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ON CYCLIC CURVES IN THE EUCLIDEAN n-SPACE

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To Werner Fenchel on his 70th birthday.

1. Introduction.

Let $Q_1,Q_2,\ldots,Q_m$, $m<n$, be m fixed linearly independent points and P a variable point in a Euclidean n-space R^n. If P traverses a curve γ, the distances $r_i=|Q_iP|$ may be considered as functions of the arclength s of the curve γ. In this paper we shall study the real curves for which the squares of the distances r_i are polynomials in s of at most degree two. It means that m equations

$$(1.1) \qquad r_i^2 = a_i s^2 + 2b_i s + c_i, \quad i=1,2,\ldots,m ,$$

are satisfied. The coefficients a_i, b_i and c_i are real numbers.

If $m=1$ the curve is called *monocyclic*. If $m=2$ and if in addition the rank of the matrix

$$M_2 = \begin{pmatrix} 1 & 1 \\ a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}$$

is equal to two, γ is called *dicyclic*. The monocyclic curves in R^2 and R^3 and the dicyclic curves in R^3 have been studied in a previous paper [1].

If $m=3$ and if in addition the rank of the matrix

$$M_3 = \begin{pmatrix} 1 & 1 & 1 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}$$

is equal to three, the curve is called *tricyclic*.

In chapter I it will be proved that independently of the number of conditions (1.1) there exist in R^n only these three kinds of curves for which (1.1) are satisfied. More than three equations implies that the considered curve is restricted to lie in a subspace whose dimension decreases for increasing values of m.

In chapter II we consider the monocyclic curves. The properties of a monocyclic curve γ depend on the coefficient a_1 to s^2 and the determinant $D_1=a_1c_1-b_1^2$. We shortly examine the well-known class of plane mono-

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cyclic curves which in addition to the epi- and hypocycloids contains the pseudocycloids, the logarithmic spiral, the involute of a circle and the straight line. Next we state some properties of monocyclic curves in R^n, $n > 2$, which are mostly generalizations of properties of curves in R^3 (see [1, p. 23–28 and 36–45]).

Chapter III deals with the dicyclic curves. According as $a_1 \neq a_2$ or $a_1 = a_2$ the curve γ is said to belong to the class C_1 or C_2. A dicyclic curve γ is lying on a hypersurface of revolution Φ. The meridian curve of Φ is a Descartes' (plane) curve or a conic according as γ belongs to C_1 or to C_2. In both cases the meridian curve may be a circle and Φ consequently a hypersphere. The involutes of γ are lying on concentric hyperspheres or in parallel hyperplanes according as $\gamma \in C_1$ or $\gamma \in C_2$. Most of the properties of dicyclic curves in R^n, $n > 3$, are generalizations of properties of dicyclic curves in R^3 (see [1, p. 46–75]).

In chapter IV we examine the tricyclic curve which exists only in spaces of four or more dimensions. A tricyclic curve γ may be regarded as a dicyclic curve in a double infinity of ways, and γ is a common element of the two classes C_1 and C_2. A tricyclic curve is a helix lying on an $(n-2)$-dimensional manifold of revolution, the intersection between a hypersphere with center O and a hyperquadric of revolution with axis m where O does not lie on m. This manifold can be generated by a Descartes' space curve which turns about its plane of symmetry. Finally, it is shown that the projection of a tricyclic curve γ on a hyperplane perpendicular to the axis m is a dicyclic curve of class C_1, and we state a simple construction of a tricyclic curve in a space of four dimensions.

Chapter I. The three kinds of cyclic curves in R^n.

2. The space of poles.

Let γ denote a curve in R^n for which the equations (1.1) are satisfied, and let $P = P(s)$ be a point of γ. A point Q is called a *pole* for the curve γ if the square of the distance $r = |QP|$ may be expressed as a polynomial in s of at most degree two, i.e. there exists an equation

$$r^2 = as^2 + 2bs + c \tag{2.1}$$

between the distance r and the arclength s of γ. We will show that not only the points Q_i but all the points in the $(m-1)$-dimensional space $\Pi = \Pi^{m-1}$, spanned by the points Q_i, are poles of γ. The space Π is called the *space of poles*.

In order to prove (2.1) and find the coefficients a, b and c we remark that a point $Q \in \Pi$ may be determined by its *barycentric coordinates* with respect to the simplex (Q_i) with vertices Q_i. We choose an arbitrary origin O in R^n and put $\boldsymbol{q}=\overrightarrow{OQ}$ and $\boldsymbol{q}_i=\overrightarrow{OQ_i}$. Now it is known that if $Q \in \Pi$ the vector $\boldsymbol{q}$ is a linear combination of the vectors $\boldsymbol{q}_i$

$$\boldsymbol{q} = \sum \lambda_i \boldsymbol{q}_i , \tag{2.2}$$

where

$$\sum \lambda_i = 1 . \tag{2.3}$$

The set (λ_i) is the set of barycentric coordinates of Q with respect to (Q_i). The coordinates are independent of the choice of O.

We then use the generalized Stewart's formula

$$r^2 = \sum \lambda_i r_i^2 - \sum \lambda_i \lambda_k q_{ik}^2 , \tag{2.4}$$

where $q_{ik} = |Q_i Q_k|$. In the first summation on the right hand side of (2.4) i assumes the integers from 1 to m, in the second one each edge in the simplex (Q_i) shall occur exactly once. A proof of the formula (2.4) and some applications of it is given in [3].

If we now replace the squares r_i^2 in (2.4) by the expressions (1.1) we obtain an equation (2.1) where

$$a = \sum \lambda_i a_i \tag{2.5a}$$

$$b = \sum \lambda_i b_i \tag{2.5b}$$

$$c = \sum \lambda_i c_i - \sum \lambda_i \lambda_k q_{ik}^2 . \tag{2.5c}$$

Hence any point $Q \in \Pi$ is a pole for γ. The space of poles will be reduced to a point, a line or a plane according as $m=1$, 2 or 3.

3. The cyclic curves in R^n.

In this section we shall prove that a curve γ in R^n for which the equations (1.1) are satisfied, is a monocyclic, a dicyclic or a tricyclic curve, lying in R^n or in a (linear) subspace of R^n. For this purpose we eliminate s between the equations (1.1) choosing m real numbers $t_1, t_2, \ldots, t_m$ such that the sum

$$\sum t_i r_i^2 = \sum t_i a_i s^2 + 2 \sum t_i b_i s + \sum t_i c_i \tag{3.1}$$

is independent of s, i.e. the numbers t_i satisfy the equations

$$\sum t_i a_i = 0 \quad \text{and} \quad \sum t_i b_i = 0 . \tag{3.2}$$

Hence (3.1) is reduced to

$$\sum t_i (r_i^2 - c_i) = 0 . \tag{3.3}$$

Putting $p = \overrightarrow{OP}$ we get $r_i = |p - q_i|$ and (3.3) may be rewritten to

$$p^2 \sum t_i - 2p \cdot \sum t_i q_i + \sum t_i (q_i^2 - c_i) = 0 . \tag{3.4}$$

The curve γ is lying on any hypersurface which is represented by the equation (3.4). If

$$\sum t_i = 0 \tag{3.5}$$

and at least one number t_i (i.e. at least two numbers t_i) is different from zero, then the vector

$$n = \sum t_i q_i = \sum_{i=1}^{m-1} t_i (q_i - q_m) = \sum_{i=1}^{m-1} t_i \overrightarrow{Q_m Q_i} \tag{3.6}$$

is not a zero vector. Hence (3.4) represents a *hyperplane* with n as a normal vector. Consequently, to any non-trivial solution of the homogeneous system

$$\sum t_i = 0, \quad \sum a_i t_i = 0, \quad \sum b_i t_i = 0 \tag{3.7}$$

corresponds a hyperplane which contains the given curve γ.

Let p denote the *rank* of the matrix

$$M = M_m = \begin{pmatrix} 1 & 1 & \dots & 1 \\ a_1 & a_2 & \dots & a_m \\ b_1 & b_2 & \dots & b_m \end{pmatrix}$$

If $p = m$ the system (3.7) has only the trivial solution $(0, 0, \dots, 0)$, and according to the definitions in section 1 the curve γ for $m = 1, 2$ or 3 is monocyclic, dicyclic or tricyclic, respectively.

If $p < m$ there exist $m - p$ linearly independent solutions to (3.7) and hence $m - p$ linearly independent normal vectors n. The corresponding hyperplanes (3.4) intersect one another at a *subspace* U of $n - (m - p) = (n - m) + p$ dimensions. The curve γ is lying in this subspace.

The poles of γ lying in U are the points of the subspace $S = \Pi \cap U$. The equation (3.6) shows that the normal vectors n belong to the vector space of Π which implies that Π contains at least one normal space to U and, consequently, S contains at least one point. The dimension of S is $(m-1) + (n - m + p) - n = p - 1$, such that the space of poles S lying in U is a single point S^0, a line S^1 or a plane S^2, corresponding to $p = 1$, 2 or 3. We consider each of the three cases separately.

$p = 1$. The curve γ has the pole S^0 and is a *monocyclic* curve lying in a subspace U of $(n - m) + 1$ dimensions. The space of poles Π is the normal space to U at S^0.

$p=2$. We remark that if Q is a pole of γ and Q' its projection on U then Q' lies on S^1, and since

$$|PQ|^2-|PQ'|^2 = |QQ'|^2 = \text{const.}$$

we get for the point Q' an equation corresponding to (2.1), where the coefficients a and b are unchanged while c is replaced by another constant.

Since $p=2$ there exists a submatrix of M, say M_2, having the rank 2. The projections Q_1' and Q_2' on U of the poles Q_1 and Q_2 are then different points on S^1 and the corresponding matrix $M_2'=M_2$ has the rank 2. Hence the curve γ is a *dicyclic* curve lying in a subspace of $(n-m)+2$ dimensions with S^1 as the line of poles.

$p=3$. The poles of γ are lying in the plane S^2. Let M_3 denote a submatrix of M with rank 3. The projections Q_1', Q_2' and Q_3' on U of the poles Q_1, Q_2 and Q_3 are different points (in S^2), since the corresponding matrix $M_3'=M_3$. We will show that the poles Q_i' are linearly independent. If Q_3' lies on the line through Q_1' and Q_2' the equations (2.3), (2.5a) and (2.5b) for $m=2$ gives the relations

$$\begin{aligned} \lambda_1+\lambda_2 &= 1 \\ a_1\lambda_1+a_2\lambda_2 &= a_3 \\ b_1\lambda_1+b_2\lambda_2 &= b_3\,. \end{aligned}$$

These equations shall be satisfied for some (λ_1,λ_2) which implies that $\det M_3=0$ contrary to the assumption of M_3 being a regular matrix.

Thus we have found three linearly independent poles for γ in the space U for which the corresponding matrix $M_3'=M_3$ has the rank 3. Consequently the curve γ is a *tricyclic* curve lying in a subspace U of $(m-m)+3$ dimensions and with S^2 as the plane of poles.

It is seen that the *kind* of the curve γ for which the conditions (1.1) are satisfied only depends on the *rank* p of the matrix M, while the *dimension* of the subspace U in which γ is lying *depends on* p *and on the number* m *of equations* (1.1).

4. Change of poles and change of parameter.

A. *Change of poles*. Let $Q^1,Q^2,\ldots,Q^m$ denote m linearly independent points in the space of poles, where Q^j is determined by its barycentric coordinates $(\lambda^j{}_1,\lambda^j{}_2,\ldots,\lambda^j{}_m)$ with respect to the simplex (Q_i). Putting $r^j=|Q^jP|$ we find from (2.4)

$$(r^j)^2 = \sum \lambda^j{}_i r_i{}^2-\sum \lambda^j{}_i\lambda^j{}_k q_{ik}{}^2\,, \tag{4.1}$$

such that $(r^j)^2$ is a linear function of the squares r_i^2. Conversely, since the matrix $\Lambda = (\lambda^j{}_i)$ is regular, the squares r_i^2 can be expressed as linear functions of the squares $(r^j)^2$ Hence we may replace the simplex (Q_i) by the simplex (Q^j), replacing at the same time the system of conditions (1.1) by the system

$$(4.2) \qquad (r^j)^2 = a^j s^2 + 2b^j s + c^j ,$$

where the coefficients a^j, b^j and c^j may be found by means of the equations (2.5), λ_i being replaced by $\lambda^j{}_i$.

To the system (4.2) corresponds a matrix M^m analogous to M_m. It is easily shown that the matrices M_m, M^m and Λ are connected by the matrix equation

$$(4.3) \qquad M^m = M_m \Lambda .$$

Since Λ is regular the *rank* p is an *invariant* under the change of *poles*.

B. *Change of parameter*. If we in the equation (2.1) make the substitution $s = s^* + k$, we find the coefficients in the new expression for r^2

$$(4.4) \qquad a^* = a, \quad b^* = b + ka, \quad c^* = ak^2 + 2bk + c .$$

If we substitute in the m equations (1.1) we find $a_i^* = a_i$, and $b_i^* = b_i + ka_i$ which shows that the *rank* p of the matrix M is an *invariant* under the change of *parameter*.

Further it is seen that

$$(4.5) \qquad a^* c^* - b^{*2} = ac - b^2 ,$$

i.e. the *determinant* $D = ac - b^2$ is an *invariant* under the change of *parameter*.

In the following three chapters we examine the monocyclic, the dicyclic and the tricyclic curves, lying in an n-space R^n, where $n \geqq 2$, $n \geqq 3$ and $n \geqq 4$, respectively. It is assumed that the curves do not lie in proper subspaces of the considered R^n.

Chapter II. Monocyclic curves.

5. Basic equations. Central development.

In this chapter we examine the monocyclic curves in R^n, $n \geqq 2$, for which only one equation

$$(5.1) \qquad r^2 = as^2 + 2bs + c$$

is given. As before r denotes the distance from a fixed point Q to a variable point $P=P(s)$ on the curve γ.

In the preceding section 4 we have seen that the coefficient a, which is called the *modul* of the curve, and the determinant $D=ac-b^2$ are invariants under the change of parameter $s \to s^*+k$. Conversely, if two polynomials in s and s^* of degree two have the same coefficient a and the same determinant D any one of them may be transformed into the other by a substitution $s=s^*+k$. If $a \neq 0$ this is seen by rewriting

$$(5.2) \qquad r^2 = a(s+b/a)^2+D/a ,$$

and for $a=0$ (and $D=-b^2$) it is obvious.

The hypersphere with center Q and the equation

$$(5.3) \qquad r^2 = D/a, \quad (a \neq 0)$$

is called the *basic hypersphere* of the curve γ and is denoted $S(Q)$. Depending on the sign of D/a it may contain an infinity of real points, one real point or no real points. In the first case the hypersphere will touch γ at a point corresponding to $s=-b/a$.

In the equation (5.1) we may replace r by the vector $\boldsymbol{r}=\overrightarrow{QP}$ and by differentiation we get

$$(5.4) \qquad \boldsymbol{r}\cdot\boldsymbol{r}' = as+b .$$

By means of (5.2) we find

$$(5.5) \qquad a\boldsymbol{r}^2 = (\boldsymbol{r}\cdot\boldsymbol{r}')^2+D .$$

Conversely, integration of (5.5) gives a solution (5.1) with constants a_1, b_1 and c_1, where $a_1=a$ and $a_1c_1-b_1{}^2=D$, such that (5.5) may replace the equation (5.1).

Now, let M denote the projection of the pole Q on the tangent p to γ at P, and put $\mu=\angle(\boldsymbol{r},\boldsymbol{r}')$ (fig. 1). Since $|\boldsymbol{r}'(s)|=1$, we get

$$(5.6) \qquad MP = \boldsymbol{r}\cdot\boldsymbol{r}' = r\cos\mu ,$$

and using (5.5) we find the following relation between r and μ

$$(5.7) \qquad (a-\cos^2\mu)r^2 = D .$$

It may be noted that differentiation of (5.4) gives the equation

$$(5.8) \qquad \boldsymbol{r}\cdot\boldsymbol{r}'' = a-1 .$$

The stated equations (5.1)–(5.8) are all independent of the dimension n of the space in which γ is lying.

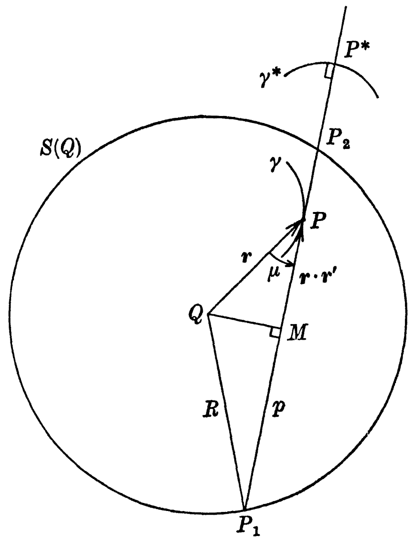

Fig. 1.

If $n \geqq 3$ the pole Q is vertex and the curve γ directrix of a conical surface which is called the *central cone* of γ. By development into a plane of this surface, the *central development*, the curve γ is mapped into a plane curve with unchanged distance r and arclength s, such that (5.1) is valid for the plane curve. Hence, the central development of a spatial monocyclic curve is a plane monocyclic curve, the two curves having the same constants a and D.

The reverse procedure may be applied to construct a monocyclic curve lying on an arbitrary cone, when a plane monocyclic curve is given. Since the plane monocyclic curves are well-known (see section 6) this may give some information on the form of a monocyclic curve in the space.

6. Plane monocyclic curves.

First we examine the kind of the curve for some special values of the modul a and the determinant D.

1°. $a=0$. The equations (5.4) and (5.6) show that $MP=b$ (fig. 1). If $b \neq 0$ the normal γ are tangents to a circle with center Q and radius $|b|$, and γ is consequently (an arc of) an *involute of a circle.* If $a=b=0$, then γ is an arc of a circle.

2°. $a=1$. From (5.7) and fig. 1 we find $QM=r\sin\mu=\sqrt{D}$, i.e. the tangents have a constant distance $\sqrt{D}$ from the pole Q. Since γ cannot lie on a circle, γ is (a segment of) a *straight line.* If $D=0$ the line passes through the pole Q.

3°. $0<a<1$, $D=0$. The equation (5.7) shows that the angle μ is constant, determined by $\cos\mu=\sqrt{a}$, and γ is a *logarithmic spiral* with Q as the pole. In this case the polynomial (5.1) may be rewritten to a perfect square such that the *distance r is a linear function of the arclength s.*

In order to determine the monocyclic plane curves for other values of a and D we remark that, according to (5.8), the vector $\boldsymbol{r}''\neq\boldsymbol{0}$, when $a\neq 1$. Hence the curvature has constant sign along the curve. In that case it is possible to determine γ by its supporting functional equation $QM=h(\theta)$, where θ is the angle from a fixed direction in the plane to the tangent to γ at P. It is well-known that $MP=h'(\theta)$, and instead of (5.5) we get the differential equation

$$(a-1)h'(\theta)^2+ah(\theta)^2 = D\,. \tag{6.1}$$

Referring to [1, p. 7] or [2, p. 30] we only write down the results of the integration of (6.1):

4°. $a<0$, $D<0$. The integral curves are *epicycloids.*

5°. $0<a<1$, $D\neq 0$. The integral curves are pseudocycloids, for $D<0$ *paracycloids*, for $D>0$ *hypercycloids.*

6°. $a>1$, $D>0$. The integral curves are *hypocycloids.*

From (5.7) it is seen that when γ is assumed to be a real curve then the combinations $a<0$, $D>0$ or $a>1$, $D<0$ cannot occur. As to the many properties of this class of curves we refer to [1, p. 1–36].

7. Monocyclic curves in R^n, $n\geqq 3$.

In this section we are concerned with A) The special monocyclic curves in R^n for which $a=0$, $a=1$ or $D=0$, B) The radius of curvature for an arbitrary monocyclic curve γ, C) The involutes of γ, and D) The connection between γ and the basic hypersphere.

A. *The special curves.*

1°. $a=0$. Now the condition $MP=b$ $(b\neq 0)$ implies that the normal hyperplanes of γ touch a hypersphere with center Q and radius $|b|$. Conversely, if the normal hyperplanes of a curve γ touch a hypersphere, then an equation (5.4) where $a=0$ and $b\neq 0$ is valid, and by integration we find an equation (5.1), where $a=0$, i.e. a monocyclic curve with modul zero. — If $a=b=0$, the curve γ is lying on a hypersphere.

2°. $a=1$. Since the central development of γ is a line (section 6.2°), γ is a *geodesic* on the central cone. The constant distance $QM=\sqrt{D}$ implies that the *tangents to γ touch a hypersphere* with center Q and radius $\sqrt{D}$ which, since $a=1$, is the *basic hypersphere* $S(Q)$. The tangent surface of γ is then circumscribed about $S(Q)$ and touches it along a curve γ^*. This curve is orthogonal to the tangents of γ and consequently an *involute* of γ. Any other involute of γ must lie on a hypersphere with center Q and radius greater than $\sqrt{D}$.

Conversely, we will show that a curve γ whose tangent surface is circumscribed about a hypersphere S is monocyclic with modul $a=1$. The tangents touch S along an involute γ^* of γ. If P and P^* denote corresponding points of γ and γ^* we may put $|PP^*|=s+c$, and putting $|QP^*|=R$, the right triangle QP^*P gives the desired equation

$$r^2 = (s+c)^2+R^2 .$$

For curves in R^3 it is known that the normal planes of a curve γ are osculating planes of another curve γ_1, the locus of the centers of the osculating spheres of γ. From our considerations above it follows that if a family of planes are tangent planes to a sphere, then the orthogonal trajectories to the planes are monocyclic curves with modul $a=0$, while the edge of regression γ_1 is a monocyclic curve with modul $a=1$.

3°. $0<a<1$, $D=0$. The curve γ cuts the generators of the central cone at the same angle μ, determined by $\cos\mu=\sqrt{a}$, and γ is a *loxodrome* on the cone. Since an arbitrary loxodrome γ on a cone can be developed into a logarithmic spiral, it is a monocyclic curve with $D=0$.

B. *The radius of curvature.*

Let $\boldsymbol{n}$ denote a unit vector of the principal normal n and ϱ the radius of curvature at a point P of an arbitrary monocyclic curve γ in R^n. The equation (5.8) may be rewritten to

$$(\boldsymbol{r}\cdot\boldsymbol{n})/\varrho = a-1 . \tag{7.1}$$

If N denotes the projection of the pole Q on the normal $\boldsymbol{n}$ we have $\boldsymbol{r}\cdot\boldsymbol{n}=NP$ and consequently

$$PN/\varrho = 1-a , \tag{7.2}$$

i.e. the length of the projection of the vector $\overrightarrow{QP}$ on the principal normal to γ at P and the radius of curvature ϱ at P have a *constant ratio* when P traverses the curve. Since (7.2) is equivalent to (5.8), this property is characteristic for the monocyclic curves.

If $a=0$ we get $PN=\varrho$, and N is the center of curvature corresponding to the point P. In this case the locus of the centers of curvature is the pedal curve of the pole with respect to the principal normals of γ.

If $a=1$ the point N lies in P and the principal normal $\boldsymbol{n}$ is perpendicular to the generator QP of the central cone. Hence $\boldsymbol{n}$ is a normal to the tangent plane of the cone at P, in accordance with γ being a geodesic on the cone.

When $D=0$ we have $a=\cos^2\mu$, and (7.2) gives $PN=\varrho\sin^2\mu$. If $n=2$ this equation leads to a well-known construction of the center of curvature of a logarithmic spiral.

C. *The involutes.*

We consider for $a\neq 0$ the involute γ^* of γ which is represented by the parametric equation

$$\overrightarrow{QP^*} = \boldsymbol{r}^* = \boldsymbol{r}(s)-(s+b/a)\boldsymbol{r}'(s) , \tag{7.3}$$

and will prove that corresponding points P and P^* on γ and γ^* are *conjugate* with respect to the *basic hypershere* $S(Q)$. For this purpose we calculate the scalar product

$$\boldsymbol{r}\cdot\boldsymbol{r}^* = \boldsymbol{r}^2-(s+b/a)(\boldsymbol{r}\cdot\boldsymbol{r}') . \tag{7.4}$$

If we in (7.4) replace $\boldsymbol{r}^2=r^2$ and $\boldsymbol{r}\cdot\boldsymbol{r}'$ by the expressions (5.1) and (5.4) we get

$$\boldsymbol{r}\cdot\boldsymbol{r}^* = D/a , \tag{7.5}$$

which proves the theorem.

If $S(Q)$ is a real hypersphere the curves γ and γ^* have the point in common at which γ touches $S(Q)$. When γ is a *plane* curve — and only in this case — the point of contact is a vertex on γ, and the involute γ^* is a monocyclic curve with the same modul as γ (see [1, p. 11]). In the case $a=1$ the tangents to γ touch $S(Q)$, and γ^* is the involute considered in A,2°. If $D=0$ the triangle PQP^* is right at Q, a property known for the logarithmic spiral.

Now, let γ_1 denote an arbitrary involute of a monocyclic curve γ and P and P_1 corresponding points on the two curves. The tangent t_1 to γ_1 at P_1 is parallel to the principal normal n to γ at P, and consequently the length of the projection N_1P_1 of $\overrightarrow{QP_1}$ on t_1 is equal to the length of the projection NP of $\overrightarrow{QP}$ on n. According to (7.2) we get

$$P_1N_1 = \varrho(1-a) .$$

If γ has *constant radius of curvature* ϱ, P_1N_1 has constant length and the normal hyperplanes to γ_1 through P_1 touch a hypersphere with center Q and radius $\varrho|1-a|$. Hence the involute γ_1 is a monocyclic curve with modul $a_1=0$. The circular helix in R^3 is an example on a curve of this kind. Its involutes are involutes of a circle, i.e. (plane) monocyclic curves with modul zero.

D. *The basic hypersphere.*

Let γ denote a monocyclic curve for which $a \neq 0$ or 1, and let P_1 and P_2 be the (real or imaginary) points where the tangent p to γ at P intersects the basic hypersphere $S(Q)$. We will prove that the *ratio* PP_1/PP_2 *is independent of the position of* P *on the curve.*

Whether P_1 and P_2 are real or imaginary we have (fig. 1)

$$|MP_1|^2 = D/a - r^2 \sin^2 \mu .$$

Replacing D by the expression (5.7) we get

$$|MP_1|^2 = (1-a^{-1})r^2 \cos^2 \mu$$

and hence by (5.6)

$$\frac{|MP_1|^2}{|MP|^2} = \frac{a-1}{a} .$$

Since M is the midpoint of the segment P_1P_2 the equation (7.6) shows that the ratio $f=PP_1/PP_2$ only depends on the modul a of the curve and is independent of the position of P on γ.

If $a>1$ or $a<0$ the points P_1 and P_2 are real points, and the ratio f which easily may be expressed by a, is a real number. If $0<a<1$ the points P_1 and P_2 are imaginary, and f is a complex number.

The common property of the monocyclic curves expressed by the theorem above is well-known for the epicycloids and the hypocycloids, and it is a characteristic property for the monocyclic curves in R^n for which $a \neq 0$ and $a \neq 1$. In order to define this class of curves in the plane and in the space R^3 the above mentioned property was heading the quoted paper [1], and herein applied to give a common treatment of the cycloids and the pseudocycloids in the plane and the corresponding curves in R^3.

Chapter III. Dicyclic curves.

8. The line of poles.

For a dicyclic curve in R^n, $n \geqq 3$, two equations

$$r_1^2 = a_1 s^2 + 2b_1 s + c_1 \tag{8.1a}$$

$$r_2^2 = a_2 s^2 + 2b_2 s + c_2 \tag{8.1b}$$

are given, where r_1 and r_2 denote the distances from the poles Q_1 and Q_2 to the variable point $P = P(s)$ on the curve γ. It is assumed that the rank of the matrix

$$M_2 = \begin{pmatrix} 1 & 1 \\ a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}$$

is equal to two.

The space of poles (section 2) is reduced to the *line of poles* through Q_1 and Q_2. To each point Q on that line there corresponds an equation

$$r^2 = as^2 + 2bs + c\ , \tag{8.2}$$

where $r = |QP|$, and a, b and c can be determined by means of the results in section 2. Let the line of poles be chosen as an x-axis and the abscissae of Q, Q_1 and Q_2 be denoted q, q_1 and q_2. Then the equation (2.2) may be replaced by

$$q = \lambda_1 q_1 + \lambda_2 q_2$$

and (2.3) by

$$\lambda_1 + \lambda_2 = 1\ . \tag{8.3}$$

Using the equations (2.5) we get

$$a = \lambda_1 a_1 + \lambda_2 a_2 \tag{8.4a}$$

$$b = \lambda_1 b_1 + \lambda_2 b_2 \tag{8.4b}$$

$$c = \lambda_1 c_1 + \lambda_2 c_2 - \lambda_1 \lambda_2 (q_2 - q_1)^2\ . \tag{8.4c}$$

The curve γ may be regarded as a monocyclic curve with an arbitrary point Q on the line of poles as its pole and with constants a and $D = ac - b^2$ which depend on Q, i.e. on (λ_1, λ_2).

Let x denote the abscissa of the projection of P on the x-axis. Then the equation

$$r_1^2 - r_2^2 = (x - q_1)^2 - (x - q_2)^2$$

is valid, and applying (8.1) we get

$$2(q_2 - q_1)x = (a_1 - a_2)s^2 + 2(b_1 - b_2)s + (c_1 - c_2) - (q_1^2 - q_2^2) \tag{8.5}$$

and hence

$$\frac{dx}{ds} = \cos v = \frac{a_1 - a_2}{q_2 - q_1} s + \frac{b_1 - b_2}{q_2 - q_1}, \tag{8.6}$$

where v denotes the angle from the x-axis to the tangent of γ at P.

If we develop the cylinder through γ whose generators are parallel to the line of poles (the x-axis), then γ is mapped into a plane curve for which the abscissa x of a point and the corresponding value of s are connected by the equation (8.5). The equation (8.6) shows that the developed curve is a simple *cycloid* if $a_1 \neq a_2$ and a *straight line* if $a_1 = a_2$.

The dicyclic curves fall into two classes C_1 and C_2 with different properties according as $a_1 \neq a_2$ or $a_1 = a_2$. In the following two sections we state the main properties of curves belonging to each of the two classes.

9. The class C_1.

The equations (8.3) and (8.4a) show that if $a_1 \neq a_2$ the coefficient a in (8.2) assumes *any value* when the pole Q traverses the line of poles. This implies that γ possesses all the properties which characterize the monocyclic curves with different moduls. Thus the central development of γ may be an arc of any plane monocyclic curve. To $a = 0$ and $a = 1$ correspond poles denoted Q' and Q''. Putting $r' = |Q'P|$ and $r'' = |Q''P|$ and normalizing s such that $b'' = 0$ we get the equations

$$r'^2 = 2b's + c' \tag{9.1a}$$

$$r''^2 = s^2 + c''. \tag{9.1b}$$

Referring to sections 7.A,1° and 7.B we find that the normal hyperplanes to γ touch a hypersphere with center Q' or go through Q' according as $b' \neq 0$ or $b' = 0$, and that the center of curvature corresponding to a point P on γ is the projection of Q' on the principal normal to γ at P.

The results in section 7.A,2° show that the tangents to γ touch the basic hypersphere $S(Q'')$ along an involute γ^* of γ and that any involute of γ is a hyperspherical curve. Moreover γ is a geodesic on the cone with vertex Q''.

Conversely, if the tangents to a curve γ touch a hypersphere and its normal hyperplanes touch another hypersphere not concentric with the first, two equations like (9.1) are valid, and γ is a dicyclic curve of class C_1.

If $b' \neq 0$ an elimination of s from the equations (9.1a–b) gives the equation

$$(r'^2 - c')^2 = 4b'^2(r''^2 - c''). \tag{9.2}$$

This equation represents a *hypersurface of revolution* Φ, on which γ is lying. The meridian curve μ is a *plane Descartes' curve* with Q' as the particular focus. The ordinary foci are the points F_i on the x-axis for which $D(\lambda_1, \lambda_2) = 0$ (cf. section 6.3°). Since this equation is of degree three in λ_1 and λ_2 there are at most three real foci, all of them lying on the segment $Q'Q''$. In case of three real foci the meridian curve μ is composed of two conjugate Descartes' ovals.

If $b' = 0$ the curve γ lies on the *hypersphere* Φ with the equation $r'^2 = c'$. Since $b' = 0$ the coefficient b in (8.2) is zero for any pole Q, and the equation $D = ac - b^2 = 0$ is reduced to $a = 0$ or $c = 0$. To $a = 0$ corresponds the center Q' of Φ, and $c = 0$ gives at most two real points F_i. — Whether $b' \neq 0$ or $b' = 0$ the curve γ is a loxodrome on the cone with vertex F_i.

10. The class C_2.

When $a_1 = a_2$ the equations (8.3) and (8.4a) show that the coefficient a in (8.2) has the same value $a = a_1 = a$ for any pole, and this number may be ascribed the dicyclic curve as its *modul*. If $a > 1$ and $a < 0$ the central developments of corresponding to different poles are arcs of similar hypo- or epicycloids, and for $0 < a < 1$ and unchanged sign of D we get arcs of similar pseudocycloids.

Since the rank of M_2 is two the condition $a_1 = a_2$ implies $b_1 \neq b_2$. The equation (8.6) shows that the angle v between the x-axis and the tangents to γ is constant and determined by

$$\cos v = (b_1 - b_2)/(q_2 - q_1) .$$

Hence γ is a *helix* with the x-axis (the line of poles) as *line of reference* [6]. Normalizing s the equation (8.5) may be written

$$x = s \cos v . \tag{10.1}$$

This equation and an equation

$$r^2 = as^2 + 2bs + c , \tag{10.2}$$

corresponding to an arbitrary pole Q, may replace (8.1a–b) such that a curve γ is dicyclic of class C_2 when (10.1) and (10.2) are satisfied. It is seen that a dicyclic curve of class C_2 can be regarded as a *monocyclic helix*.

Now we consider A) The hypersurface Φ on which γ is lying, B) The projection γ' of γ on a hyperplane perpendicular to the line of reference for γ, C) The special curves for which the modul $a = 0$ or $a = 1$.

A. *The hypersurface Φ.*

Let the origin O be the pole corresponding to (10.2), and let H denote the hyperplane $x=0$. The projection of a point P on H is called P' and we put $|OP'|=r'$. Since $P'P=x$ we have

$$r^2 = r'^2+x^2 . \tag{10.3}$$

If we eliminate s between (10.1) and (10.2) and make use of (10.3) we find an equation

$$r'^2 = Ax^2+2Bx+C , \tag{10.4}$$

where

$$A = \frac{a}{\cos^2 v}-1, \quad B = \frac{b}{\cos v}, \quad C = c . \tag{10.5}$$

(10.4) represents a hypersurface of revolution Φ with the x-axis as axis. A meridian μ in an xy-plane (through the x-axis) has the equation

$$y^2 = Ax^2+2Bx+C . \tag{10.6}$$

Hence Φ is a *hyperquadric of revolution* on which γ is lying. As above the foci of μ, situated on the x-axis, are determined by $D(\lambda_1,\lambda_2)=0$. Since a is a constant the equation is of degree two.

Conversely, if a helix γ belongs to a hyperquadric of revolution Φ with axis m such that m is a line of reference for γ, then γ is a dicyclic curve of class C_2. Let m be the x-axis and γ the helix determined by (10.1), and let Φ be given by the equation (10.4). Using (10.3) and (10.1) we return to an equation like (10.2) where the constants a, b and c may be found by means of (10.5). Hence γ is dicyclic of class C_2. Thus we have proved

THEOREM 1. *A helix γ is a dicyclic curve of class C_2 if and only if it lies on a hyperquadric of revolution Φ whose axis is a line of reference for γ.*

B. *The projection γ' of γ.*

We will prove that the projection γ' of the dicyclic curve γ (given by (10.1) and (10.2)) on the hyperplan H is a monocyclic curve. The equations (10.1-3) give

$$r'^2 = (a-\cos^2 v)s^2+2bs+c , \tag{10.7}$$

and since corresponding arclengths s and s' on the helix γ and its projection γ' are connected by the relation

$$s' = s\sin v \tag{10.8}$$

(see [6]), we find

$$r'^2 = a's'^2 + 2b's' + c' , \tag{10.9}$$

where

$$a' = \frac{a-\cos^2 v}{\sin^2 v}, \quad b' = \frac{b}{\sin v}, \quad c' = c . \tag{10.10}$$

Hence γ' is a monocyclic curve with the pole O and modul a'.

Conversely, if the projection γ' of a helix γ on a hyperplane perpendicular to the lines of reference for γ is a monocyclic curve, then γ is a dicyclic curve of class C_2. Let (10.9) be valid for γ'. By means of (10.8) we find r'^2 as function of s, and using (10.3) and (10.1) we recover an equation like (10.2) where the constants a, b and c may be found from (10.10). Hence γ is a dicyclic curve of class C_2 with the normal to H through O as the line of poles.

Corresponding to theorem 1 we have proved

THEOREM 2. *A helix γ with a line of reference m is a dicyclic curve of class C_2 if and only if its projection γ' on a hyperplane perpendicular to m is a monocyclic curve.*

The equation (10.5) and (10.10) show that the constants belonging to the dicyclic curve γ, its projection γ' and the hyperquadric Φ are connected by the relations

$$A \cos^2 v = a' \sin^2 v = a - \cos^2 v \tag{10.11a}$$

$$B \cos v = b' \sin v = b \tag{10.11b}$$

$$C = c' = c . \tag{10.11c}$$

If $A=0$ and $B \neq 0$ the meridian (10.6) is a parabola and (10.11a) gives $a'=0$ (and $a=\cos^2 v$). Thus we have obtained a generalization of the well-known theorem that the projection of a helix, lying on a paraboloid of revolution, on a plane orthogonal to the axis of the surface is an involute of a circle, when the axis is line of reference for the helix. If $A \neq 0$ the hyperplane H may be chosen as hyperplane of symmetry for Φ and we get in this case

$$B = b' = b = 0 .$$

C. *The special curves.*

1° $a=0$. We find $A=-1$ and $a'=-\cot^2 v$. Since $A=-1$ the meridian μ is a circle and Φ is a *hypersphere*. The central developments of γ are arcs of involutes of a circle apart from the case where the pole is placed in the center Q_0 of Φ. According to 7.A,1° the normal hyperplanes of γ touch an infinity of hyperspheres with centres on the line of poles and pass through Q_0.

For $n=3$ the curve is a spherical helix which is identical with a spherical involute of a circle. Since $a' = -\cot^2 v$ the projection γ' of γ is an arc of an epicycloid. For a dicyclic curve in R^n, $n>3$, with $a=0$ the projection γ' is a monocyclic curve whose central development is an arc of an epicycloid.

2°. $a=1$. According to 7.A,2° the curve is a geodesic on any of the central cones. Hence the principal normal at a point P of γ must be a common normal to the tangent planes at P to these cones, i.e. the planes through the tangent t to γ at P and the points on the line of poles. If $n=3$ this implies that γ is a straight line, and, if $n \geqq 4$, that the principal normal at P is perpendicular to the 3-dimensional subspace which is spanned by t and the line of poles.

For $a=1$ we have $a'=1$ and $A=\tan^2 v$. Only in this case the curve γ and its projection γ' have the same modul. γ' is a geodesic on its central cone which is the projection of any of the central cones of γ.

Since $A=\tan^2 v$ the lines on the hyperquadric Φ will be parallel to the line of its asymptotic hypercone for which a generator in the xy-plane has the equation $y = x \tan v$. The lines on Φ may be considered as singular dicyclic curves of class C_2 with modul $a=1$. For $n=3$ the surface Φ is an ordinary hyperboloid of revolution with one sheet (or a cone of revolution), and there exist no other dicyclic curves on Φ with $a=1$ than these lines, but if $n \geqq 4$ there exist ordinary (non-linear) dicyclic curves with modul $a=1$ lying on Φ. This will be proved in the next chapter.

Again referring to 7.2° we note that any tangent to γ is a common tangent to all the basic hyperspheres belonging to γ, and the points of contact with a hypersphere $S(Q)$ is an involute γ^* of γ. Since γ is a helix the involute γ^* is also lying in a hyperplane H perpendicular to a line of reference [6], and consequently γ^* lies on the $(n-2)$-dimensional sphere $S'(Q)$ in which H intersects $S(Q)$. In the hyperplane H the curve γ^* will be an involute of the projection γ' of γ on H.

Chapter IV. Tricyclic curves.

11. The plane of poles.

A curve γ in R^n, $n \geqq 4$, is called *tricyclic*, when three equations

(11.1a) $$r_1^2 = a_1s^2+2b_1s+c_1$$

(11.1b) $$r_2^2 = a_2s^2+2b_2s+c_2$$

(11.1c) $$r_3^3 = a_3s^2+2b_3s+c_3$$

are valid. r_1, r_2 and r_3 denote the distances from three linearly independent points Q_1, Q_2 and Q_3 to the point $P=P(s)$ on γ. It is assumed that the rank of the matrix

$$M_3 = \begin{pmatrix} 1 & 1 & 1 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}$$

is equal to three.

The space of poles is reduced to the *plane of poles* Π through the points Q_i, and the simplex (Q_i) is the triangle $Q_1Q_2Q_3$. To each point $Q \in \Pi$ corresponds an equation

(11.2) $$r^2 = as^2+2bs+c\ ,$$

where $r=|QP|$ and a, b and c can be determined by means of the results in section 2. For $n=3$ the equation (2.2) and (2.3) give

(11.3) $$\boldsymbol{q} = \lambda_1\boldsymbol{q}_1+\lambda_2\boldsymbol{q}_2+\lambda_3\boldsymbol{q}_3$$

and

(11.4) $$\lambda_1+\lambda_2+\lambda_3 = 1\ ,$$

where $\boldsymbol{q}$ is the position vector and $(\lambda_1, \lambda_2, \lambda_3)$ the barycentric coordinates of Q with respect to the triangle $Q_1Q_2Q_3$. The equation (2.5) yields the desired expressions for a, b and c

(11.5a) $$a = \lambda_1a_1+\lambda_2a_2+\lambda_3a_3$$

(11.5b) $$b = \lambda_1b_1+\lambda_2b_2+\lambda_3b_3$$

(11.5c) $$c = \lambda_1c_1+\lambda_2c_2+\lambda_3c_3-\lambda_2\lambda_3q_{23}^2-\lambda_3\lambda_1q_{31}^2-\lambda_1\lambda_2q_{12}^2$$

where $q_{ik}=|Q_iQ_k|$.

To $a=b=0$ corresponds a point O with the barycentric coordinates $(\lambda_i)=(d_i/\det M_3)$, where d_i denote the cofactors of the elements in the first row of M_3. The equation (11.2) corresponding to the pole O is reduced to $r^2=c$, i.e. the *tricyclic curve* γ *is lying on a hypersphere* Ψ *with center* O.

To $a=0$ corresponds a line through O which is called the *principal line* (with respect to γ) and denoted p_0. To $a=1$ corresponds a line p_1 *parallel* to p_0. The equations (11.5) show that the points Q where $D=ac-b^2=0$ lie on a *cubic* φ. In order to study the properties of φ we introduce rectangular coordinates in Π and express a, b and c as functions of the coordinates (x,y) of Q.

We choose the point Q_3 at O, the vector $\overrightarrow{Q_3Q_1} = \overrightarrow{OQ_1} = \boldsymbol{q}_1$ as a unit vector on p_0 and $\overrightarrow{Q_3Q_2} = \overrightarrow{OQ_2} = \boldsymbol{q}_2$ as a unit vector orthogonal to $\boldsymbol{q}_1$ (fig. 2). We then find $a_3 = b_3 = 0$, $a_1 = 0$ and $a_2 \neq 0$. Since rank $M_3 = 3$ we get $b_1 \neq 0$ and normalizing s we may obtain $b_2 = 0$. Consequently we have

$$(11.6) \qquad a_1 = a_3 = 0, \quad a_2 \neq 0; \quad b_2 = b_3 = 0, \quad b_1 \neq 0 .$$

Since Q_3 is lying at O the vector $\boldsymbol{q}_3 = \mathbf{0}$, and (11.3) may be written

$$\boldsymbol{q} = \overrightarrow{OQ} = \lambda_1 \boldsymbol{q}_1 + \lambda_2 \boldsymbol{q}_2 ,$$

such that λ_1 and λ_2 may be regarded as rectangular coordinates of Q. Thus we can put

$$(11.7) \qquad \lambda_1 = x, \quad \lambda_2 = y, \quad \lambda_3 = 1 - x - y .$$

Moreover we find in the triangle $Q_3Q_1Q_2$ the sides

$$(11.8) \qquad q_{13} = q_{23} = 1, \quad q_{12} = \sqrt{2} .$$

By means of (11.8,7,6) we get for a, b and c the expressions

$$(11.9) \qquad a = a_2 y, \quad b = b_1 x, \quad c = x^2 + y^2 + 2c'x + 2c''y + c''' ,$$

where c', c'' and c''' are new constants.

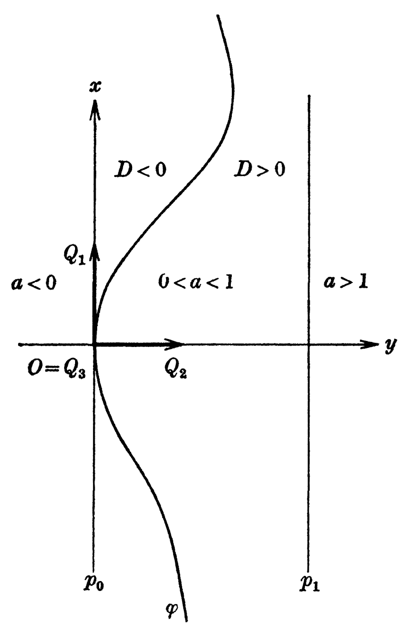

fig. 2.

The principal line p_0 is the x-axis while the line p_1 has the equation $y = 1/a_2$. Further we get

$$D = D(x,y) = a_2 y(x^2+y^2+2c'x+2c''y+c''')-b_1{}^2x^2 . \tag{11.10}$$

The equation (11.10) shows that the curve φ, determined by $D=0$, is a *circular cubic*. It is called the *focal cubic*. The center O lies on φ with the x-axis as tangent at O. Since $D=0$ never will occur when $a>1$ or $a<0$ (see e.g. equation (6.1)) the cubic lies in the domain limited by the lines p_0 and p_1, and its asymptote is parallel to these lines (fig. 2). The cubic may be bipartite. It is seen that $D(x,y)<0$ when $y=0$ and $x \neq 0$. Hence D is negative for any point Q apart from O, in the domain of the plane which is bounded by φ and contains the x-axis, and positive in the other part of the plane. If φ is bipartite, consisting of a branch of order three and and an oval, D must be negative in the interior of the oval.

The lines p_0 and p_1 and the curve φ divide the plane into domains corresponding to $a<0$, $D<0$, $a>1$, $D>0$ and $0<a<1$ with $D>0$ or $D<0$ (fig. 2). For any choice of the pole Q in one of these domains the central development of γ is an arc of a monocyclic curve for which the kind is determined in section 6. It may be noted that γ is a geodesic on any central cone with vertex on p_1 and a loxodrome on any central cone with vertex on φ.

12. Tricyclic curves regarded as dicyclic. Descartes' manifold.

It is easily proved that a tricyclic curve γ may be considered as dicyclic with an *arbitrary line m in Π* as line of poles.

Consider the line m through the poles Q_1 and Q_2 for which the equations (11.1a, b) are valid. Since the matrix M_3 has the rank 3 the submatrix consisting of the two first columns of M_3 has the rank 2, and γ may be regarded as a dicyclic curve with m as line of poles. Now, let m denote an arbitrary line in Π. The three poles Q_1, Q_2 and Q_3 may be replaced by any other three linearly independent poles in Π, the rank of M_3 being unchanged, and choosing Q_1 and Q_2 on m, it is obvious that γ is dicyclic with m as line of poles.

If m *intersects* then principal line p_0 the coefficient a in (11.2) assumes any real value, when the pole Q traverses m, and γ will belong to the class C_1 of dicyclic curves. If m is *parallel* to p_0 the coefficient a is constant, when Q traverses m, and γ belongs to the class C_2; any number may be considered as the modul of the curve. For the dicyclic curves in R^3 the classes C_1 and C_2 are disjoint, but for the dicyclic curves in R^n, $n \geq 4$, the classes are not disjoint, the tricyclic curve being a *common* *element of the two classes.* Consequently the tricyclic curve γ has *all the geometric properties* concerning its tangents, normal hyperplanes, involutes etc., which belong to the two classes of dicyclic curves as stated in sections 9 and 10, including the special properties mentioned in section 10.C,1° and 2°.

We will especially be concerned with the hypersurfaces on which γ is lying. As dicyclic curve of class C_1 the curve lies on a *double infinity of hypersurfaces of revolution* with axes *intersecting* p_0 and for which the meridians are (plane) *Descartes' curves.* For a given meridian μ the points at which the corresponding axis m intersects the focal curve φ are ordinary foci of μ, while m cuts p_0 at the particular focus. Moreover, as dicyclic curve of class C_2 the curve γ lies on a *single infinity of hyperquadrics of revolution* with axes *parallel* to p_0. The points at which m intersects φ are foci of the corresponding meridian conic μ. Finally the curve γ lies on the *hypersphere* Ψ with center O.

All these hypersurfaces have an $(n-2)$-dimensional algebraic manifold Δ in common on which γ is lying. If s is regarded as an arbitrary parameter, the equations (11.1) are parametric equations of Δ in tripolar coordinates. If r_1, r_2 and r_3 are restricted to denote distances in a 3-dimensional space R^3 through the plane of poles, the equations (11.1) are parametric equations of a space curve δ. The manifold Δ can be generated by *rotation of* δ about the plane Π. During this movement any point of δ traverses an $(n-3)$-dimensional sphere in an $(n-2)$-dimensional space normal to Π.

In order to examine the manifold Δ and the curve δ we choose the poles Q_1, Q_2 and Q_3 as in section 11 such that the equations (11.6) are valid. We then find

(12.1a) $$r_1^2 = 2b_1 s + c_1$$

(12.1b) $$r_2^2 = a_2 s^2 + c_2$$

(12.1c) $$r_3^3 = c_3 \,.$$

Let x denote the abscissa of the projection of the point $P=P(s)$ on the x-axis (the line p_0). Since

$$r_3^2 - r_1^2 = x^2 - (x-1)^2 = 2x - 1 \,,$$

it is seen that x is a linear function of s. Hence the equations (12.1) may be replaced by the system

(12.2a) $$x = s \cos v$$

(12.2b) $$r_2^2 = as^2 + 2bs + c$$

(12.2c) $$r_3^2 = R^2 \,,$$

where a new normalization of s has taken place and new notations of the constants have been introduced. A curve γ is tricyclic if the equations (12.2) are satisfied.

Elimination of s between (12.2a) and (12.2b) gives, as proved p. 244, the equation of a hyperquadric of revolution Φ with axis m through Q_2 and parallel to p_0. The manifold Δ is the intersection of Φ and the hypersphere Ψ with center O, corresponding to (12.2c). Hence the meridian curve δ, lying in the mentioned R^3, is the intersection between a *quadric of revolution* and a *sphere* with center O, where O does not lie on the axis m of the quadric. The curve δ is known as a *Descartes' space curve* (see [5], [7] and [8]), and the manifold Δ will be called a *Descartes' manifold* of revolution. The principal line p_0 (for γ) is said to be *principal line for the manifold* Δ and for its meridian δ. The line p_0 may, in relation to Δ, be characterized as the line through the center O of the hypersphere Ψ being parallel to the axis m of the hyperquadric Φ.

As dicyclic curve of class C_2 the tricyclic curve γ is a helix. It lies on a Descartes' manifold Δ with principal line p_0 where p_0 is a line of reference for γ. Conversely, we will prove, that if a helix γ lies on a manifold Δ such that the principal line for Δ is a line of reference for γ, then γ is a tricyclic curve.

Let Δ be the intersection between a hyperquadric of revolution Φ with axis m and a hypersphere Ψ with center O, where O does not lie on m. The axis m is parallel to the principal line for Δ and consequently a line of reference for γ. According to the Theorem 1 the curve γ is dicyclic of class C_2 with m as line of poles, and equations like (12.2a) and (12.2b) may be stated. Moreover γ is lying on Ψ such that an equation like (12.2c) holds. Hence γ is a tricyclic curve having the plane through the center O and the line m as plane of poles.

Corresponding to theorem 1 we have proved

THEOREM 3. *A helix γ is a tricyclic curve if and only if it lies on a Descartes' manifold of revolution Δ whoses principal line is a line of reference for γ.*

13. Projection and construction of a tricyclic curve.

Let γ' denote the projection of the tricyclic curve γ on the hyperplane H with the equation $x=0$, i.e. the hyperplane through O and perpendicular to the principal line p_0. The hyperplane H intersects the plane of poles Π at the y-axis (fig. 2). Since γ may be regarded as a dicyclic curve with an arbitrary line m parallel to p_0 as line of poles we find (section

10.B) that γ' is monocyclic with any point on the y-axis as pole. Now the equation (10.10) shows that the coefficient a', corresponding to γ', assumes any real value when the coefficient a, corresponding to γ, traverses the real numbers. Hence the projection γ' is a dicyclic curve of class C_1.

Conversely we prove that if the projection γ' of a helix γ on a hyperplane perpendicular to the lines of reference for γ is a dicyclic curve of class C_1, then the helix γ is a tricyclic curve.

Let m denote a line of reference, H the hyperplane containing the projection γ' of γ, and q the line of poles for γ'. Again referring to section 10.B we find that γ may be considered as a dicyclic curve of class C_2 with any line through a point of q and parallel to m as line of poles. Since γ' is assumed to be of class C_1, then a' assumes any real value when a pole traverses q, and the same property holds for the coefficient a. To $a=0$ corresponds a line p_0, and, as shown (section 10.C,1°), the curve γ lies on a hypersphere Ψ with center O on p_0. To $a=1$ corresponds a line p_1 which is the axis of a hyperquadric of revolution Φ on which γ lies. Hence γ is a helix lying on the intersection between Ψ and Φ, i.e. on a Descartes' manifold Δ, and using theorem 3 we find, that γ is a tricyclic curve.

Thus we have proved the following theorem 4 which is the analogue to theorem 2:

THEOREM 4. *A helix γ with a line of reference m is a tricyclic curve if and only if its projection γ' on a hyperplane perpendicular to m is a dicyclic curve of class C_1.*

By means of theorem 4 we state a simple construction of a tricyclic curve in a fourdimensional space R^4: Let γ' denote a dicyclic curve of class C_1 in a hyperplane R^3, and let $P'=P'(s')$, where s' denotes the arclength on γ', be a variable point on γ'. Now we lay out on the normal at P' to R^3 the segment $P'P=ks'$, where k is an arbitrary constant. When P' traverses γ', the point P will traverse a helix γ in R^4 with the normals to R^3 as lines of reference [6]. According to theorem 4 the helix is a tricyclic curve.

It may be noted that if γ' is a dicyclic curve of class C_2 then the construction does not lead to a tricyclic curve in R^4, but to another dicyclic curve of class C_2 lying in a hyperplane R'^3 and affinely connected with γ'. It is a consequence of the property of a dicyclic curve of class C_2 that the curve itself is a helix. This is a special case of a more general theorem proved in [6].

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