MATH. SCAND. 36 (1975), 174–198

UNIQUE REDUCIBILITY OF SUBSETS OF COMMUTATIVE TOPOLOGICAL GROUPS AND SEMIGROUPS

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To Werner Fenchel on his 70th birthday.

Introduction.

As the term is used here, a *reduction* of a set is a direct sum decomposition into a finite number of indecomposable summands. Our main goal is to find conditions under which reductions are "unique", and it is important at the outset to distinguish two kinds of uniqueness–uniqueness "up to isomorphism" and strict uniqueness. These notions are formalized later but an example should make the distinction clear: R^n reduces as the direct sum of n lines through the origin, but these lines may be chosen in infinitely many different ways; on the other hand, $\mathsf{R}_+{}^n$ reduces as the direct sum of n half-lines in only one way. Our first goal is to study the latter sort of uniqueness. Previous results on unique reducibility in R^n were obtained by Isbell [7] in studying factorizations of Banach spaces and by Heller [5] in studying decompositions of stochastic transformations. Isbell proved unique reducibility for compact convex sets symmetric about the origin 0, Heller for pointed polyhedral cones with apex 0. One of our main results is a rather broad generalization of these, asserting that any set $X \subset \mathsf{R}^n$ is uniquely reducible if $0 \in X$ and X or $\bar{X}$ has an *extreme point* (convexity turns out to be irrelevant here). Here extreme point is taken in the very general sense of a point that is not midway between two others; that is, x is extreme in X if there do not exist distinct points y and z of X such that $x+x=y+z$. Note that this definition makes sense in any commutative group G, and indeed our theorem holds in that more general context. Thus, for example (as a very special case), if $0 \in X \subset G$, where X is finite and G is a torsion-free commutative group, then X is uniquely reducible.

As to unique reducibility up to isomorphism, it is conjectured that all convex sets in R^n have this property and it is proved for a very wide

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Received October 28, 1974.

class of convex sets. Indeed, all the examples in R^n in which this sort of uniqueness fails seem to be related to some examples of commutative groups discovered by Jónsson [8].

The problem of unique reducibility is of course a classical one in many contexts. The fundamental theorem of commutative groups and the Krull–Schmidt theorem are probably the best known instances, but there are many others concerned with rings, modules, lattices and so on. (For example, the fundamental theorem of arithmetic is a special case of a unique reduction theorem for pointed semigroups established here.) The method of proof used in the present study exploits the key geometric notion of extreme point in conjunction with combinatorial methods involving the notion of the *refinement property.*

Section headings of this paper are as follows: 1. Some basic definitions; 2. The direct sum; 3. Semigroup preliminaries; 4. Extreme points; 5. The refinement property; 6. Unique reducibility in semigroups; 7. Group preliminaries; 8. Unique reducibility in groups; 9. Applications; 10. Additional comments. The main results on unique reducibility appear in sections 6, 8 and 9; the reader who wants to scan them should first consult the definitions in sections 1–2 and the standing hypotheses in sections 2 and 7. The notation $(^k)$ refers to comments in section 10.

The first author acknowledges support from the National Science Foundation and a helpful comment from Ilan Adler; the second acknowledges support from the Office of Naval Research and helpful comments from Ross Beaumont, Isaac Namioka, Robert Phelps and Robert Warfield.

1. Some basic definitions.

Suppose that a commutative and associative direct sum $\oplus$ has been defined for the subsets of a commutative semigroup G with neutral element 0. A finite collection $\{X_i\}_1^n$ of subsets of G is then said to be a *decomposition* of a set X and the individual X_i's to be *summands* of X if

$$X = X_1 \oplus \ldots \oplus X_n \,. \tag{1-1}$$

A set is *decomposable* if it admits a decomposition into two or more (and hence by associativity into precisely two) nonsingleton summands. A *reduction* of X is a decomposition of X into one or more indecomposable nonsingleton summands, or into X itself when X is a singleton. A set is *reducible* (resp. *uniquely reducible*) if it is empty or admits at least (resp. precisely) one reduction. A reduction (1-1) is *basic* if each summand of X is of the form

(1-2) $\sum_{k\in K} X_k$ for some $K \subset \{1,\ldots,n\}$,

where the sum (1-2) is interpreted as $\{0\}$ when K is empty.

Though our main goal is to find conditions under which reducibility implies unique reducibility, we also consider weaker sorts of uniqueness. When $\mathscr{T}$ is a group of one-to-one transformations of G onto itself, a reducible set X is said to be *$\mathscr{T}$-uniquely reducible* if the members of any two reductions of X are paired by $\mathscr{T}$-equivalence and *strongly $\mathscr{T}$-uniquely reducible* if the pairing can always be effected by a single member of $\mathscr{T}$. A reduction (1-1) is *$\mathscr{T}$-basic* if each summand of X is $\mathscr{T}$-equivalent to a set of the form (1-2).

A set $X \subset G$ has the *refinement property* if

(1-3) $$X = Y_1 \oplus Y_2 = Z_1 \oplus Z_2$$

implies there exist sets $X_{ij} \subset G$ satisfying the four equalities of the diagram

$$\begin{array}{ccccc} X_{11} & \oplus & X_{12} & = & Y_1 \\ \oplus & & \oplus & & \\ X_{21} & \oplus & X_{22} & = & Y_2 \\ \| & & \| & & \\ Z_1 & & Z_2 & & \end{array}$$

which is henceforth abbreviated as

(1-4) $$\oplus \begin{array}{cc|c} X_{11} & X_{12} & Y_1 \\ X_{21} & X_{22} & Y_2 \\ \hline Z_1 & Z_2 & \end{array}$$

An *extreme point* of a set $X \subset G$ is a point $x \in X$ such that $x=y=z$ is the only solution of $x+x=y+z$ with $y,z \in X$.

2. The direct sum.

Our attention is restricted to definitions of $\oplus$ for which (1-1) implies

(2-1) $$X = X_1 + \ldots + X_n .$$

Decompositions of commutative groups into direct sums of their subsets have been studied extensively (see Fuchs [4, Chap. XV]), defining (1-1) by (2-1) in conjunction with

(2-2) for each $x \in X$ there are unique $x_i \in X_i$ such that $x = x_1 + \ldots + x_n$.

However, that definition does not yield the desired results on unique reducibility. For example, if G is an infinite cyclic group and $\oplus$ is as indicated then

$$X = \{0,1\}\oplus\{0,2,4\} = \{0,3\}\oplus\{0,1,2\}$$

provides two different reductions of the set $X=\{0,1,2,3,4,5\}$. In contrast, the definition of $\oplus$ adopted below assures the unique reducibility of X whenever $0 \in X \subset G, X$ is finite, and G is a torsion-free commutative group. (1)

In preparation for the definition of $\oplus$ to be employed here, we adopt the following

STANDING HYPOTHESES AND NOTATION:
G is a commutative topological semigroup with neutral element 0;
$\mathscr{S}$ is a lattice of closed subsemigroups of G such that $G \in \mathscr{S}$, every member of $\mathscr{S}$ includes 0, *$S+S' \in \mathscr{S}$ for all $S,S' \in \mathscr{S}$* (2), *and $\bigcap \mathscr{R} \in \mathscr{S}$ for each $\mathscr{R} \subset \mathscr{S}$*;
for each $X \subset G$, $\langle X \rangle$ is the smallest member of $\mathscr{S}$ containing X.

All that follows is relative to $\mathscr{S}$, but specific mention of $\mathscr{S}$ is suppressed when there is no danger of confusion.

Our direct sum is defined by (2-1) in conjunction with

(2-3) the natural homomorphism of $\langle X_1 \rangle \times \ldots \times \langle X_n \rangle$ into G is a homeomorphism onto $\langle X \rangle$.

When all the X_i's belong to $\mathscr{S}$ that becomes the usual definition of internal direct sum in topological semigroups.

Note that, in the presence of (2-1), (2-3) is equivalent to the conjunction of

(2-4) for each $x \in \langle X \rangle$ there are unique $x_i \in \langle X_i \rangle$ such that $x = x_1 + \ldots + x_n$

and

(2.5) for each i the natural projection $\pi_i \colon \langle X \rangle \to \langle X_i \rangle$ is continuous.

Of course the conjunction of (2-1) and (2-4) implies (2-2), but (2-4) requires much more "independence" of the X_i's than (2-2).

Our direct sum $\oplus$ is plainly commutative. For associativity it suffices to show that if $X=(A\oplus B)\oplus C$ and $Z=B+C$ then $Z=B\oplus C$ and $X=A\oplus Z$. That follows from a standard argument when the topology is discrete (use 3.1 below) and is easily verified in the general case with the aid of (2-5).

3. Semigroup preliminaries.

The results of this section are required for later use.

3.1 $\langle X\rangle+\langle X'\rangle=\langle X+X'\rangle$ *for all* $X,X'\subset G$.

3.2 *If* $X=X_1\oplus\ldots\oplus X_n$ *then* $\bar{X}=\bar{X}_1\oplus\ldots\oplus\bar{X}_n$.

3.3 *If* $X=X_1\oplus\ldots\oplus X_n$ *then* $0\in X$ *implies* $0\in\bigcap_{i=1}^n X_i$, *and* $0\in\bigcap_{i=1}^n X_i$ *implies* $\bigcup_{i=1}^n X_i\subset X$.

3.4. *If* $X=Y_1\oplus Y_2=Z_1\oplus Z_2$ *and* $Y_1\in\mathscr{S}$ *or* $\langle Y_1\rangle$ *is a group then*

$$Y_1\supset\langle Y_1\rangle\cap Z_1+\langle Y_1\rangle\cap Z_2 .$$

PROOF. 3.1 follows from the assumption that $\mathscr{S}$ is closed under addition, 3.2 from a standard property of product topologies and the fact that each member of $\mathscr{S}$ is closed. 3.3 is immediate from the definition of $\oplus$.

3.4 is obvious when $Y_1\in\mathscr{S}$. Suppose, on the other hand, that $\langle Y_1\rangle$ is a group, and consider arbitrary points $z_j\in\langle Y_1\rangle\cap Z_j$. Then $z_1+z_2\in X$ and hence there exist $y_i\in Y_i$ with $z_1+z_2=y_1+y_2$. Thus

$$y_2=z_1+z_2-y_1\in\langle Y_1\rangle\cap Y_2=\{0\}$$

and

$$z_1+z_2=y_1\in Y_1 .$$

3.5 THEOREM. *Every finite subset of* G *is reducible, and if the lattice* $\mathscr{S}$ *satisfies the descending chain condition then every subset of* G *is reducible.*

PROOF. If (1-1) holds and no X_i is a singleton then the cardinality of X is $\geqq 2^n$. If X is finite and of cardinality >1 there is a decomposition of the described sort for which n is maximum, and it is a reduction of X.

If X is not reducible it admits an infinite sequence of decompositions,

$$X=X_1^j\oplus\ldots\oplus X_{n(j)}^j\quad(j=1,2,\ldots)$$

such that no X_i^j is a singleton and for each j the $(j+1)$th decomposition is obtained from the jth one by replacing a single set of the latter with two of its summands. Since $\langle Y\rangle\supsetneqq\langle Z\rangle$ wherever Z is a summand of Y other than Y itself, it follows with the aid of König's lemma that there is an infinite sequence $i(1),i(2),\ldots$ of indices such that

$$\langle X_{i(1)}^1\rangle\supsetneqq\langle X_{i(2)}^2\rangle\supsetneqq\cdots,$$

contradicting the descending chain condition.

4. Extreme points.

The following result is the key to the relevance of extreme points in our investigation.

4.1 THEOREM. *If* $X = Y_1 \oplus Y_2 = Z_1 \oplus Z_2$, 0 *is an extreme point of* X, *and for each* $W \in \{Y_1, Y_2, Z_1, Z_2\}$ *it is true that* $W \in \mathscr{S}$ *or* $\langle W \rangle$ *is a group, then*

$$\oplus \begin{array}{cc|c} Y_1 \cap Z_1 & Y_1 \cap Z_2 & Y_1 \\ Y_2 \cap Z_1 & Y_2 \cap Z_2 & Y_2 \\ \hline Z_1 & Z_2 & \end{array}$$

PROOF. To prove

$$Y_1 \subset (Y_1 \cap Z_1) + (Y_1 \cap Z_2)\,, \tag{4-1}$$

consider an arbitrary point $y_1 \in Y_1$. Since $y_1 \in X$ by 3.3, there exist $z_j \in Z_j$ with $y_1 = z_1 + z_2$, and since $z_j \in X$ there exist $y_{ij} \in Y_i$ with $z_j = y_{1j} + y_{2j}$. But then

$$y_1 = (y_{11} + y_{12}) + (y_{21} + y_{22})\,,$$

where $y_{i1} + y_{i2} \in \langle Y_i \rangle$ and consequently $y_1 = y_{11} + y_{12}$ while $0 = y_{21} + y_{22}$. But 0, being an extreme point of X, is also by 3.3 an extreme point of Y_2, whence $0 = y_{21} = y_{22}$ and $z_j = y_{1j} \in Y_1 \cap Z_j$. That establishes (4-1), and with the aid of 3.4 we conclude that

$$Y_1 = (Y_1 \cap Z_1) + (Y_1 \cap Z_2)\,.$$

But then

$$\langle Y_1 \rangle = \langle Y_1 \cap Z_1 \rangle + \langle Y_1 \cap Z_2 \rangle$$

by 3.1, and $\langle Y_1 \rangle$ is the image of $\langle Y_1 \cap Z_1 \rangle \times \langle Y_1 \cap Z_2 \rangle$ under the natural homomorphism of the product into G. To see that the homomorphism is a homeomorphism, recall (2-5) and note that the present hypotheses imply the continuity of the appropriate projections. Thus

$$Y_1 = (Y_1 \cap Z_1) \oplus (Y_1 \cap Z_2)\,.$$

The other equalities of 4.1 follow by symmetry.

5. The refinement property.

5.1 THEOREM. *For each* $X \subset G$ *the following five conditions are equivalent* (3):

(a) *X has the refinement property*;

(b) *each summand of X has the refinement property*;

(c) *whenever* $X = Y_1 \oplus \ldots \oplus Y_m = Z_1 \oplus \ldots \oplus Z_n$ *there exist* $X_{ij} \subset G$ *such that*

$$\oplus \begin{array}{ccc|c} X_{11} & \ldots & X_{1n} & Y_1 \\ \vdots & & \vdots & \vdots \\ X_{m1} & \ldots & X_{mn} & Y_m \\ \hline Z_1 & \ldots & Z_n & \end{array}$$

(d) *whenever* $X = Y_1 \oplus Y_2 = Z_1 \oplus Z_2$ *there exist* $X_{ij} \subset G$ *such that*

$$+ \begin{array}{cc|c} X_{11} & X_{12} & Y_1 \\ X_{21} & X_{22} & Y_2 \\ \hline Z_1 & Z_2 & \end{array}$$

(e) *whenever* $X = Y_1 \oplus Y_2 = Z_1 \oplus Z_2$ *there exist* $S_{ij} \in \mathscr{S}$ *such that*

$$\oplus \begin{array}{cc|c} S_{11} & S_{12} & \langle Y_1 \rangle \\ S_{21} & S_{22} & \langle Y_2 \rangle \\ \hline \langle Z_1 \rangle & \langle Z_2 \rangle & \end{array}$$

PROOF. Plainly (b) $\Rightarrow$ (a), (c) $\Rightarrow$ (a) and (a) $\Rightarrow$ (d). We shall prove (a) $\Rightarrow$ (b), (a) $\Rightarrow$ (c), (d) $\Rightarrow$ (e), and (e) $\Rightarrow$ (a).

If (a) holds, $X = A \oplus B$ and $A = U_1 \oplus U_2 = V_1 \oplus V_2$, then

$$X = U_1 \oplus (U_2 \oplus B) = V_1 \oplus (V_2 \oplus B)$$

and there are sets $X_{ij} \subset G$ such that

$$\oplus \begin{array}{cc|c} X_{11} & X_{12} & U_1 \\ X_{21} & X_{22} & U_2 \oplus B \\ \hline V_1 & V_2 \oplus B & \end{array}$$

If $\pi \colon \langle X \rangle \to \langle A \rangle$ denotes the natural projection then

$$\oplus \begin{array}{cc|c} X_{11} & X_{12} & U_1 \\ X_{21} & \pi X_{22} & U_2 \\ \hline V_1 & V_2 & \end{array}$$

and hence (a) $\Rightarrow$ (b).

Now let $A(k,n)$ denote the assertion that (c) holds whenever $m \leqq k$ and X has the refinement property. Then $A(2,2)$ is given by (a) and $A(1,n)$ is obvious for all n. If $A(k,n)$ is known and

$$X = Y_1 \oplus \ldots \oplus Y_{k+1} = Z_1 \oplus \ldots \oplus Z_n$$

then from (a) and $A(k,n)$ there follows the existence of sets W_{ij} such that

$$\oplus\begin{array}{ccc|c} W_{11} & \dots & W_{1n} & Y_1 \\ \vdots & & \vdots & \vdots \\ W_{(k-1)1} & \dots & W_{(k-1)n} & Y_{k-1} \\ W_{k1} & \dots & W_{kn} & Y_k \oplus Y_{k+1} \\ \hline Z_1 & \dots & Z_n & \end{array}$$

while the existence of sets T_{ij} such that

$$\oplus\begin{array}{ccc|c} T_{11} & \dots & T_{1n} & Y_k \\ T_{21} & \dots & T_{2n} & Y_{k+1} \\ \hline W_{k1} & \dots & W_{kn} & \end{array}$$

follows from (b) and $A(2,n)$. With $X_{ij}=W_{ij}$ for $1 \leqq i < k$, $X_{kj}=T_{ij}$ and $X_{(k+1)j}=T_{2j}$, the sets X_{ij} are as desired in (c) and hence it follows by induction that (a) $\Rightarrow$ (c).

To see that (d) $\Rightarrow$ (e) let all sets be as in (d) and define $S_{ij}=\langle X_{ij}\rangle$. Then

$$+\begin{array}{cc|c} S_{11} & S_{12} & \langle Y_1\rangle \\ S_{21} & S_{22} & \langle Y_2\rangle \\ \hline \langle Z_1\rangle & \langle Z_2\rangle & \end{array}$$

by 3.1 and our task is to replace $+$ by $\oplus$. In the discrete case it suffices to note that

$$S_{i1} \cap S_{i2} \subset \langle Z_1\rangle \cap \langle Z_2\rangle = \{0\}$$

and hence each point of $\langle Y_i\rangle$ admits a unique expression as the sum of a point in S_{i1} and a point in S_{i2}. To handle the general case, recall (2-5) and note that the natural projection of $\langle Y_i\rangle$ onto S_{ij} is the restriction to $\langle Y_i\rangle$ of the natural projection of $\langle X\rangle$ onto Z_j. Thus $\langle Y_i\rangle = S_{i1}\oplus S_{i2}$. By symmetry, $\langle Z_j\rangle = S_{1j}\oplus S_{2j}$ and hence (d) $\Rightarrow$ (e).

To show, finally, that (e) $\Rightarrow$ (a), let the notation be as in (e), so that

$$\langle X\rangle = S_{11}\oplus S_{12}\oplus S_{21}\oplus S_{22}\,,$$

and let $\pi_{ij}\colon \langle X\rangle \to S_{ij}$ denote the natural projection. We claim (1-4) holds with $X_{ij}=\pi_{ij}X$, and first show

$$\text{(5-1)} \qquad X_{i1}+X_{i2} \subset Y_i \quad \text{for } i=1,2\,.$$

Consider an arbitrary pair of points x, $x' \in X$ with

$$x = x_{11}+x_{12}+x_{21}+x_{22},\ x' = x_{11}'+x_{12}'+x_{21}'+x_{22}'$$

and x_{ij}, $x_{ij}' \in S_{ij}$. Since

$$x_{1j}+x_{2j} \in S_{1j}\oplus S_{2j} = \langle Z_j\rangle\,,$$

$x_{1j}+x_{2j}$ is the image of x under the natural projection of $\langle X\rangle$ onto $\langle Z_j\rangle$ and hence $x_{1j}+x_{2j}\in Z_j$. Similarly, $x_{1j}'+x_{2j}'\in Z_j$. But then

$$x_{11}+x_{21}+x_{12}'+x_{22}'\in Z_1+Z_2 = X$$

and there exists $\tilde{x}$ such that

$$x_{11}+x_{12}'+x_{21}+x_{22}' = \tilde{x}\in X .$$

Since

$$x_{i1}+x_{i2}'\in S_{i1}\oplus S_{i2} = \langle Y_i\rangle ,$$

$x_{i1}+x_{i2}'$ is the image of $\tilde{x}$ under the natural projection of $\langle X\rangle$ onto $\langle Y_i\rangle$ and therefore $x_{i1}+x_{i2}'\in Y_i$. That proves (5-1), and equality in (5-1) then follows from the observation that

$$X\subset X_{11}+X_{12}+X_{21}+X_{22}\subset Y_1\oplus Y_2 = X .$$

The proof that $X_{i1}\oplus X_{i2}=Y_i$ is completed as was the proof of 4.1, and the equalities $X_{1j}\oplus X_{2j}=Z_j$ follow by symmetry.

5.2 Corollary. *If the closure* $\bar{X}$ *has the refinement property then so has* X. (4)

Proof. If $X=Y_1\oplus Y_2=Z_1\oplus Z_2$ then by 3.2

$$\bar{X} = \bar{Y}_1\oplus\bar{Y}_2 = \bar{Z}_1\oplus\bar{Z}_2 .$$

Since $\bar{X}$ has the refinement property and (a) $\Rightarrow$ (e) there exist $S_{ij}\in\mathscr{S}$ with

$$\oplus\begin{array}{cc|c} S_{11} & S_{12} & \langle\bar{Y}_1\rangle \\ S_{21} & S_{22} & \langle\bar{Y}_2\rangle \\ \hline \langle\bar{Z}_1\rangle & \langle\bar{Z}_2\rangle & \end{array}$$

But $\langle\overline{W}\rangle=\langle W\rangle$ for all $W\subset G$, because all members of $\mathscr{S}$ are closed, and the desired conclusion then follows from the fact that (e) $\Rightarrow$ (a).

6. Unique reducibility in semigroups.

6.1 Theorem. *Consider the following conditions on the set* X:

(a) *X is reducible and has the refinement property*;
(b) *X admits a basic reduction*;
(c) *X is uniquely reducible.*

Then (b) *implies* (a) *and* (c) *for all* $X\subset G$, *while* (a) *implies* (b) *if* $0\in X$ *or singletons in* X *are closed and* $0\in\bar{X}$.

PROOF. We may assume X is not a singleton, as that case is easily handled.

To see that (b) $\Rightarrow$ (a), let

$$X = X_1 \oplus \ldots X_n \tag{6-1}$$

be a basic reduction of X and consider two decompositions $X = Y_1 \oplus Y_2 = Z_1 \oplus Z_2$ of X. Then there are subsets K_1, K_2, L_1 and L_2 of $\{1, \ldots, n\}$ such that

$$Y_i = \sum_{h \in K_i} X_h \quad \text{and} \quad Z_j = \sum_{h \in L_j} X_j , \tag{6-2}$$

where plainly

(6-3) K_1 and K_2 are complementary in $\{1, \ldots, n\}$, as are L_1 and L_2. But then

$$\oplus \begin{array}{cc|c} \sum_{h \in K_1 \cap L_1} X_h & \sum_{h \in K_1 \cap L_2} X_h & Y_1 \\ \sum_{h \in K_2 \cap L_1} X_h & \sum_{h \in K_2 \cap L_2} X_h & Y_2 \\ \hline Z_1 & Z_2 & \end{array}$$

To see that (b) $\Rightarrow$ (c), let (6-1) be as above and consider another reduction $X = Y_1 \oplus \ldots \oplus Y_m$ of X. Since (6-1) is basic and each Y_i is an indecomposable nonsingleton, each Y_i belongs to $\{X_i\}_1^n$ and it then follows easily that $(Y_1, \ldots, Y_m)$ is merely a permutation of $(X_1, \ldots, X_n)$.

Suppose, finally, that (a) holds, and consider an arbitrary reduction (6-1) of X and summand Y_1 of X. Let Y_2 be such that $X = Y_1 \oplus Y_2$, whence by 5.1(c) there exist sets $X_{ij} \subset G$ such that

$$\oplus \begin{array}{ccc|c} X_{11} & \ldots & X_{1n} & Y_1 \\ X_{21} & \ldots & X_{2n} & Y_2 \\ \hline X_1 & \ldots & X_n & \end{array} \tag{6-4}$$

But then

(6-5) each column of the matrix (X_{ij}) includes a unique nonsingleton, for each set X_h is an indecomposable nonsingleton. If $0 \in X$ it follows from 3.3 that 0 belongs to each set appearing in (6-4) and hence

(6-6) each singleton in (6-2) is equal to $\{0\}$.

If $0 \in \overline{X}$ it follows from 3.2 and 3.3 that 0 belongs to the closure of each set appearing in (6-4) and hence (6-5) continues to hold if singletons in G are closed. From (6-2), (6-5) and (6-6) it follows that $Y = \sum_{h \in H} X_h$ with $H = \{j : X_{1j} \neq \{0\}\}$.

6.2 COROLLARY. *Suppose that singletons in G are closed, 0 is an extreme point of $\overline{X}$, and $\mathscr{S}$ includes $\overline{X}$ and also all summands of its own members. If the set X is reducible it has the properties* (a)–(c) *of* 6.1.

PROOF. The set X has the refinement property by 4.1, whence X has the refinement property by 5.2 and the desired conclusion then follows from 6.1.

A *pointed semigroup* is a semigroup having 0 as an extreme point. The following result may be regarded as a generalization of the uniqueness part of the fundamental theorem of arithmetic.

6.3 COROLLARY. *If G's topology is discrete and $\mathcal{S}$ consists of all pointed semigroups in G then every reducible member of $\mathcal{S}$ has the properties* (a)–(c) *of* 6.1.

PROOF. Use 6.2 and the fact that each summand of a pointed semigroup is itself a pointed semigroup.

6.4 COROLLARY. *Suppose that all members of $\mathcal{S}$ are groups, and* 0 *is an extreme point of X or singletons in G are closed and* 0 *is an extreme point of $\bar{X}$. If the set X is reducible it has properties* (a)–(c) *of* 6.1.

PROOF. Apply 6.1 after using 4.1 and 5.2 to see that X has the refinement property.

The main goal of the next two sections is to reduce the role of the special point 0 and thus obtain relatives of 6.1 and 6.4 that are more nearly translation-invariant.

7. Group preliminaries.

In sections 7–9 the standing hypotheses of section 2 are strengthened by adding the

SUPPLEMENTARY STANDING HYPOTHESES AND NOTATION:

all members of $\mathcal{S}$ are groups, and $\{0\} \in \mathcal{S}$;
G's topology is Hausdorff;
for each $X \subset G$, $|X|$ is the smallest $\mathcal{S}$-coset containing X.

We have the following results.

7.1 $|X| + p = |X + p|$ *for all $X \subset G$ and $p \in G$.*

7.2 $\langle X - x\rangle \subset \langle X\rangle$ *for all $X \subset G$ and $x \in \langle X\rangle$.*

7.3 $\langle X - x\rangle = |X| - x$ *for all $X \subset G$ and $x \in |X|$.*

7.4 *If* $x \in |X|$ *then the following are equivalent*: $|X| = \langle X \rangle$; $0 \in |X|$; $\langle X - x \rangle = \langle X \rangle$.

7.5 *If* $X = X_1 \oplus \ldots \oplus X_n$ *then* $0 \in |X|$ *implies* $0 \in \bigcap_{i=1}^n |X_i|$.

7.6 *If* $X = X_1 \oplus \ldots \oplus X_n$ *and* $x = x_1 + \ldots + x_n$ *with* $x_i \in \langle X_i \rangle$ *then*

$$X - x = (X_1 - x) \oplus \ldots \oplus (X_n - x_n) . \tag{7-1}$$

7.7 *If* $Y_1 \oplus \ldots \oplus Y_m = Z_1 \oplus \ldots \oplus Z_m$, *where no* Y_i *is a singleton, and if for each* $j \in \{1,\ldots,n\}$ *the point* $t_j \in G$ *and set* $M_j \subset \{1,\ldots,m\}$ *are such that* $Z_j = t_j + \sum_{i \in M_j} Y_i$ *then the sets* M_j $(1 \leqq j \leqq n)$ *are pairwise disjoint and their union is* $\{1,\ldots,m\}$.

Proof. 7.1–7.3 are routine, as is the first equivalence in 7.4. For the second equivalence, suppose $0 \in |X|$ and note that if $S \in \mathscr{S}$ with $S \supset X - x$ then $S + x \supset X$, whence $S + x \supset |X| \ni 0$ and $S + x$ is a group. Since $x \in |X|$ we have $S + x \supset X + x$ and $S \supset X$, showing that $\langle X - x \rangle \supset \langle X \rangle$. Conversely, suppose $\langle X - x \rangle = \langle X \rangle$ and consider $S \in \mathscr{S}$ and $p \in G$ with $S + p \supset X$. Then $S + p - x \supset |X| - x \ni 0$, so $S + p - x$ is a group and therefore belongs to $\mathscr{S}$. Since $S + p - x \supset X - x$ it follows that

$$S + p - x \supset \langle X - x \rangle = \langle X \rangle$$

and hence $S + p - x \ni -x$. But then $S + p \ni 0$ and we conclude $0 \in |X|$.

To prove 7.5, suppose $x \in X = X_1 \oplus \ldots \oplus X_n$ and let $x_i \in X_i$ be such that $x = x_1 + \ldots + x_n$. If $0 \in |X|$ we see with the aid of 7.4, 3.1 and 7.2 that

$$\begin{aligned} \langle X \rangle = \langle X - x \rangle &= \langle X_1 - x_1 \rangle + \ldots + \langle X_n - x_n \rangle \\ &\subset \langle X_1 \rangle \oplus \ldots \oplus \langle X_n \rangle = \langle X \rangle , \end{aligned}$$

whence $\langle X_i - x_i \rangle = \langle X_i \rangle$ for all i and $0 \in |X_i|$ by 7.4.

Suppose next that the hypotheses of 7.6 are satisfied, whence plainly (7-1) holds with $\oplus$ replaced by $+$. But $\langle X - x \rangle \subset \langle X \rangle$ and $X_i - x_i \subset \langle X_i \rangle$ for all i, so it is clear that for each $y \in \langle X - x \rangle$ there are unique $y_i \in \langle X_1 - x_i \rangle$ such that $y = y_1 + \ldots + y_n$, and the natural projection of $\langle X - x \rangle$ onto $\langle X_i - x_i \rangle$ is the restriction to $\langle X - x \rangle$ of the natural projection of $\langle X \rangle$ onto $\langle X_i \rangle$. The desired conclusion then follows with the aid of (2-5).

Turning now to 7.7, we note that if there exist h, j and k such that $h \in M_j \cap M_k$, and if u_1 and u_2 are distinct points of Y_h, then it is easy to find points p_j and p_k of G such that

$$t_j + p_j + u_i \in Z_j \quad \text{and} \quad t_k + p_k + u_i \in Z_k \quad (i = 1, 2) ,$$

whence

$$0 \neq u_1 - u_2 \in \langle Z_j \rangle \cap \langle Z_k \rangle = \{0\} ,$$

an impossibility.

To complete the proof of 7.7, let $M = \bigcup_{j=1}^{n} M_j$ and $M' = \{1,\ldots,m\} \setminus M$; we want to show $M' = \emptyset$. Suppose, on the contrary, that $h \in M'$, let v_1 and v_2 be distinct points of Y_h, and let $t = t_1 + \ldots + t_n$. Since

$$\textstyle\sum_{i \in M} Y_i + \sum_{i \in M} Y_i' = \sum_{j=1}^{n} Z_j = t + \sum_{i \in M} Y_i ,$$

there exist $u \in G$ and $w_1, w_2 \in \sum_{i \in M} Y_i$ such that $u + v_i = t + w_i$ for $i = 1, 2$. But then $v_1 - w_1 = t - u = v_2 - w_2$ and $v_1 - v_2 = w_1 - w_2$, whence

$$\langle Y_h \rangle \cap \langle \textstyle\sum_{i \in M} Y_i \rangle \neq \{0\} ,$$

a contradiction completing the proof.

When $X \subset G$, a property $\mathscr{P}$ is said to be *invariant for* X if either all translates of X have $\mathscr{P}$ or all lack $\mathscr{P}$; and $\mathscr{P}$ is *fully invariant in* G if it is invariant for all $X \subset G$. Analogously, when $0 \in \bar{X}$ and $\mathscr{T}_0(X)$ is the family of all translates Y of X such that $0 \in \bar{Y}$, the property $\mathscr{P}$ is 0-*invariant* for X if either all members of $\mathscr{T}_0(X)$ have $\mathscr{P}$ or all lack $\mathscr{P}$; and $\mathscr{P}$ is *fully* 0-*invariant* in G if it is 0-invariant for all $X \subset G$ such that $0 \in \bar{X}$.

Decomposability need not be fully invariant under our definition, even when the topology is discrete and $\mathscr{S}$ is the lattice of all subgroups of G.[5] For example, if G is the direct sum of two infinite cyclic groups, $X = \{(0,0),(0,1),(2,1),(2,2)\}$ and $t = (1,1)$, then X is decomposable but $X + t$ is not. However, we do have the following results.

7.8 *If X is indecompsable and $0 \in \bar{X}$ then all translates of X are indecomposable.*

7.9 *If X is decomposable so is $X + t$ for all $t \in \langle X \rangle$.*

7.10 *If X has the refinement property then so does $X + t$ whenever $t \in \langle X + t \rangle$. If $0 \in |X|$ that is true of all $t \in G$.*

7.11 *Decomposability and the refinement property are fully invariant for all $X \subset G$ such that $|X| = G$.*

7.12 *Decomposability, the refinement property, reducibility, basic reducibility and unique reducibility are all fully 0-invariant in G.*[6]

We say *the situation is pleasant* if

(7-2) $X + t = (X_1 \oplus t) \oplus X_2$ *whenever* $X \subset G$, $X = X_1 \oplus X_2$, *and* $t \in G \setminus \langle X \rangle$.

7.13 *If the situation is pleasant then decomposability, the refinement property, reducibility, translation-basic reducibility, and translation-unique reducibility are all fully invariant in G, while basic reducibility and unique reducibility are invariant for all $X \subset G$ such that $|X| = G$.*

PROOF. For 7.8 note that if $0 \in \bar{X}$ and $X - t = X_1 \oplus X_2$ then

(7-3) there exist $x_i \in \langle X_i \rangle$ with $t = x_1 + x_2$, whence

$$X = (X_1 + x_1) \oplus (X_2 + x_2)$$

by 7.6. (The example preceding 7.8 shows it does not suffice to assume $0 \in |X|$.)

For 7.9 note that if $X = X_1 \oplus X_2$ and $t \in \langle X \rangle$ then (7-3) holds and

$$X + t = (X_1 + x_1) \oplus (X_2 + x_2)$$

by 7.6.

Now suppose that X has the refinement property, consider two decompositions

$$X + t = Y_1 \oplus Y_2 = Z_1 \oplus Z_2$$

of $X + t$, and let the points $y_i \in \langle Y_i \rangle$ and $z_j \in \langle Z_j \rangle$ be such that

$$t = y_1 + y_2 = z_1 + z_2 \, .$$

Since

$$X = (Y_1 - y_1) \oplus (Y_2 - y_2) = (Z_1 - z_1) \oplus (Z_2 - z_2)$$

there exist sets X_{ij} such that

$$\oplus \begin{array}{cc|c} X_{11} & X_{12} & Y_1 - y_1 \\ X_{21} & X_{22} & Y_2 - y_2 \\ \hline Z_1 - z_1 & Z_2 - z_2 & \end{array}$$

and with

$$+ \begin{array}{cc|c} x_{11} & x_{12} & y_1 \\ x_{21} & x_{22} & y_2 \\ \hline z_1 & z_2 & \end{array}$$

we have

$$+ \begin{array}{cc|c} X_{11} + x_{11} & X_{12} + x_{12} & Y_1 \\ X_{21} + x_{21} & X_{22} + x_{22} & Y_2 \\ \hline Z_1 & Z_2 & \end{array}$$

It then follows with the aid of 5.1(d) that $X + t$ has the refinement property.

To complete the proof of 7.10 note that if $0 \in |X|$ and $t \in G$ then

$$t \in |X| + t = |X + t| \subset \langle X + t \rangle \, .$$

7.11 is an immediate consequence of 7.9 and 7.10.

For 7.12 it suffices to show that if $\mathscr{P}$ is any of the listed properties, X is a subset of G having property $\mathscr{P}$, and $\{0, t\} \subset \bar{X}$, then the set $X - t$ also has property $\mathscr{P}$. For decomposability and the refinement property that is immediate from 7.9 and 7.10 respectively. For reducibility, note that if $X = X_1 \oplus \ldots \oplus X_n$ is a reduction of X and the points $x_i \in \overline{X}_i$ are such that $t = x_1 + \ldots + x_n$, then

$$X - t = (X_1 - x_1) \oplus \ldots \oplus (X_n - x_n) \tag{7-4}$$

by 7.6. And since $0 \in \bigcap_{i=1}^{n} \overline{X}_i$ by 3.2 and 3.3, it follows from the full 0-invariance of decomposability that (7-4) is a reduction of $X - t$.

The remaining two properties in 7.12 are left to the reader, for their full 0-invariance is not actually used here.(7)

The proof of 7.13 is similar to that of 7.12. For example, if X is decomposable the decomposability of $X + t$ follows from 7.9 when $t \in \langle X \rangle$ and (7-2) when $t \notin \langle X \rangle$. If X is reducible the reducibility of $X + t$ follows from the full invariance of decomposability in conjunction with 7.6 when $t \in \langle X \rangle$ and (7-2) when $t \notin \langle X \rangle$. If X has the refinement property, so does $X + t$ by 7.10 when $t \in \langle X + t \rangle$, while if $t \notin \langle X + t \rangle$ and $X + t = Y_1 \oplus Y_2 = Z_1 \oplus Z_2$ it follows from (7-2) that

$$X = (Y_1 - t) \oplus Y_2 = (Z_1 - t) \oplus Z_2 .$$

But then there are sets X_{ij} such that

$$\oplus \begin{array}{cc|c} X_{11} & X_{12} & Y_1 - t \\ X_{21} & X_{22} & Y_2 \\ \hline Z_1 - t & Z_2 & \end{array}$$

whence

$$+ \begin{array}{cc|c} X_{11} + t & X_{12} & Y_1 \\ X_{21} & X_{22} & Y_2 \\ \hline Z_1 & Z_2 & \end{array}$$

and $X + t$ has the refinement property. The remaining properties are left to the reader.(7)

8. Unique reducibility in groups.

Assuming the supplementary as well as the original standing hypotheses, this section extends the reasoning of 6.1 and 6.4 to establish analogues of those results that are more nearly translation-invariant.

8.1 THEOREM. *If* $0 \in |X|$ *then* (a) *and* (b) *are equivalent and imply* (c), *where the conditions are as follows*:

(a) *X is reducible and has the refinement property*;
(b) *X admits a basic reduction*;
(c) *X is uniquely reducible.*

PROOF. It suffices in view of 6.1 to prove (a) $\Rightarrow$ (b). Let the notation be as in the last paragraph of the proof of 6.1. Then (6-4) and (6-5) are still valid and it remains only to establish (6-6). Since $0 \in |X|$ it follows with the aid of 7.5 that $0 \in |W|$ for each set W appearing in (6-4). Then recall that $\{0\} \in \mathscr{S}$, whence $|W| = W$ whenever W is a singleton.

The simplest examples show that 8.2 may fail when $0 \notin |X|$. In particular, let G be R^3 with its usual topology, let $\mathscr{S}$ be the lattice of all subspaces of R^3, and let $X = Y + Z + p$ where p is the origin and Y and Z are the segments joining the origin to the points $(1,0,0)$ and $(0,1,0)$ respectively. Then X is reducible and has the refinement property, but X is not uniquely reducible because it admits the two reductions

$$X = (Y+p) \oplus Z = Y \oplus (Z+p) .$$

However, we are able to prove the following analogue of 6.1 and 8.1.

8.2 THEOREM. *For all* $X \subset G$, (a) *and* (d) *are equivalent and imply* (e), *where* (a) *is as in* 8.1 *and the other conditions are as follows*:

(d) X *admits a translation-basic reduction*;
(e) X *is translation-uniquely reducible.*

PROOF. The proof parallels that of 6.1, and we may assume as in 6.1 that X is not a singleton.

To see that (d) $\Rightarrow$ (a), let $X = X_1 \oplus \ldots \oplus X_n$ be a translation-basic reduction of X and consider two decompositions $X = Y_1 \oplus Y_2 = Z_1 \oplus Z_2$. Then there are subsets K_1, K_2, L_1 and L_2 of $\{1, \ldots, n\}$ and points p_1, p_2, q_1, and q_2 of G such that

$$Y_i = p_i + \textstyle\sum_{h \in K_i} X_h \text{ and } Z_j = q_j + \textstyle\sum_{h \in L_j} X_h .$$

It follows from 7.7 that (6-3) is still valid, whence with

$$+\begin{array}{cc|c} r_{11} & r_{12} & p_1 \\ r_{21} & r_{22} & p_2 \\ \hline q_1 & q_2 & \end{array}$$

we have

$$+\begin{array}{cc|c} r_{11} + \sum_{h \in K_1 \cap L_1} X_h & r_{12} + \sum_{h \in K_1 \cap L_2} X_h & Y_1 \\ r_{21} + \sum_{h \in K_2 \cap L_1} X_h & r_{22} + \sum_{h \in K_2 \cap L_2} X_h & Y_2 \\ \hline Z_1 & Z_2 & \end{array}$$

and the desired conclusion ensues.

Now suppose (a) holds, let $X = X_1 \oplus \ldots \oplus X_n$ be a reduction of X, and consider a decomposition $X = Y_1 \oplus Y_2$. As in the proof of 6.1 there exist sets X_{ij} for which (6-4) and (6-5) are valid. For $i = 1, 2$, let

$$P_i = \{j : X_{ij} \text{ is a singleton}\} \text{ and } p_i = \sum_{j \in P_i} X_{ij} .$$

Then X_{2j} is a singleton for all $j \notin P_1$, whence

$$Y_1 = p_1 + \sum_{j \notin P_1} X_{1j} = p_1 + \sum_{j \notin P_1} (X_j - X_{2j}) = (p_1 - p_2) + \sum_{j \notin P_1} X_j .$$

It follows, therefore, that (a) and (d) are equivalent and imply that every reduction of X is translation-basic.

To see, finally, that (d) $\Rightarrow$ (e), let $X = Y_1 \oplus \ldots \oplus Y_m$ be a reduction of X for which the number m of summands is minimum, and consider an arbitrary reduction $X = Z_1 \oplus \ldots \oplus Z_n$. If the sets M_j are as in 7.7, it follows from 7.7 and the minimality of m that each M_j consists of a single member of $\{1, \ldots, m\}$, whence the reductions $\{Y_i\}_1^m$ and $\{Z_j\}_1^n$ are translation-equivalent and it follows that X is translation-uniquely reducible.

The following is the major result of the paper.([8])

8.3 THEOREM. *Suppose that X is reducible and X or $\bar{X}$ has an extreme point. Then*

(a) *if $0 \in \bar{X}$, X admits a basic reduction;*

(b) *if the situation is pleasant, X admits a translation-basic reduction, and a basic reduction when $0 \in |X|$.*

PROOF. If x is an extreme point of X or $\bar{X}$, then 0 is an extreme point of the set $X - x$ or its closure, whence it follows from 4.1 and 5.2 that $X - x$ has the refinement property. Then X has the refinement property by 7.12 or 7.13 and the stated conclusions follow from 8.1 and 8.2.

The remaining results of this section assure the applicability of 8.3(b).

8.4 PROPOSITION. *Suppose that G is a vector space over a field Φ, and G is topologized in such a way that all standing hypotheses are satisfied by G together with the lattice $\mathscr{S}$ of all closed subspaces. Suppose that for each $S \in \mathscr{S}$ and $t \in G \setminus S$ the subspace S_t generated by $S \cup \{t\}$ belongs to $\mathscr{S}$ and the natural projections of S_t onto S and onto Φt are continuous. Then the situation is pleasant.*

PROOF. Suppose that $X = X_1 \oplus X_2$ and $t \in G \setminus \langle X \rangle$. Then

$$\langle X \cup \{t\} \rangle = \langle X \rangle_t = (\langle X_1 \rangle \oplus \langle X_2 \rangle)_t ,$$

so each point $y \in \langle X \cup \{t\} \rangle$ admits a unique expression in the form

$$y = \pi_t(y) + \pi(y) \text{ with } \pi_t(y) \in \Phi t, \ \pi(y) \in \langle X \rangle .$$

The natural projections $\pi_t \colon \langle X \rangle_t \to \Phi t$ and $\pi \colon \langle X \rangle_t \to \langle X \rangle$ are continuous by hypothesis, as are the natural projections $\pi_i \colon \langle X \rangle \to \langle X_i \rangle$. Now note that

(8-1) $$X + t = (X_1 + t) + X_2$$

and

(8-2) $$\langle X + t \rangle = \langle X_1 + t \rangle + \langle X_2 \rangle ,$$

where

$$\langle X_1 + t \rangle \subset \langle X_1 \cup \{t\} \rangle = \langle X_1 \rangle_t$$

and hence

$$\langle X_1 + t \rangle \cap \langle X_2 \rangle = \{0\} .$$

It follows that the sums in (8-1) and (8-2) are direct when the topology is discrete, so to establish (7-2) in the general case it remains only to show the natural projections $\xi \colon \langle X + t \rangle \to \langle X_1 + t \rangle$ and $\eta \colon \langle X + t \rangle \to \langle X_2 \rangle$ are continuous. But note that if $\varphi(y) = \pi_t(y) + \pi_1(\pi(y))$ then

(8-3) $$y \in \langle X + t \rangle \text{ implies } \varphi(y) \in \langle X_1 + t \rangle ,$$

an implication that is obvious when $y \in X + t$, follows by linearity of φ and $\langle X_1 + t \rangle$ when y belongs to the subspace generated by $X + t$, and then follows for all $y \in \langle X + t \rangle$ by the continuity of φ and closedness of $\langle X_1 + t \rangle$. Since $y = \varphi(y) + \pi_2(\pi(y))$ for all $y \in \langle X + t \rangle \subset \langle X \rangle_t$, and $\pi_2(\pi(y)) \in \langle X_2 \rangle$, it follows with the aid of (8-3) that $\xi = \varphi$ and $\eta = \pi_2 \pi$, thus establishing the desired continuity and completing the proof.

8.5 COROLLARY. *If G is a vector space over a field Φ and $\mathscr{S}$ is the lattice of all subspaces of G, then each of the following implies the situation is pleasant:*

(a) *the topology is discrete;*

(b) *G is finite-dimensional, Φ is an ordered field or a complete non-discrete valued field, and the topology is the usual "product" topology.*

8.6 COROLLARY. *If G is a torsion-free divisible group, the topology is discrete, and $\mathscr{S}$ is the lattice of all subgroups of G, then the situation is pleasant.*

PROOF. 8.5(a) is an immediate consequence of 8.4. For 8.6, recall that a torsion-free divisible group G is isomorphic with a (possibly infinite) direct sum of full rational groups (see Fuchs [4, p. 64]) and hence has the additive structure of a rational vector space. With $\mathcal{S}$ (resp. $\mathcal{S}'$) denoting the lattice of all subgroups (resp. subspaces) of G, it is not hard to verify that direct sums in G relative to $\mathcal{S}$ are the same as those relative to $\mathcal{S}'$, and hence the desired conclusion follows from 8.5(a).

For the case of a valued field in 8.5(b), see Bourbaki [1, p. 28]. The case of an ordered field is handled similarly.

9. Applications.

For the sake of simplicity, this section proceeds under the assumption that $0 \in |X|$ (9), the most interesting cases being those in which $0 \in X$ or $|X| = G$.

9.1 THEOREM. *Let G be a torsion-free commutative group with discrete topology, $\mathcal{S}$ the lattice of all subgroups of G, and X a finite subset of G. Then X admits a basic reduction (and hence is uniquely reducible) if*

(a) $0 \in X$

or

(b) *G is divisible* (10) *and* $0 \in |X|$.

PROOF. In view of 3.5, 8.3 and 8.5 it suffices to show X has an extreme point. The smallest subgroup of G containing X is for some k a free commutative group on k generators and hence is isomorphic to the subgroup of R^k consisting of all lattice points. But every finite subset of R^k has an extreme point (take any extreme point of its convex hull).

It follows from 8.3 and 8.4, in conjunction with various extreme point theorems of functional analysis (11), that if G is a certain sort of topological vector space, $\mathcal{S}$ is the lattice of *all* subspaces (2) of G, and $\mathcal{U}$ is a second topology such that $(G, \mathcal{U})$ is a topological group and all members of $\mathcal{S}$ are $\mathcal{U}$-closed, then relative to $(\mathcal{U}, \mathcal{S})$ certain kinds of subsets of G admit at most one reduction. However, these results are not very satisfactory except when G is finite-dimensional and the two topologies can be made to coincide.

9.2 THEOREM. *Let G be R^k with the usual topology, $\mathcal{S}$ the lattice of all subspaces of G, and X a subset of G whose affine hull includes the origin. If the convex hull of X is line-free (and hence, in particular, if X is bounded), then X admits a basic reduction and hence is uniquely reducible.*(12)

PROOF. In view of 3.5, 8.3 and 8.5 it suffices to show $\bar{X}$ has an extreme point. If the convex hull C of X contains no line it is easily verified that $\bar{C}$ is also line-free. It then follows from results of Klee [9] that $\bar{C}$ has an extreme point x and $x \in \bar{X}$.

When X is a subset of R^k whose closure has no extreme point, X may fail even to be affine-uniquely reducible when $k \geqq 3$.(13) For $k=3$ that can be seen from an example of Jónsson [8] as simplified by Fuchs [4, p. 154]. Let P and Q be disjoint infinite sets of primes with $5 \notin P \cup Q$, let P^π (resp. Q^π) be the set of all squarefree products of members of P (resp. Q), and let A, B, C and D denote the additive subgroups of R^3 having respectively the following generating systems:

$$\{(1/r,0,0) : \ r \in P^\pi\};$$
$$\{(0,1/r,0) : \ r \in P^\pi\} \cup \{(0,0,1/s) : \ s \in Q^\pi\} \cup \{(0,1/5,1/5)\};$$
$$\{(8/r,3/r,0) : \ r \in P^\pi\};$$
$$\{(5/r,2/r,0) : \ r \in P^\pi\} \cup \{(0,0,1/s) : \ s \in Q^\pi\} \cup \{(3,6/5,1/5)\}\ .$$

Fuchs shows B and D are indecomposable but not isomorphic, and of course A and C are indecomposable. But

$$A+B \ = \ C+D \text{ and } \langle A\rangle \cap \langle B\rangle \ = \ \{0\} \ = \ \langle C\rangle \cap \langle D\rangle\ ,$$

so

$$X \ = \ A \oplus B \text{ and } X \ = \ C \oplus D$$

are two reductions of the set X that are not affine-equivalent. For a striking extension of this example to higher-dimensional spaces and several summands, see Corner [2].

We believe that theorem 9.4 below remains valid when the set X is assumed merely to be convex, but the proof given here relies heavily on the following additional assumption about X:

(9-1) X contains every line that lies in $\bar{X}$ and intersects X.

9.3 LEMMA. *Suppose that E is a finite-dimensional real vector space, the origin is interior to the convex subset X of E, X satisfies* (9-1), *and L is the union of all lines through* 0 *in X. Then L is a subspace and $X = L \oplus (X \cap S)$ for each subspace S supplementary to L in E. Every representation of S as the direct sum of a subspace and a line-free set is of this form. If S_1 and S_2 are subspaces supplementary to L and the linear transformations $U_j: E \to L$ and $V_j: E \to S_j$ are defined by the condition that $I = U_j + V_j$ (where I is the identity transformation on E), then X is carried onto itself by the linear transformation $W = I + U_1 - U_2$.*

PROOF. Parts of 9.3 are well-known when X is closed; see, for example, Klee [9] and Hirsch and Hoffman [6]. Using the methods of those papers it is a routine matter to prove 9.3 in the form stated above. Details are left to the reader.

9.4 THEOREM. *Let G be R^k with the usual topology, $\mathscr{S}$ the lattice of all subspaces of G, and X a convex subset of G whose affine hull includes the origin. If the set X satisfies* (9-1) *then it is strongly affine-uniquely reducible. That is, for any two reductions*

$$X = Y_1 \oplus \ldots \oplus Y_m = Z_1 \oplus \ldots \oplus Z_n$$

of X there is a nonsingular affine transformation T of G onto itself that carries the Y_i's onto the Z_j's.

PROOF. A routine argument based on 7.5, 7.6, 7.13 and 8.5 shows that it suffices to consider the case in which the origin 0 is interior to X. We may also assume without loss of generality that there are integers m_0 and n_0 such that Y_i (resp. Z_j) is line-free if and only if $i > m_0$ (resp. $j > n_0$). It follows with the aid of 9.3 that the summands Y_i for $i \leqq m_0$ and Z_j for $j \leqq n_0$ are all lines through 0, $\sum_1^{m_0} Y_i$ and $\sum_1^{n_0} Z_j$ are both equal to the lineality space L of 9.3, and there are supplements S_1 and S_2 of L such that

$$X \cap S_1 = Y_{m_0+1} \oplus \ldots \oplus Y_m \text{ and } X \cap S_2 = Z_{n_0+1} \oplus \ldots \oplus Z_n .$$

The linear transformation W of 9.3 carries $X \cap S_1$ onto $X \cap S_2$ and hence by 9.2 induces a pairing of the Y_i's for $i > m_0$ with the Z_j's for $j > n_0$. Let T be a linear transformation of G onto itself such that T has the same restriction to S_1 as W has and T's restriction to L induces a pairing of the lines Y_i for $i \leqq m_0$ with the lines Z_j for $j \leqq n_0 = m_0$.(14)

10. Additional comments.

(1) When G is a torsion group of rank > 1, ambiguity of reduction of finite sets persists no matter what reasonable definition of $\oplus$ is chosen. Consider, for example, the three different reductions of the Klein four group.

(2) It would be nice to avoid this condition and hence cover the case in which G is an infinite-dimensional topological linear space and $\mathscr{S}$ is the lattice of all closed subspaces. However, the additive closedness of $\mathscr{S}$ is required in the present treatment to justify the often-used identity 3.1 and even to show that $\oplus$ is associative.

(3) Other equivalent conditions result from replacing $\oplus$ by $+$ in the diagrams of (c) and (e).

(4) The converse of 5.2 is false. For example, let G be R^k with the usual topology, let $\mathscr{S}$ be the lattice of all subspaces of G, and let Y be a subset of G such that $0 \in Y$ and each point of Y is a condensation point. Then even though Y itself may be closed and lack the refinement property (for example, take $Y = \mathsf{R}^k$ when $k > 1$) it is easy to produce a countable dense subset X of Y such that 0 is an extreme point of X and hence X has the refinement property by 4.1.

(5) It would be of interst to know the structure of the commutative groups G in which decomposability is fully invariant when $\mathscr{S}$ is as described. See 8.6 for a sufficient condition.

(6) The stronger refinement property of 4.1 is also of interest. The set X is said to have the *strong refinement property* if

$$\oplus \begin{array}{cc|c} Y_1 \cap Z & Y_1 \cap Z_2 & Y_1 \\ Y_2 \cap Z_1 & Y_2 \cap Z_2 & Y_2 \\ \hline Z_1 & Z_2 & \end{array}$$

whenever $X = Y_1 \oplus Y_2 = Z_1 \oplus Z_2$. We do not know whether X must have this property if $0 \in X$ and $\bar{X}$ has an extreme point, nor whether the property is fully 0-invariant in the weaker sense obtained by deleting all closure operators in the former definition.

(7) The characterizations in 8.1 and 8.2 can be used to deal with basic and translation-basic reducibility.

(8) The following additional result on unique reducibility can be derived from 6.4. *If decomposability is fully invariant in G, X is reducible and X or $\bar{X}$ has an extreme point, then X is translation-uniquely reducible and is uniquely reducible when* $0 \in |X|$.

(9) Without that assumption it would be necessary in 9.1(b) and 9.2 to replace basic by translation-basic reducibility, and in 9.4 to replace strong affine-unique by affine-unique reducibility.

(10) Perhaps the assumption of divisibility can be deleted, at least when $|X| = G$.

([11]) If X is a closed subset of a topological vector space E, each of the following conditions implies X has an extreme point:

(a) E is locally convex and X is compact (use the Krein–Milman theorem and Milman's converse — see Day [3, pp. 78–80];

(b) E is a metrizable locally convex space and X lies in a weakly compact subset of E (use a renorming theorem of Troyanski [13] and a theorem of Lindenstraus [11] on strongly exposed points);

(c) E is a Banach space with the Radon–Nikodym property (in particular, a separable conjugate space or a reflexive space) and X is bounded (Phelps [12]).

([12]) By using a theorem of Klee [10] it is possible to establish a close relative of this theorem in a finite-dimensional vector space over an arbitrary ordered field.

([13]) With $\mathscr{S}$ as in 9.2, X is of course uniquely reducible when $k=1$ and probably affine-uniquely reducible when $k=2$, though we have not actually settled the latter case.

([14]) There is independent interest in the problem, which we encountered in an earlier approach to 9.4, of finding geometric conditions on a convex set X that assure the validity of a converse of 3.2. The result established below involves the following strengthening of (9-1):

(10-1) $]p,q[\subset X$ whenever $p\in X$ and $q\in \bar{X}$.

Condition (a) below is quite restrictive and it would be of interest to find a weaker one. For example, if $\bar{X}$ is a d-polytope P with $d\geqq 3$ then X satisfies (a) if and only if $X=P$ or there is a set $\mathscr{F}$ of pairwise disjoint facets of P such that X is the union of the relative interior of P with the relative interiors of the various members of $\mathscr{F}$.

As the term is used in (b) below, a *face* of X is a convex set $Y\subset X$ such that Y contains every segment $]u,v[$ for which $u,v\in X$ and $]u,v[$ intersects Y.

PROPOSITION. *If* $0\in X\subset \mathsf{R}^k$ *then* (a) *and* (b) *are equivalent and imply* (c), *where the conditions are as follows*:

(a) X *satisfies* (10-1), *and whenever two segments* $]p,x[$ *and* $]q,x[$ *in* X *are not collinear then* $x\in X$ *or* X *contains a third segment that starts on one of the first two and crosses the other*;

(b) X *satisfies* (10-1), *and* $\bar{Y}\cap\bar{Z}\subset X$ *for each pair of proper faces* Y *and* Z *of* $\bar{X}$;

(c) *for each decomposition* $\bar{X}=C_1\oplus\ldots\oplus C_n$ *of* $\bar{X}$ *there exists a decomposition* $X=X_1\oplus\ldots\oplus X_n$ *of* X *such that* $\bar{X}_i=C_i$ *for all* i.

PROOF. Since (a) $\Leftrightarrow$ (b) is straightforward, we prove only (a) $\Rightarrow$ (c). We first establish

(10-2) if $0\in X$, (a) holds and $\bar{X}=A\oplus B$, then

$$X = (\langle A\rangle\cap X)\oplus(\langle B\rangle\cap X) .$$

To prove (10-2) we show that if $x\in\bar{X}$, $a\in A$, $b\in B$ and $x=a+b$ then $x\in X$ if and only if $\{a,b\}\subset X$. Indeed, if $\{a,b\}\subset X$ but $x\notin X$ it follows from several applications of (10-1) that

$$\{(\alpha,\beta):\ \alpha\geqq 0,\ \beta>0,\ \alpha a+\beta b\in X\} = ([0,1]\times]0,1])\setminus\{(1,1)\} ,$$

while if $x\in X$ but $a\notin X$ it follows from (10-1) that

$$\{(\alpha,\beta):\ \alpha\geqq 0,\ \beta<1,\ \alpha a+\beta b\in X\} = ([0,1]\times[0,1[)\setminus\{(1,0)\};$$

in both cases the second part of (a) is contradicted.

To prove (a) $\Rightarrow$ (c) it suffices, in view of 3.2, to show that if (a) holds and $\bar{X}=C_1\oplus\ldots\oplus C_n$ then

$$X = (\langle C_1\rangle\cap X)\oplus\ldots\oplus(\langle C_n\rangle\cap X) .$$

The case in which $n=2$ is settled by (10-2) and the general case follows by induction. For if $A=\sum_1^n C_i$, (10-2) yields

$$X = (\langle A\rangle\cap X)\oplus(\langle C_n\rangle\cap X) ,$$

it can be seen with the aid of (a) that

$$\overline{\langle A\rangle\cap X} = \langle A\rangle\cap\bar{X} = A ,$$

and since the set $\langle A\rangle\cap X$ also satisfies condition (a) it follows from the inductive hypothesis that

$$\begin{aligned}\langle A\rangle\cap X &= (\langle C_1\rangle\cap(\langle A\rangle\cap X))\oplus\ldots\oplus(\langle C_n\rangle\cap(\langle A\rangle\cap X))\\ &= (\langle C_1\rangle\cap X)\oplus\ldots\oplus(\langle C_n\rangle\cap X) .\end{aligned}$$

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