$$d(x,A\cap B) \leqq \frac{\varrho + \operatorname{diam} A \cap B}{\varrho - \alpha}(d(x,A) + d(x,B)). \tag{18}$$

PROOF. In view of (16) there exists $a \in A$ satisfying with ϱ the hypothesis of the proposition. On the other hand, similarly to inequality (3),

$$\|x-a\| \leqq \operatorname{diam} A \cap B + d(x, A \cap B) .$$

Therefore inequality (12) ensures

$$d(x, A\cap B) \leqq \left(1+\varrho^{-1}(\operatorname{diam} A\cap B + d(x, A\cap B))\right)(d(x,A)+d(x,B)) .$$

Denoting by s the left member of (17) and by r the left member of (18), this is equivalent to

$$\varrho r \leqq (\varrho + \operatorname{diam} A\cap B + r)s .$$

If x satisfies (17) one has $\varrho - s \geqq \varrho - \alpha > 0$, hence

$$r \leqq \frac{\varrho + \operatorname{diam} A \cap B}{\varrho - s} s \leqq \frac{\varrho + \operatorname{diam} A \cap B}{\varrho - \alpha} s ,$$

which is (18).

REMARK. When E is a Hilbert or pre-Hilbert space some more precise inequalities may be obtained by using trigonometrical arguments (see [10]); they may be of use, for instance, in the study of multifunctions with discontinuous finite variation.

6. First semi-continuity.

PROPOSITION. *Let T denote a topological space and let $t \mapsto A_t$ and $t \mapsto B_t$ be two multifunctions from T into the normed space E, with convex values. Let $s \in T$ be such that*

$$\operatorname{diam} A_s \cap B_s < +\infty ,$$

(19) $$A_s \cap \operatorname{int} B_s \neq \varnothing ,$$

(20) $$\lim_{t\to s} e(A_t, A_s) = 0 ,$$

(21) $$\lim_{t\to s} e(B_t, B_s) = 0 .$$

Then

(22) $$\lim_{t\to s} e(A_t \cap B_t,\, A_s \cap B_s) = 0$$

and the two functions $t \mapsto \operatorname{diam} A_t \cap B_t$ and $t \mapsto e(A_t, E\setminus B_t)$ are upper-semi-continuous at the point s.

PROOF. In view of assumption (19) there exist two real numbers α and ϱ such that $0<\alpha<\varrho<e(A_s, E\setminus B_s)$. By assumptions (20) and (21) there exists a neighborhood V of s in T such that, for every t in V one

has $e(A_t,A_s) \leqq \alpha/2$ and $e(B_t,B_s) \leqq \alpha/2$. Hence, if $t \in V$ and $x \in A_t \cap B_t$ one has $d(x,A_s)+d(x,B_s) \leqq \alpha$. Therefore, using (18) we get

$$d(x,A_s \cap B_s) \leqq \frac{\varrho + \operatorname{diam} A_s \cap B_s}{\varrho - \alpha}(e(A_t,A_s)+e(B_t,B_s)) .$$

Taking the supremum for x ranging over $A_t \cap B_t$, one obtains an upper bound for $e(A_t \cap B_t, A_s \cap B_s)$ which shows that (22) follows from (20) and (21).

The upper semi-continuity of the function $t \mapsto \operatorname{diam} A_t \cap B_t$ is a consequence of the inequality (2), namely

$$\operatorname{diam} A_t \cap B_t \leqq \operatorname{diam} A_s \cap B_s + 2e(A_t \cap B_t, A_s \cap B_s) .$$

The upper semi-continuity of the function $t \mapsto e(A_t, E \setminus B_t)$ (trivial in the case where $e(A_s, E \setminus B_s) = +\infty$) follows from the proposition of section 4, which implies

$$e(A_t,E \setminus B_t) \leqq e(A_s,E \setminus B_s)+e(A_t,A_s)+e(B_t,B_s)$$

Remark. Concerning, for instance the multifunction $t \mapsto A_t$, observe that assumption (20) holds in particular when this multifunction is *upper semi-continuous* at the point s in the following classical sense: For every open set Ω containing A_s, there exists a neighborhood V of s in T such that $t \in V$ implies $A_t \subset \Omega$. Furthermore, under the additional assumption that A_s is compact this upper semi-continuity is equivalent to (20).

7. Second semi-continuity.

Proposition. *In the same framework as in the preceding section, let the hypotheses* (20) *and* (21) *be replaced by*

$$\lim_{t \to s} e(A_s,A_t) = 0 , \tag{23}$$

$$\lim_{t \to s} e(B_s,B_t) = 0 . \tag{24}$$

Then

$$\lim_{t \to s} e(A_s \cap B_s, A_t \cap B_t) = 0 \tag{25}$$

and the two functions $t \mapsto \operatorname{diam} A_t \cap B_t$ *and* $t \mapsto e(A_t, E \setminus B_t)$ *are lower semi-continuous at the point* s.

Proof. As the assumption $A_s \cap \operatorname{int} B_s \neq \emptyset$ still holds, there exist $r > 0$ and $a \in A_s$ such that $d(a, E \setminus B_s) > r$. The assumptions (23) and (24) yield a neighborhood V of s in T, such that, for every $t \in V$,

$$e(A_s,A_t) < \tfrac{1}{4}r, \quad e(B_s,B_t) < \tfrac{1}{4}r .$$

Thus, for each $t \in V$, one has $d(a, A_t) < \frac{1}{4}r$ and this ensures the existence of a point a_t in A_t such that $\|a - a_t\| < \frac{1}{4}r$. On the other hand, inequality (6), rewritten as

$$d(a, E \setminus B_s) \leqq d(a, E \setminus B_t) + e(B_s, B_t)$$

implies that, if $t \in V$,

$$d(a, E \setminus B_t) > r - e(B_s, B_t) > \tfrac{3}{4}r .$$

On the other hand

$$d(a, E \setminus B_t) \leqq \|a - a_t\| + d(a_t, E \setminus B_t)$$

thus $d(a_t, E \setminus B_t) \geqq \frac{1}{2}r$. Therefore, inequality (12) yields, for every $t \in V$,

$$\forall x \in E:\ d(x, A_t \cap B_t) \leqq (1 + 2r^{-1}\|x - a_t\|)\big(d(x, A_t) + d(x, B_t)\big) .$$

If $x \in A_s \cap B_s$ one has

$$\|x - a_t\| \leqq \|x - a\| + \|a - a_t\| \leqq \operatorname{diam} A_s \cap B_s + \tfrac{1}{4}r$$

hence finally, for every $t \in V$,

$$e(A_s \cap B_s,\ A_t \cap B_t) \leqq (\tfrac{3}{2} + 2r^{-1} \operatorname{diam} A_s \cap B_s)\big(e(A_s, A_t) + e(B_s, B_t)\big) .$$

This shows that (25) follows from (23) and (24).

The lower semi-continuity of $t \mapsto \operatorname{diam} A_t \cap B_t$ results from inequality (2) rewritten as

$$\operatorname{diam} A_s \cap B_s \leqq \operatorname{diam} A_t \cap B_t + 2e(A_s \cap B_s,\ A_t \cap B_t) .$$

The lower semi-continuity of $t \mapsto e(A_t, E \setminus B_t)$ results from inequality (10) rewritten as

$$e(A_s, E \setminus B_s) \leqq e(A_t, E \setminus B_t) + e(A_s, A_t) + e(B_s, B_t) .$$

Remark. Concerning, for instance, the multifunction $t \mapsto A_t$, observe that assumption (23) implies that this multifunction is *lower semi-continuous* in the following classical sense: For every open set Ω meeting A_s, there exists a neighborhood V of s in T such that $t \in V$ implies $A_t \cap \Omega \neq \emptyset$.

8. Variation over a compact interval.

Proposition. *Let K be a compact interval of R; let $t \mapsto A_t$ and $t \mapsto B_t$ be two multifunctions from K into the normed space E, with convex values; it is supposed that, for every $t \in K$,*

$$\operatorname{diam} A_t \cap B_t < +\infty, \quad A_t \cap \operatorname{int} B_t \neq \emptyset .$$

Then, if both multifunctions are of continuous finite variation in K (resp. are absolutely continuous, resp. are Lipschitzian), so is also the multifunction $t \mapsto A_t \cap B_t$.

PROOF. The assumption involves that both multifunctions are continuous in the sense of Hausdorff distance h which majorizes the écart e; therefore the propositions of sections 6 and 7 can be applied. This shows that the finite-valued function $t \mapsto \operatorname{diam} A_t \cap B_t$ is upper semi-continuous at every point of the compact interval K, thus majorized by some finite constant D. Similarly the function $t \mapsto e(A_t, E \setminus B_t)$, with strictly positive values, is lower semi-continuous on K, thus minorized by some constant $\varrho > 0$; let us choose α in $]0,\varrho[$.

The indefinite variations of the two multifunctions are continuous finite numerical functions on K, hence uniformly continuous. Consequently there exists $p > 0$ such that for any two points σ and τ of K, the condition $|\sigma - \tau| \leqq p$ ensures that $h(A_\sigma, A_\tau)$ and $h(B_\sigma, B_\tau)$ are majorized by $\frac{1}{2}\alpha < \frac{1}{2}\varrho$. When these majorations hold, by putting $k = \varrho + D/\varrho - \alpha$, inequality (18) yields

$$\forall x \in A_\sigma \cap B_\sigma\colon\ d(x, A_\tau \cap B_\tau) \leqq k\big(d(x, A_\tau) + d(x, B_\tau)\big)\,.$$

After taking the suprema for x ranging over $A_\sigma \cap B_\sigma$, one obtains an inequality concerning the écart e; the same holds when σ and τ are exchanged; hence finally

$$h(A_\sigma \cap B_\sigma,\ A_\tau \cap B_\tau) \leqq k\big(h(A_\sigma, A_\tau) + h(B_\sigma, B_\tau)\big)\,.$$

In view of this, the definition of the variation of multifunctions over a compact subinterval $[s,t]$ of K will be applied, under the precaution of considering only sufficiently fine subdivisions of this subinterval, in order that the distance between successive points be less than p. This yields

$$\operatorname{var}(A \cap B; s,t) \leqq k\big(\operatorname{var}(A; s,t) + \operatorname{var}(B; s,t)\big)$$

which establishes the proposition.

A useful special case consists in taking as B_t a fixed open ball β:

COROLLARY. *Let K be a compact interval of R; let $t \mapsto A_t$ be a multifunction from K into the normed space E, with convex values and let β denote a fixed open ball such that, for every t in K, one has $A_t \cap \beta \neq \varnothing$. Then, if $t \mapsto A_t$ is of finite continuous variation (resp. is absolutely continuous, resp. is Lipschitzian) so is also the multifunction $t \mapsto A_t \cap \beta$.*

REMARK. For an interval I possibly non-compact, the above proposition may be used from a *local* standpoint: If the multifunctions $t \mapsto A_t$ and $t \mapsto B_t$ are of finite continuous variation in I and if $s \in I$ is such that

$$\operatorname{diam} A_s \cap B_s < +\infty \quad \text{and} \quad A_s \cap \operatorname{int} B_s \neq \varnothing\,,$$

sections 6 and 7 yield a compact interval, neighborhood of s in I, to which the proposition can be applied.

9. Distance function.

Let x be a fixed point of E. By elementary inequalities, the semi-continuity properties of the multifunction $t \mapsto A_t \cap B_t$ possibly established by applying the foregoing propositions, as well as the properties concerning the variation of this multifunction, imply similar properties for the numerical function $t \mapsto d(x, A_t \cap B_t)$. The purpose of the present section is to show that the conclusions concerning this numerical function may actually be obtained in a slightly more general framework, free from the boundedness assumption precedingly made about the intersection $A_t \cap B_t$.

PROPOSITION. *Let T denote a topological space and let $t \mapsto A_t$ and $t \mapsto B_t$ be two multifunctions from T into the normed space E, with convex values. Let $s \in T$ such that $A_s \cap \operatorname{int} B_s \neq \emptyset$ and*

$$\lim_{t \to s} e(A_t, A_s) = 0 \quad (\text{respectively } \lim_{t \to s} e(A_s, A_t) = 0)$$

$$\lim_{t \to s} e(B_t, B_s) = 0 \quad (\text{respectively } \lim_{t \to s} e(B_s, B_t) = 0)\,.$$

Then, for every $x \in E$, the function $t \mapsto d(x, A_t \cap B_t)$ is lower semi-continuous (resp. upper semi-continuous) at the point s.

PROOF OF THE LOWER SEMI-CONTINUITY. As $A_s \cap B_s$ is nonempty, $d(x, A_s \cap B_s)$ is finite. We have to prove that, for every $\varepsilon > 0$, there exists a neighborhood V of s in T ensuring the implication

$$t \in V \;\Rightarrow\; d(x, A_t \cap B_t) \geqq d(x, A_s \cap B_s) - \varepsilon\,. \tag{26}$$

Let us choose an open ball β, with center x and large enough to meet $A_s \cap \operatorname{int} B_s$. To establish (26) it suffices to proves the similar implication where the considered sets are replaced by their intersections with β. In fact, if $A_t \cap B_t \cap \beta \neq \emptyset$, one has

$$d(x, A_t \cap B_t \cap \beta) = d(x, A_t \cap B_t)$$

and if $A_t \cap B_t \cap \beta = \emptyset$, the distance $d(x, A_t \cap B_t)$ is greater than or equal to the radius of β which itself is greater than $d(x, A_s \cap B_s)$. Put $B_t' = B_t \cap \beta$ and $B_s' = B_s \cap \beta$; the triangle inequality concerning the écart e, applied to the three sets $\{x\}$, $A_t \cap B_t'$ and $A_s \cap B_s'$ yields

$$d(x, A_s \cap B_s') \leqq d(x, A_t \cap B_t') + e(A_t \cap B_t', A_s \cap B_s')\,. \tag{27}$$

The results of section 6 hold for the pair of multifunctions $t \mapsto B_t$ and $t \mapsto \beta$, thus

$$\lim_{t\to s} e(B_t', B_s') = 0 .$$

As these results hold again for the pair of multifunctions $t \mapsto A_t$ and $t \mapsto B_t'$, one has

$$\lim_{t\to s} e(A_t \cap B_t', A_s \cap B_s') = 0 .$$

In view of (27) this proves the expected implication.

PROOF OF THE UPPER SEMI-CONTINUITY. The arguments of section 6, at their beginning, do not make use of the boundedness of $A_s \cap B_s$: In the present case they still yield a real number $r > 0$, a point $a \in A_s$, a neighborhood V of s in T such that for every $t \in V$ there exists $a_t \in A_t$ with $\|a - a_t\| < \frac{1}{4}r$ and $d(a_t, E \setminus B_t) \geqq \frac{1}{2}r$. The point a_t belongs to $A_t \cap B_t$, thus

$$d(x, A_t \cap B_t) \leqq \|x - a_t\| \leqq \|x - a\| + \|a - a_t\| < \|x - a\| + \tfrac{1}{4}r$$

which means that the open ball β with center x and radius $\|x - a\| + \frac{1}{4}r$ meets $A_t \cap B_t$. Consequently, for every $t \in V$,

$$d(x, A_t \cap B_t) = d(x, A_t \cap B_t \cap \beta) . \tag{28}$$

The results of section 6 apply to the pair of multifunctions $t \mapsto B_t$ and $t \mapsto \beta$, thus by putting $B_t' = B_t \cap \beta$

$$\lim_{t\to s} e(B_s', B_t') = 0 .$$

The same results apply also to the pair $t \mapsto A_t$ and $t \mapsto B_t'$, with the conclusion

$$\lim_{t\to s} e(A_s \cap B_s', A_t \cap B_t') = 0 .$$

Then, in view of (28), the upper semi-continuity of $t \mapsto d(x, A_t \cap B_t)$ follows from the triangle inequality

$$d(x, A_t \cap B_t') \leqq d(x, A_s \cap B_s') + e(A_s \cap B_s', A_t \cap B_t') .$$

As an application of this proposition, let us indicate:

COROLLARY. *Let K be a compact interval of* $\mathbb{R}$; *let $t \mapsto A_t$ and $t \mapsto B_t$ be two multifunctions from K into the normed space E, with convex values; let x be a point of E. It is supposed that for every $t \in K$ one has $A_t \cap \operatorname{int} B_t \neq \varnothing$. Then, if both multifunctions are of finite continuous variation on K (resp. are absolutely continuous, respectively are Lipschitzian) so is also the function $t \mapsto d(x, A_t \cap B_t)$.*

PROOF. The above proposition ensures that this function, with finite values, is upper semi-continuous on K, thus strictly majorized by some constant $D > 0$. For every t in K, the open ball β with center x and radius D meets $A_t \cap B_t$, thus meets the interior of B_t. The results of section 8 apply to the pair of multifunctions $t \mapsto A_t$ and $t \mapsto B_t' = B_t \cap \beta$; therefore the multifunction $t \mapsto A_t \cap B_t'$ is of continuous finite variation (resp. is absolutely continuous, resp. is Lipschitzian). Then the conclusions concerning the function $t \mapsto d(x, A_t \cap B_t) = d(x, A_t \cap B_t')$ follow from the inequality

$$|d(x, A_\tau \cap B_\tau') - d(x, A_\sigma \cap B_\sigma')| \leqq h(A_\tau \cap B_\tau', A_\sigma \cap B_\sigma')$$

for every σ and τ in K.

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