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ON THE "THREE SPACE PROBLEM"

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To Werner Fenchel on his 70th birthday.

1. Introduction.

Let X be a Banach space and Y a closed subspace of X. In this paper we will study the so-called three space problem: if one has information about two of the Banach spaces X, Y and X/Y, what can be said about the third one. In the sequel we shall have information about Y and X/Y and draw conclusions about X. For other aspects of the problem cf. [7].

It is a classical result [1, II.4, p. 19–20] that if Y and X/Y are reflexive then X is also reflexive. D. P. Giesy has shown [3, Th. II.9] that if Y and X/Y are B-convex then X is also B-convex. We prove below that if Y and X/Y are super-reflexive then X is also super-reflexive.

We also solve the following problem (apparently due to Palais): if each of the spaces Y and X/Y is isomorphic to a Hilbert space, is X isomorphic to a Hilbert space? We prove that X is in a certain sense close to being isomorphic to a Hilbert space, but that it need not be isomorphic to a Hilbert space.

2. Some inequalities.

In this section, we obtain information on the behavior of Rademacher series (resp. of martingales) with values in X knowing the corresponding information for Y and X/Y. We denote by (ε_n) the Rademacher system on the interval $[0,1]$. Let $(\Omega,\mathscr{A},P)$ be a probability space, a sequence of random variables $(X_n)_{n\geqq 0}$ on $(\Omega,\mathscr{A},P)$ with values in a Banach space is called a martingale if there exists an increasing sequence $(\mathscr{A}_n)_{n\geqq 0}$ of sub-σ-algebras of $\mathscr{A}$ such that $\forall n \geqq 0,\ \ X_n = E^{\mathscr{A}_n}(X_{n+1})$.

In this paper we shall say briefly „martingale" meaning „martingale defined on some probability space $(\Omega,\mathscr{A},P)$"; moreover if Z is a Banach space valued random variable on a probability space $(\Omega,\mathscr{A},P)$ we shall write simply $\|Z\|_2$ for $(\int \|Z(\omega)\|^2 dP(\omega))^{\frac{1}{2}}$.

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When we wish to distinguish the norms of the Banach spaces involved, we will write $\| \|_X$, $\| \|_Y, \ldots$ for the norms in the spaces $X, Y, \ldots$. In particular recall the definition of the norm in X/Y: let π denote the canonical projection from X onto X/Y, by definition we have

$$\forall x \in X,\ \|\pi(x)\|_{X/Y} = \inf\{\|x+y\| \mid y \in Y\}.$$

Definition/Notation. Let X be a Banach space and n an integer. We define $a_n{}^X$ as the smallest positive number a such that:

$$\left(\int \|\textstyle\sum_{i=1}^{i=n} \varepsilon_i(t)x_i\|^2 dt\right)^{\frac{1}{2}} \leqq a\left(\textstyle\sum_{i=1}^{i=n} \|x_i\|^2\right)^{\frac{1}{2}}$$

for all n-tuples $(x_i)_{1\leqq i\leqq n}$ of elements of X.

We define $\alpha_n{}^X$ as the smallest positive number α such that:

$$\|X_n\|_2 \leqq \alpha\left(\textstyle\sum_{i=1}^{i=n} \|X_i - X_{i-1}\|_2^2\right)^{\frac{1}{2}}$$

for all martingales $(X_n)_{n\geqq 0}$ with values in X and such that $X_0 = 0$. Obviously we have $a_n{}^X \leqq \alpha_n{}^X \leqq \sqrt{n}$ for all integers n. The following theorem motivates the preceding definitions.

Theorem 1. *Let X be a Banach space and Y be a closed subspace of X. The following inequalities hold for all integers n and k:*

$$a_{nk}{}^X \leqq a_n{}^Y a_k{}^X + a_n{}^Y a_k{}^{X/Y} + a_n{}^X a_k{}^{X/Y} \tag{1}$$

$$\alpha_{nk}{}^X \leqq \alpha_n{}^Y \alpha_k{}^X + 2\alpha_n{}^Y \alpha_k{}^{X/Y} + 2\alpha_n{}^X \alpha_k{}^{X/Y}. \tag{2}$$

Proof. We start with (1): let $(x_j)_{j\leqq nk}$ be a nk-tuple in X. $\forall \theta \in [0,1]$, let $X_i(\theta)$ denote $\sum_{(i-1)k<j\leqq ik} \varepsilon_j(\theta)x_j$. Let π denote the canonical projection from X onto X/Y. Then $\forall \theta \in [0,1]$, $\forall i = 1,2,\ldots,n$, $\forall \gamma > 0$ there exists $Y_i(\theta)$ in Y such that: $\|X_i(\theta) + Y_i(\theta)\|_X \leqq \|\pi(X_i(\theta))\|_{X/Y} + \gamma$. Let $A(\theta)$ be the integral

$$\left(\int \|\textstyle\sum_{i=1}^{i=n} \varepsilon_i(t)X_i(\theta)\|^2 dt\right)^{\frac{1}{2}};$$

by convexity of the norm, we have:

$$A(\theta) \leqq \left(\int \|\textstyle\sum_{i=1}^{n} \varepsilon_i(t)Y_i(\theta)\|^2 dt\right)^{\frac{1}{2}} + \left(\int \|\textstyle\sum_{i=1}^{i=n} \varepsilon_i(t)(X_i(\theta) + Y_i(\theta))\|^2 dt\right)^{\frac{1}{2}},$$

so that by the definitions of $a_n{}^Y$ and $a_n{}^X$:

$$A(\theta) \leqq a_n{}^Y\left(\textstyle\sum_{i=1}^{i=n} \|Y_i(\theta)\|^2\right)^{\frac{1}{2}} + a_n{}^X\left(\textstyle\sum_{i=1}^{i=n} \|X_i(\theta) + Y_i(\theta)\|^2\right)^{\frac{1}{2}};$$

but on one hand $\|Y_i(\theta)\| \leqq \|X_i(\theta)\| + \|Y_i(\theta) + X_i(\theta)\|$ and on the other hand $\|X_i(\theta) + Y_i(\theta)\| \leqq \|\pi(X_i(\theta))\| + \gamma$, so that we have:

$$A(\theta) \leqq a_n{}^Y\left(\textstyle\sum_{i=1}^{i=n} \|X_i(\theta)\|^2\right)^{\frac{1}{2}} + (a_n{}^Y + a_n{}^X)\left[\left(\textstyle\sum_{i=1}^{i=n} \|\pi(X_i(\theta))\|^2\right)^{\frac{1}{2}} + \gamma\sqrt{n}\right]$$

which gives after integration:

$$\begin{aligned}(\textstyle\int A(\theta)^2 d\theta)^{\frac{1}{2}} &\leqq a_n^Y(\sum_{i=1}^{i=n} \|X_i\|_2^2)^{\frac{1}{2}} + (a_n^Y + a_n^X)[(\sum_{i=1}^{i=n} \|\pi(X_i)\|_2^2)^{\frac{1}{2}} + \gamma\sqrt{n}] \\ &\leqq [a_n^Y a_k^X + (a_n^Y + a_n^X) a_k^{X/Y}](\sum_{i=1}^{i=nk} \|x_j\|^2)^{\frac{1}{2}} + (a_n^Y + a_n^X)\gamma\sqrt{n}.\end{aligned}$$

By an easy argument of symmetry one finds that

$$(\textstyle\int A(\theta)^2 d\theta)^{\frac{1}{2}} = (\int \|\sum_{j=1}^{j=nk} \varepsilon_j(t) x_j\|^2 dt)^{\frac{1}{2}};$$

the result then follows since $\gamma > 0$ is arbitrary.

Let us now prove (2): Let $(X_m)_{m \geqq 0}$ be a martingale with values in X such that $X_0 = 0$, adapted to a sequence of σ-algebras $(\mathscr{A}_m)_{m \geqq 0}$. We write $\Delta_i = X_{ik} - X_{(i-1)k}$ for $i = 1, 2, \ldots, n$, and set $Z_0 = 0$ and $\forall \lambda = 1, 2, \ldots$, $Z_\lambda = \sum_{1 \leqq i \leqq \lambda} \Delta_i$. Obviously $(Z_\lambda)_{\lambda \geqq 0}$ is a martingale with respect to the sequence of σ-algebras $(\mathscr{A}_{\lambda k})_{\lambda \geqq 0}$. Now, as easily seen, $\forall \gamma > 0$, $\forall i = 1, 2, \ldots$, there exists an $\mathscr{A}_{ik}$-measurable random variable Y_i with values in Y such that:

$$\|\Delta_i + Y_i\|_{L^2(X)} \leqq \|\pi(\Delta_i)\|_{L^2(X/Y)} + \gamma\,. \tag{3}$$

We define a martingale $(U_\lambda)_{\lambda \geqq 0}$ with values in Y by setting $U_0 = 0$ and $\forall \lambda = 1, 2, \ldots$,

$$U_\lambda = \sum_{1 \leqq i \leqq \lambda} E^{\mathscr{A}_{ik}}(Y_i) - E^{\mathscr{A}_{(i-1)k}}(Y_i);$$

$(U_\lambda)_{\lambda \geqq 0}$ is a martingale adapted to the sequence of σ-algebras $(\mathscr{A}_{\lambda k})_{\lambda \geqq 0}$.

We notice that: $\forall i = 1, 2, \ldots, n$

$$U_i - U_{i-1} = E^{\mathscr{A}_{ik}}(Y_i) - E^{\mathscr{A}_{(i-1)k}}(Y_i)\,,$$

and (since $E^{\mathscr{A}_{(i-1)k}}(\Delta_i) = 0$) that:

$$\Delta_i + U_i - U_{i-1} = E^{\mathscr{A}_{ik}}(\Delta_i + Y_i) - E^{\mathscr{A}_{(i-1)k}}(\Delta_i + Y_i);$$

by the triangle inequality:

$$\|U_i - U_{i-1}\|_2 \leqq \|\Delta_i\|_2 + \|\Delta_i + U_i - U_{i-1}\|_2$$

so that, by the continuity of the conditional expectations on $L^2(X)$, we have:

$$\|\Delta_i + U_i - U_{i-1}\|_2 \leqq 2\|\Delta_i + Y_i\|_2\,, \tag{4}$$

hence:

$$\|U_i - U_{i-1}\|_2 \leqq \|\Delta_i\|_2 + 2\|\Delta_i + Y_i\|_2\,. \tag{5}$$

Now, using the definition of α_n^X, we get:

$$\begin{aligned}\|X_{nk}\|_2 &= \|Z_n\|_2 \leqq \|U_n\|_2 + \|Z_n + U_n\|_2 \\ &\leqq \alpha_n^Y(\sum_{i=1}^{i=n} \|U_i - U_{i-1}\|_2^2)^{\frac{1}{2}} + \alpha_n^X(\sum_{i=1}^{i=n} \|\Delta_i + U_i - U_{i-1}\|_2^2)^{\frac{1}{2}} \\ &\leqq \alpha_n^Y(\sum_{i=1}^{i=n} \|\Delta_i\|_2^2)^{\frac{1}{2}} + 2(\alpha_n^Y + \alpha_n^X)[\sum_{i=1}^{i=n} \|\Delta_i + Y_i\|_2^2]^{\frac{1}{2}};\end{aligned} \tag{6}$$

the last inequality being deduced from (4) and (5).

Now fix i between 1 and n, and set $V_0{}^i=0$ and $\forall\lambda=1,2,\ldots$, $V_\lambda{}^i=\sum_{(i-1)k<j\leq[(i-1)k+\lambda]}X_j-X_{j-1}$, so that $(V_\lambda{}^i)_{\lambda\geq 0}$ is a martingale adapted to the sequence of σ-algebras $(\mathscr{A}_{(i-1)k+\lambda})_{\lambda\geq 0}$; we have therefore:

$$\|\Delta_i\|_2 = \|V_k{}^i\|_2 \leqq \alpha_k{}^X(\textstyle\sum_{(i-1)k<j\leq ik}\|X_j-X_{j-1}\|_2^2)^{\frac{1}{2}}$$

and:

$$\begin{aligned}\|\pi(\Delta_i)\|_2 = \|\pi(V_k{}^i)\|_2 &\leqq \alpha_k{}^{X/Y}(\textstyle\sum_{(i-1)k<j\leq ik}\|\pi(X_j)-\pi(X_{j-1})\|_2^2)^{\frac{1}{2}}\\ &\leqq \alpha_k{}^{X/Y}(\textstyle\sum_{(i-1)k<j\leq ik}\|X_j-X_{j-1}\|_2^2)^{\frac{1}{2}}\,.\end{aligned}$$

With (3), (6) and the inequalities above, we finally obtain:

$$\begin{aligned}\|X_{nk}\|_2 \leqq (\alpha_n{}^Y\alpha_k{}^X+2\alpha_n{}^Y\alpha_k{}^{X/Y}+2\alpha_n{}^X\alpha_k{}^{X/Y})(\textstyle\sum_{j=1}^{nk}\|X_j-X_{j-1}\|_2^2)^{\frac{1}{2}}+\\ +2\gamma\sqrt{n}(\alpha_n{}^Y+\alpha_n{}^X)\,,\end{aligned}$$

and this concludes the proof of (2) since $\gamma>0$ is arbitrary.

3. Applications.

We first recall some definitions:

A Banach space X is called B-convex if there exist an integer n and $\varepsilon>0$ such that

$$\inf\|\textstyle\sum_{i=1}^{i=n}\varepsilon_i x_i\| \leqq n(1-\varepsilon)$$

for all n-tuples $(x_1,x_2,\ldots x_n)$ in the unit ball of X and the infimum is over all choices of n-signs $(\varepsilon_1,\ldots,\varepsilon_n)$ in $\{-1,1\}^n$.

Following James, we say that a Banach space Z is finitely representable in a Banach space X if for all $\varepsilon>0$ and any finite dimensional subspace M of Z there exist a subspace N of X and an isomorphism T from M onto N such that

$$\|T\|\,\|T^{-1}\| \leqq 1+\varepsilon\,.$$

A Banach space X is called super-reflexive if all the Banach spaces Z which are finitely representable in X are reflexive.

R. C. James has recently produced [4] an example of a B-convex Banach space which is not super-reflexive. It is proved in [2] that a Banach space is super-reflexive if and only if there is an equivalent norm on X for which the space is uniformly convex, i.e. $\forall\varepsilon\in(0,2)$

$$\delta(\varepsilon) = \inf\{1-\|\tfrac{1}{2}(x+y)\| \mid \|x\|=\|y\|=1,\ \|x-y\|\geqq\varepsilon\} > 0\,.$$

Moreover (cf. [9]) it is possible to choose a renorming for which the modulus of convexity $\delta(\varepsilon)$ is greater than $K\varepsilon^q$ for some constant $K>0$ and some $q<\infty$.

D. P. Giesy proved [3, th. II.9] that if Y and X/Y are B-convex then X is also B-convex; actually this also follows from (1) since it is known that a Banach space X is B-convex iff $a_n{}^X < \sqrt{n}$ for some integer n or iff $a_n{}^X/\sqrt{n}$ tends to 0 when n tends to infinity (cf. [8, exp. VII. p. 12–13]. The situation is quite similar in the case of super-reflexivity; the following proposition is used and discussed in [10].

Proposition 1: *Let X be a Banach space; the following conditions are equivalent:*

(i) *X is super-reflexive.*
(ii) *$\alpha_n{}^X < \sqrt{n}$ for some integer n.*
(iii) *$\alpha_n{}^X/\sqrt{n} \to 0$ when $n \to \infty$.*
(iv) *There exists a real number $p > 2$ such that*

$$\alpha_n{}^X/n^{1/p} \to 0 \quad \text{when } n \to \infty .$$

Remark 1. The equivalence of (ii), (iii) and (iv) can be easily deduced from (2) which becomes, when taking $Y = X$, $\alpha_{nk}{}^X \leqq \alpha_n{}^X \alpha_k{}^X$ (since $\alpha_n{}^{\{0\}} = 0$ for all $n \in \mathsf{N}$).

Theorem 2. *If a Banach space X has a closed subspace Y such that both Y and X/Y are super-reflexive then X is super-reflexive.*

Proof. From (2) we deduce: (since obviously $\alpha_n{}^X \leqq \sqrt{n}$ for all $n \in \mathsf{N}$)

$$\forall n \in \mathsf{N},\ \alpha_{n^2}{}^X \leqq n[\alpha_n{}^Y/\sqrt{n} + 2\alpha_n{}^Y/\sqrt{n}\cdot \alpha_n{}^{X/Y}/\sqrt{n} + 2\alpha_n{}^{X/Y}/\sqrt{n}] .$$

If Y and X/Y are super-reflexive, then (proposition 1) $\alpha_n{}^Y/\sqrt{n} \to 0$ and $\alpha_n{}^{X/Y}/\sqrt{n} \to 0$ when $n \to \infty$; hence when n is sufficiently large we must have $\alpha_{n^2}{}^X < n$ which implies (proposition 1) that X itself is super-reflexive.

We will now focus our attention on the case where both Y and X/Y are isomorphic to a Hilbert space. The sequences $(a_n{}^X)_{n \geqq 1}$ and $(\alpha_n{}^X)_{n \geqq 1}$ give information on the isomorphic structure of the Banach space X. For instance, S. Kwapien has proved in [5] that $\sup_{n \geqq 1} a_n{}^X a_n{}^{X^*}$ is finite if and only if the Banach space X is isomorphic to a Hilbert space. Also, it is proved in [10] (see also [9]) that $\sup_{n \geqq 1} \alpha_n{}^X$ is finite if and only if the Banach space X has an equivalent norm $|\ |$ for which the modulus of smoothness

$$\varrho(t) = \sup \{\tfrac{1}{2}(|x+ty| + |x-ty|) - 1 \mid x,y \in X,\ |x| = |y| = 1\}$$

satisfies $\varrho(t) \leqq Kt^2$ for all $t > 0$, for some constant K.

The Banach spaces X for which $\sup_{n\geqq 1} a_n^X$ is finite are usually referred to as spaces of type 2.

THEOREM 3. *Let X be a Banach space and Y a closed subspace of X.*

a) *If both spaces Y and X/Y are of type 2 (i.e. both $\sup_{n\geqq 1} a_n^Y$ and $\sup_{n\geqq 1} a_n^{X/Y}$ are finite) then there exist constants c and α such that:*

$$\forall n \geqq 2, \quad a_n^X \leqq c(\operatorname{Log} n)^\alpha .$$

b) *If both $\sup_{n\geqq 1} \alpha_n^X$ and $\sup_{n\geqq 1} \alpha_n^{X/Y}$ are finite then there exist constants c and α such that:*

$$\forall n \geqq 2, \quad \alpha_n^X \leqq c(\operatorname{Log} n)^\alpha .$$

PROOF. Let $c_1 = \sup_{n\geqq 1} a_n^Y$, $c_2 = \sup_{n\geqq 1} a_n^{X/Y}$; the inequality (1) yields:

$$\forall n, k \in \mathsf{N}, \quad a_{nk}^X \leqq c_1 a_k^X + c_1 c_2 + a_n^X c_2 ;$$

since obviously (unless $X = \{0\}$), $1 \leqq a_n^X$ for all integers n, we obtain in particular:

$$\forall n \in \mathsf{N},\ a_{n^2}^X \leqq (c_1 + c_1 c_2 + c_2) a_n^X . \tag{7}$$

Let α be such that $2^\alpha = c_1 + c_1 c_2 + c_2$ and set $b_n = a_n^X / (\operatorname{Log} n)^\alpha$ for all $n = 2, 3, \ldots$; then (7) becomes:

$$\forall n \in \mathsf{N}, \quad b_{n^2} \leqq b_n . \tag{8}$$

Let n be an integer, $n \geqq 2$; there exist $k \geqq 0$ such that:

$$N_k = 2^{2^k} \leqq n < 2^{2^{k+1}} = N_k^2 .$$

Since $(a_n^X)_{n\geqq 1}$ is clearly increasing, we can write:

$$b_n = a_n^X / (\operatorname{Log} n)^\alpha \leqq a_{N_k^2}^X / (\operatorname{Log} n)^\alpha \leqq 2^\alpha a_{N_k^2}^X / (\operatorname{Log} N_k^2)^\alpha = 2^\alpha b_{N_k^2} ;$$

from (8) it follows that $\forall k \geqq 0$ $b_{N_k} \leqq b_{N_0} = b_2$, hence $b_n \leqq 2^\alpha b_2$ for all integers $n \geqq 2$; this completes the proof of (a). It is clear that the proof of (b) is entirely similar.

COROLLARY 1. *In the situation of theorem 2, if each of Y and X/Y is isomorphic to a Hilbert space, then for all $p < 2$ there exists a constant $c_p > 0$ such that:*

$$c_p^{-1} \left(\sum \|x_n\|^{p'}\right)^{1/p'} \leqq \left\|\sum \varepsilon_n x_n\right\|_2 \leqq c_p \left(\sum \|x_n\|^p\right)^{1/p}$$

for all finite sequences (x_n) in X.

PROOF. The assumptions imply that Y and X/Y are of type 2, and also that $X^*/Y^\perp$ and $Y^\perp$ are of type 2. By theorem 3.a we have:

$$\forall n \in \mathbb{N}:\ a_n^X \leqq c(\mathrm{Log}\, n)^\alpha,\ a_n^{X^*} \leqq c(\mathrm{Log}\, n)^\alpha ,$$

for some constants c and α. By a known argument (see [8, exp. 7, p. 5]) one can prove that for all $p<2$ there exists a constant c_p such that:

$$\|\sum \varepsilon_n x_n\|_2 \leqq c_p(\sum \|x_n\|^p)^{1/p}$$

for all finite sequences (x_n) in X and

$$\|\sum \varepsilon_n x_n^*\|_2 \leqq c_p(\sum \|x_n^*\|^p)^{1/p}$$

for all finite sequences (x_n^*) in X^*. The conclusion follows then from an argument of duality.

COROLLARY 2. *If each of the spaces Y and X/Y is isomorphic to a Hilbert space, then for all $p<2$ there exists an equivalent renorming of X for which the modulus of smoothness ϱ satisfies $\forall t>0$: $\varrho(t) \leqq K_p t^p$, for some constant K_p; moreover, for all $q>2$ there exists an equivalent renorming of X for which the modulus of convexity δ satisfies $\forall \varepsilon \leqq 2$: $\delta(\varepsilon) \geqq K_q \varepsilon^q$, for some constant $K_q>0$.*

PROOF. The assumptions imply, using theorem 3.b, that there exist constants c and α such that:

$$\forall n \in \mathbb{N}:\ \alpha_n^X \leqq c(\mathrm{Log}\, n)^\alpha \quad \text{and} \quad \alpha_n^{X^*} \leqq c(\mathrm{Log}\, n)^\alpha .$$

As proved in [10], [9], this is sufficient to imply the conclusions of corollary 2.

REMARK 2. It is proved in [6] that if a Banach space X has an equivalent norm for which the modulus of smoothness ϱ satisfies $\forall t>0$: $\varrho(t) \leqq Kt^2$ and an equivalent norm for which the modulus of convexity δ satisfies $\forall \varepsilon \in (0,2)$: $\delta(\varepsilon) \geqq L\varepsilon^2$, for some constants K and $L>0$, then X is isomorphic to a Hilbert space.

REMARK 3. A Banach space is called of type p if there exists a constant c such that:

$$\left(\int \|\sum \varepsilon_n(t) x_n\|^p\, dt\right)^{1/p} \leqq c(\sum \|x_n\|^p)^{1/p}$$

for all finite sequences (x_n) in X; let us call briefly p-smooth a Banach space for which there is an equivalent norm such that the modulus of smoothness ϱ satisfies $\forall t>0$: $\varrho(t) \leqq Kt^p$, for some constant K.

If in the definitions of a_n^X and α_n^X we replace 2 by a number p in $(1,2)$, then clearly Theorem 1 is still valid. This can be used to prove in an entirely similar way as the preceding lines: If X has a closed subspace Y such that both Y and X/Y are of type p (respectively are p-smooth) then X is of type q (respectively is q-smooth) for all $q<p$.

Remark 4. C. Stegall proved that if both $[X/Y]^*$ and Y^* have the Radon–Nikodym property then X^* also has the Radon–Nikodym property [11, corollary 6]. We mention this result because (cf. [9]) super-reflexivity happens to be equivalent to the super-Radon–Nikodym property.

4. The counterexample to Palais' problem.

We turn now to a construction of an example which shows that if Y and X/Y are both Hilbert spaces X itself need not be a Hilbert space.

We start by mentioning an elementary numerical inequality which we shall need in the sequel. Let t and s be real numbers and consider the complex numbers $u=1+is$, $v=1+it$. Then

$$\begin{aligned}|t(1+t^2)^{-\frac12}-s(1+s^2)^{-\frac12}|^2 &= (\mathrm{Imag}(u/|u|-v/|v|))^2\\ &\leqq |u/|u|-v/|v||^2 = 2-2\,\mathrm{Rea}\,u\bar{v}/|u|\cdot|v|\\ = 2((1+t^2)^{\frac12}(1+s^2)^{\frac12}-(1+ts))/|u|\cdot|v| &\leqq 2((1+t^2)^{\frac12}(1+s^2)^{\frac12}-(1+ts))\,.\end{aligned} \tag{9}$$

We define now a class B_n of functions from the n dimensional real Hilbert spâce $l^2{}_n$ into the infinite-dimensional Hilbert space l^2. These functions are defined so as to resemble linear operators. The main point in the construction below is to show that if n is large there are however functions in B_n whose distance (in a natural definition of such a notion) from the set of linear operators is large.

Definition. Let n be an integer. A function $f\colon l^2{}_n \to l^2$ is said to belong to the class B_n if

$$f(\lambda x) = \lambda f(x), \quad x\in l^2{}_n,\ \lambda \text{ real} \tag{10}$$

and

$$\|\textstyle\sum_{i=1}^k f(x_i)\| \leqq \sum_{i=1}^k \|x_i\| \tag{11}$$

whenever $\{x_i\}_{i=1}^k \subset l^2{}_n$ are such that $\sum_{i=1}^k x_i = 0$.

Clearly every linear operator belongs to B_n. The next lemma enables an inductive construction of members of B_n whose non-linearity increases with n.

LEMMA 1. *Let n be a positive integer and let $f \in B_n$. Then the map g: $l^2{}_{2n} \to l^2$ defined by*

$$(12) \qquad g(x,y) = (f(x), f(y), x \cdot \|y\|/(\|x\|^2 + \|y\|^2)^{\frac{1}{2}}), \quad x,y \in l^2{}_n$$

belongs to B_{2n}.

In (12) the pair (x,y) denotes an element in $l^2{}_{2n} = l^2{}_n \oplus l^2{}_n$ (the direct sum in the Hilbert sense). Similarly the element in the right hand side of (12) determines in an obvious way an element in l^2. The third component in the right hand side of (12) is taken as 0 if $x=y=0$.

PROOF. It is trivial that g satisfies (10) and thus we have only to check (11). Let $\{x_i\}_{i=1}^k$ and $\{y_i\}_{i=1}^k$ be elements in $l^2{}_n$ such that

$$(13) \qquad \sum_{i=1}^k x_i = \sum_{i=1}^k y_i = 0 .$$

Put

$$(14) \qquad \alpha_i = \|y_i\|/(\|x_i\|^2 + \|y_i\|^2)^{\frac{1}{2}} \quad i = 1 \ldots . k$$

(we assume as we clearly can that $\|x_i\|^2 + \|y_i\|^2 > 0$). By (11) (for the given f) and (13) we get that for any choice of the scalar c

$$(15) \qquad \begin{aligned} \|\textstyle\sum_i g(x_i, y_i)\|^2 &= \|(\textstyle\sum_i f(x_i), \sum_i f(y_i), \sum_i \alpha_i x_i - c \sum_i x_i)\|^2 \\ &= \|\textstyle\sum_i f(x_i)\|^2 + \|\sum_i f(y_i)\|^2 + \|\sum_i (\alpha_i - c) x_i\|^2 \\ &\leqq (\textstyle\sum_i \|x_i\|)^2 + (\sum_i \|y_i\|)^2 + (\sum_i |\alpha_i - c| \, \|x_i\|)^2 . \end{aligned}$$

Put now $c = \sum_i \alpha_i \|x_i\| / \sum_i \|x_i\|$. Then

$$(16) \qquad \begin{aligned} (\textstyle\sum_i |\alpha_i - c| \, \|x_i\|)^2 &\leqq \textstyle\sum_i \|x_i\| \sum_i \|x_i\| (\alpha_i - c)^2 \\ &= \textstyle\sum_i \|x_i\| \sum_i \|x_i\| (\alpha_i{}^2 + c^2 - 2\alpha_i c) \\ &= \textstyle\sum_i \|x_i\| \sum_i \|x_i\| \alpha_i{}^2 - (\sum_i \|x_i\|)^2 c^2 \\ &= \tfrac{1}{2} \textstyle\sum_i \sum_j \|x_i\| \, \|x_j\| (\alpha_i - \alpha_j)^2 . \end{aligned}$$

By (9), (14), (15) and (16) we get that

$$\begin{aligned} \|\textstyle\sum_i g(x_i, y_i)\|^2 &\leqq (\textstyle\sum_i \|x_i\|)^2 + (\sum_i \|y_i\|)^2 + \\ &\quad + \textstyle\sum_i \sum_j \left[((\|x_i\|^2 + \|y_i\|^2)(\|x_j\|^2 + \|y_j\|^2))^{\frac{1}{2}} - (\|x_i\| \, \|x_j\| + \|y_i\| \, \|y_j\|) \right] \\ &= \textstyle\sum_i \|x_i\|^2 + \sum_i \|y_i\|^2 + \sum_{i,j,\, i \neq j} ((\|x_i\|^2 + \|y_i\|^2)(\|y_i\|^2 + \|y_j\|^2))^{\frac{1}{2}} \\ &= (\textstyle\sum_i (\|x_i\|^2 + \|y_i\|^2)^{\frac{1}{2}})^2 = (\sum_i \|(x_i, y_i)\|)^2 \end{aligned}$$

and this concludes the proof of the lemma.

We introduce next a natural notion of the distance of a function $f: l^2_n \to l^2$ from the set of linear operators.

Definition. Let $f: l^2_n \to l^2$ be a bounded function. Put

$$D_n(f) = \inf_T \sup_{\|x\|=1} \|f(x) - Tx\| ,$$

where the infimum is taken over all linear operators $T: l^2_n \to l^2$. Put also

$$D_n = \sup\{D_n(f) \,;\, f \in B_n\} .$$

(Observe that every $f \in B_n$ is automatically bounded).

From Lemma 1 it is easy to get an estimate from below on the growth of D_n.

Lemma 2. *For every integer n we have*

$$D^2_{2n} \geqq D^2_n + 1/16 .$$

Proof. Let $\varepsilon > 0$ and let $f \in B_n$ be such that $D_n(f) > D_n - \varepsilon$. Let $g \in B_{2n}$ be the function given by (12). Let T be a linear operator from l^2_{2n} into l^2. In accordance with the decomposition of l^2_{2n} and l^2 into direct summands appearing in (12) we define six linear operators from l^2_n into l^2 by the relations

$$T(x,0) = (U_1 x, U_2 x, U_3 x)$$
$$T(0,y) = (V_1 y, V_2 y, V_3 y) .$$

By the definition of $D_n(f)$ there are x_0 and y_0 in l^2_n, both of norm 1, so that

$$(17) \qquad \|U_1 x_0 - f(x_0)\| > M_n - \varepsilon, \quad \|V_2 y_0 - f(y_0)\| > M_n - \varepsilon .$$

By considering the point $(x_0, 0)$ in l^2_{2n} we get that

$$(18) \quad D^2_{2n} \geqq D^2_{2n}(g) \geqq \|U_1 x_0 - f(x_0)\|^2 + \|U_3 x_0\|^2 \geqq (M_n - \varepsilon)^2 + \|U_3 x_0\|^2 .$$

Consider next the points $(x_0, \pm y_0)/\sqrt{2}$ in l^2_{2n}. We have by (10) and (12) that

$$g(x_0/\sqrt{2}, \pm y_0/\sqrt{2}) - T(x_0/\sqrt{2}, \pm y_0/\sqrt{2})$$
$$= (f(x_0) - U_1 x_0,\ -U_2 x_0,\ x_0/\sqrt{2} - U_3 x_0)/\sqrt{2} \mp$$
$$\mp (V_1 y_0, V_2 y_0 - f(y_0), V_3 y_0)/\sqrt{2} .$$

Since for every two vectors z and w in l_2 there is a sign θ such that $\|z+\theta w\|^2 \geqq \|z\|^2+\|w\|^2$ we get that

$$(19) \quad D^2_{2n} \geqq D^2_{2n}(g) \geqq \tfrac{1}{2}(\|f(x_0)-U_1x_0\|^2+\|U_2x_0\|^2+\|x_0/\sqrt{2}-U_3x_0\|^2+\|V_2y_0\|^2+ + \|V_2y_0-f(y_0)\|^2+\|V_3y_0\|^2) \geqq (M_n-\varepsilon)^2+\|\tfrac{1}{2}x_0-U_3x_0/\sqrt{2}\|^2 .$$

One of the numbers $\|U_3x_0\|$ and $\|\frac{1}{2}x_0-U_3x_0/\sqrt{2}\|$ must be larger than $\frac{1}{4}$. Since ε was arbitrary the lemma follows by comparing (18) with (19).

Corollary. *There is a constant $C>0$ so that $D_n \geqq C(\operatorname{Log} n)^{\frac{1}{2}}$.*

For the construction below it is convenient and also of interest to note that the fact that the range of the functions in B_n was allowed to be the infinite-dimensional Hilbert space l^2 was not really used. We could just as well have defined B_n by considering maps from $l^2{}_n$ into $l^2{}_{n^2}$.

Let n be an integer, let $f\colon l^2{}_n \to l^2{}_{n^2}$ be an element of B_n and let $\|\ \|$ denote the usual inner product norm in $l^2{}_n$ and $l^2{}_{n^2}$. In the direct sum $Z_n = l^2{}_n \oplus l^2{}_{n^2}$ we introduce a norm $|||\cdot|||$ by taking as its unit ball the closed convex hull of all the points of the form $(0,y)$ with $\|y\|\leqq 1$ and all the points of the form $(x,f(x))$ witn $\|x\|\leqq 1$. The subspace of Z_n of all the points of the form $(0,y)$ is denoted by Y_n. With these notations we have

Proposition 2. *The space Y_n is isometric to $l^2{}_{n^2}$. The space Z_n/Y_n is isometric to $l^2{}_n$. Any linear projection of Z_n onto Y_n has norm $\geqq D_n(f)$.*

Proof. Whenever $\|y\|\leqq 1$ the point $(0,y)$ is in the unit ball of Z_n and hence $|||(0,y)|||\leqq 1$. Assume conversely that $|||(0,y)|||<1$. Then there is a $y_0 \in l^2{}_{n^2}$ and $\{x_i\}_{i=1}^n \in l^2{}_n$ so that

$$\textstyle\sum_i x_i = 0, \quad y_0+\sum_i f(x_i) = y, \quad \|y_0\|+\sum_i \|x_i\| \leqq 1 .$$

Hence, by (11)

$$\|y\| \leqq \|y_0\|+\|\textstyle\sum_i f(x_i)\| \leqq \|y_0\|+\sum_i \|x_i\| \leqq 1 .$$

This proves the first statement of the proposition.

Consider now Z_n/Y_n. For every $x\in l^2{}_n$ we have

$$\inf_{y\in Y_n} |||(x,0)+y||| \leqq |||(x,f(x))||| \leqq \|x\| .$$

Also assume that $|||(x,0)+Y_n|||<1$. Since the first (i.e. the $l^2{}_n$) coordinate of the points in the unit ball of Z_n has $\|\cdot\|$ less or equal to 1 we get that $\|x\|\leqq 1$. This proves the second statement in the proposition.

Finally let P be a bounded linear projection from Z_n onto Y_n. Then $P(x,0) = (0, Tx)$ for some linear operator T from $l^2{}_n$ to $l^2{}_{n^2}$. Hence

$$P(x,f(x)) = P(x,0) + P(0,f(x)) = (0,Tx) + (0,f(x))$$

and thus

$$|||P||| \geqq \sup_{\|x\|=1} |||P(x,f(x))||| = \sup_{\|x\|=1} \|Tx + f(x)\| \geqq D_n f\,.$$

THEOREM 4. *There exists a Banach space Z and a subspace Y of Z so that Y and Z/Y are both isometric to l_2 but Z is not isomorphic to l_2.*

PROOF. By the Corollary to Lemma 2 we may choose for every integer n a map from $l^2{}_n$ to $l^2{}_{n^2}$ so that if $Z_n \supset Y_n$ are the spaces constructed above any projection from Z_n onto Y_n will have norm $\geqq C(\operatorname{Log} n)^{\frac{1}{2}}$. The spaces

$$Z = (\textstyle\sum_n \oplus Z_n)_2 \supset Y = (\textstyle\sum_n \oplus Y_n)_2$$

have the properties required in the statement of the theorem.

REMARK. If $1 < p < \infty$ the spaces $Z_p = (\sum_n \oplus Z_n)_p$ and $Y_p = (\sum_n \oplus Y_n)_p$ are examples of spaces such that Y_p and Z_p/Y_p are both isomorphic to l_p but Z_p is not an $\mathscr{L}_p$ space.

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